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COMPARATIVE ANALYSIS OF ML AND MAP DETECTORS FOR PAM CONSTELLATIONS IN AWGN CHANNEL

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Abstract. *In this paper we perform a comparative performances analysis of "maximum a posteriori" (MAP) and "maximum likelihood" (ML) detectors for one-dimensional constellation in the adaptive white Gaussian noise (AWGN) channel. More precisely, error probabilities per symbol for the aforementioned detectors are compared for the case when the pulse amplitude modulation (PAM) constellation with the equidistant and non-equiprobable constellation points is used as one-dimensional constellation. We perform analysis for different distributions of the constellation point probabilities and different values of the signal-to-noise ratio (SNR). The analysis indicates which detector can be adequate choice for the certain distribution of constellation point probabilities and the SNR. Besides this, for the straightforward performance assessment of the MAP detector we derive a formula for the symbol error probability. Our analysis also points out that the nonuniform distribution of the constellation points probabilities does not necessarily improve the symbol error probability. With the aim to decrease the symbol error probability we propose a method for defining constellation point probabilities. The presented results show that PAM constellation designed by utilizing the method we propose significantly outperforms the conventional PAM constellation in terms to the symbol error probability.*

Key words: *PAM constellation, AWGN channel, ML detector, MAP detector, symbol error probability*

1. INTRODUCTION

One classical issue in digital communications is estimation of the symbol error probability after transmission of digital signal through the additive white Gaussian noise (AWGN) channel [1]-[17]. This error probability mainly depends on the decision rule, that is, on the type of the detector. The decision rule that minimizes the probability of decision error is the "maximum a posteriori" (MAP) decision rule. However, the detector based on MAP decision rule requires the knowledge of a priori probabilities of symbols

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and also it has considerable complexity. From these reasons, the simpler detectors are desirable. One such simpler detector is the “maximum likelihood” (ML) detector that does not take into consideration the a priori probabilities of symbols [1], [2]. Although the topic on MAP and ML detection has been studied for a long time, it has not still completely investigated largely because of the complexity of the MAP detection. Thus, the comparison of ML and MAP detectors for multidimensional constellations was recently performed in [3] pointing that the topic is still actual. Besides this, as there have been an increasing number of studies on the constellations with the non-equiprobable symbols [3]-[17], the research we perform here can be meaningful in detection of these constellations.

In this paper, for PAM constellations with the equidistant and non-equiprobable symbols we estimate and compare the symbol error probabilities of the MAP and ML detectors in order to obtain answer on question which detector is suitable for certain scenario. Also, we derive a formula for the symbol error probability of MAP detector. The formula we propose is useful for the performances estimation when signal is modulated with the non-equiprobable and equidistant PAM constellation and is transmitted through the AWGN channel.

The choice of the constellation point probabilities considerably influences the symbol error probabilities whereby one should notice that the the unadequate choice of the constellation point probabilities can considerably increase the symbol error probability. This issue motivates us to develop a method for determining constellation points probabilities, that lead to the significantly decreased symbol error probability. We will demonstrate that the PAM constellation based on method we propose outperforms some known PAM constellations in terms of the symbol error probability.

In order to present our research in a distinct and concise manner, in follows we firstly define the one-dimensional constellation. Then we focus on the signal reception by considering in detail the ML and MAP detectors. Finally, we propose the method for designing PAM constellation with the equidistant and non-equiprobable constellation points.

2. Symbol Error Probability For PAM Constellation in AWGN Channel

As already mentioned in the Introduction we investigate the performances of ML and MAP detectors for PAM constellations in AWGN channel. Before we define constellation, we briefly explain the considered scenario. Namely, the useful signal sent into the channel is $s(t)$. Due to transmission through the communication channel the signal is corrupted by the additive white Gaussian noise $n(t)$. Then, on the channel output, that is, at the detector input there is the signal $r(t)$ which represents the sum of the useful signal and the noise. Based on the sample of the received signal $r = r(kT) = s(kT) + n(kT) = a_i + n$, $k \in N_0$, the detector makes decision about the transmitted signal.

2.1. PAM constellation parameters

We consider M-ary PAM constellation with constant distance d between the amplitudes of adjacent constellation points a_i, $i = 1, 2,..., M$. We also assume that constellation is symmetric, which enables us to focus on the positive part of the constellation. By following these assumptions, we formulate the expression for the constellation point amplitude as follows:

$$a_i = \left(i - \frac{1}{2}\right)d \quad i = 1, ..., \frac{M}{2} . \tag{1}$$

The parameter we also define is the probability of constellation point marked with $P_i = P(a_i)$. We assume PAM constellation with non-equiprobable symbols, so that it holds

$$P_i \neq const , \tag{2}$$

under the constraint that the sum of probabilities of all constellation points is equal to 1:

$$2\sum_{i=1}^{M/2} P_i = 1 . \tag{3}$$

Finally, we define the average energy per symbol E_s in the following manner [1]:

$$E_s = 2\sum_{i=1}^{M/2} a_i^2 P_i . \tag{4}$$

After that we formulate expressions for the average energy per bit E_b [1]:

$$E_b = \frac{E_s}{\log_2 M} \tag{5}$$

and the signal-to-noise ratio per bit SNR [1]:

$$SNR = 10 \log \frac{E_b}{N_0} , \tag{6}$$

where N_0 is the spectral power density of the AWGN.

2.2. MAP and ML decision rules

The sample of the received signal is expressed analytically in the following manner:

$$r = a_i + n , \tag{7}$$

where a_i is the sample of the useful signal $s(t)$ and n is the sample of the AWGN noise $n(t)$ having Gaussian probability density function with zero-mean value and variance $\sigma_n^2 = N_0/2$. From (7) follows that for given a_i the sample r has also the Gaussian probability density function with the same variance as the noise variance, but with the mean value a_i [1]:

$$p(r|a_i) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(r - a_i)^2}{N_0}\right\}, \; -\infty < r < \infty . \tag{8}$$

The MAP rule that minimizes the error probability reads [1], [2]:

$$D(r) = a_i, \text{ if } \; p(a_i|r) \geq p(a_m|r) \text{ for } \forall m . \tag{9}$$

By applying Bayesian rule on (9) we obtain a simpler form for the decision rule:

$$p(r|a_i)P_i \geq p(r|a_m)P_m \text{ for } \forall m. \tag{10}$$

By using (10), (8) and (1) we derive the decision thresholds of MAP detector:

$$\begin{aligned} &m_i = (i-1)d + \frac{N_0}{2d}\ln\frac{P_{i-1}}{P_i},\ i = 2,...,\frac{M}{2}. \\ &m_1 = 0,\ m_{M/2+1} = +\infty \end{aligned} \tag{11}$$

Let us now define the ML decision rule [1], [2]:

$$D(r) = a_i,\ \text{if}\ \ p(r|a_i) \geq p(r|a_m) \text{ for } \forall m. \tag{12}$$

By substituting (8) and (1) into (12) we derive the decision thresholds of ML detector:

$$\begin{aligned} &m_i = (i-1)d,\ i = 2,...,\frac{M}{2}. \\ &m_1 = 0,\ m_{M/2+1} = +\infty \end{aligned} \tag{13}$$

By comparing equations (10) and (11) with equations (12) and (13) for $P_i = 1/M$ one can notice the well-known fact that in the case of equiprobable symbols the MAP and ML decision rules are equivalent. After determining decision thresholds the following general formulation for decision rules can be assumed

$$D(r) = a_i,\ \text{if}\ \ m_i \leq r \leq m_{i+1}. \tag{14}$$

To stress the difference between the ML and MAP decision rules, we graphically illustrate the decision thresholds in Fig. 1.

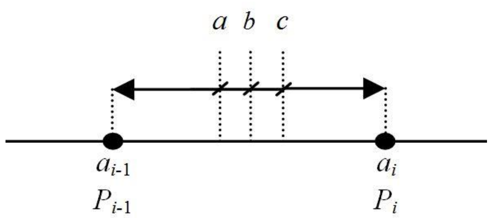

Fig. 1 The decision thresholds for: 1) MAP detector $m_i = a$ for $P_{i-1} < P_i$, $m_i = b$ for $P_{i-1} = P_i$, $m_i = c$ for $P_{i-1} > P_i$; 2) ML detector $m_i = b$.

2.3. Formulas for symbol error probability

During transmission through the channel the channel noise affects the digital signal. Due to that, there is a probability that the transmitted symbol is not accurately detected. The probability that the wrong symbol is detected is called the error probability per symbol (P_e). The general form for the symbol error probability of a symmetric PAM constellation is [1], [2]:

$$P_e = 2\sum_{i=1}^{M/2} P_i P(e|a_i)\,, \tag{15}$$

where $P(e|a_i)$ represents conditional error probability per symbol:

$$P(e|a_i) = P\left[r \notin [m_i, m_{i+1})|a_i\right] = 1 - P\left[r \in [m_i, m_{i+1})|a_i\right] = 1 - \int_{m_i}^{m_{i+1}} p(r|a_i)dr\ . \tag{16}$$

By substituting (16), (8), (11) and (1) into (15) we derive an analytical expression for the symbol error probability:

$$\begin{aligned} P_e = {} & P_1\left[\operatorname{erfc}\left(\frac{d}{2\sqrt{N_0}}\right) + \operatorname{erfc}\left(\frac{d}{2\sqrt{N_0}} + \frac{\sqrt{N_0}}{2d}\ln\frac{P_1}{P_2}\right)\right] + \\ & \sum_{i=2}^{M/2-1} P_i\left[\operatorname{erfc}\left(\frac{d}{2\sqrt{N_0}} - \frac{\sqrt{N_0}}{2d}\ln\left(\frac{P_{i-1}}{P_i}\right)\right) + \operatorname{erfc}\left(\frac{d}{2\sqrt{N_0}} + \frac{\sqrt{N_0}}{2d}\ln\left(\frac{P_i}{P_{i+1}}\right)\right)\right] + \\ & P_{M/2}\operatorname{erfc}\left(\frac{d}{2\sqrt{N_0}} - \frac{\sqrt{N_0}}{2d}\ln\frac{P_{M/2-1}}{P_{M/2}}\right). \end{aligned} \tag{17}$$

where erfc(·) is the complementary error function:

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{+\infty} \exp\{-t^2\}dt\ . \tag{18}$$

One can notice that in the case of the ML detection, the expression for the symbol error probability gets much simpler form:

$$P_e = (1 - P_{M/2})\operatorname{erfc}\left(\frac{d}{2\sqrt{N_0}}\right). \tag{19}$$

Assuming in (19) that symbols are equiprobable, we derive the known expression for the symbol error probability of the conventional PAM constellation [1]:

$$P_e = \frac{M-1}{M}\operatorname{erfc}\left(\frac{d}{2\sqrt{N_0}}\right). \tag{20}$$

3. Numerical Results and Discussion

In order to be able to quantify and compare the performances of the ML and MAP detectors for the previously defined scenario, for a given number of constellation points M we change other parameters of PAM constellation and calculate SNR and P_e. We assume the number of constellation points $M = 8$. We define the probabilities of the constellation points on several manners. Actually, we observe four cases illustrated in Fig. 2. In the first case in the Fig. 2 the probabilities of the constellation points are equal and amount $P_i = 0.125$ (the uniform distribution). In the case given in the Fig. 2.b the probability of the constellation point decreases with the increase of the amplitude of the constellation point (decreasing distribution). In the third case we observe the PAM constellation whose probability of constellation point increases with the constellation amplitude, so we name it the increasing distribution. Graphical representation of the third case is shown in Fig. 2.c. The last case is characterized with the probabilities of constellation points arranged in 'zigzag' form, that is, the distribution of constellation points does not change monotonically with the constellation amplitude. The last case can be seen in Fig. 2.d.

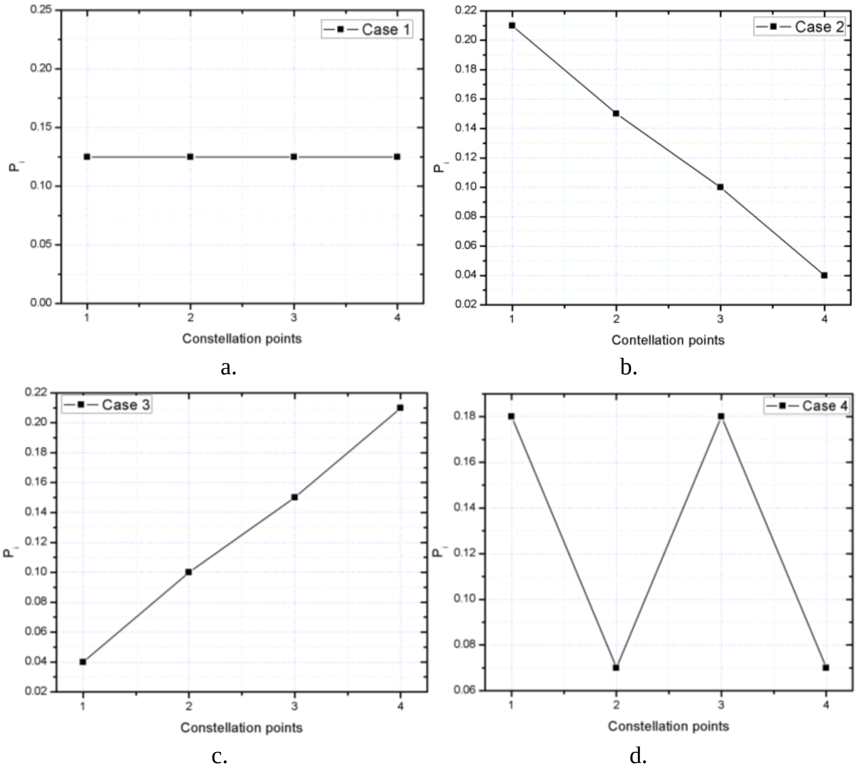

Fig. 2 Distribution of the constellation point probabilities for 8-PAM constellation:
a) uniform distribution $\mathbf{P}_i$ = [0.125, 0.125, 0.125, 0.125];
b) decreasing distribution $\mathbf{P}_i$ = [0.21, 0.15, 0.1, 0.04];
c) increasing distribution $\mathbf{P}_i$ = [0.04, 0.1, 0.15, 0.21];
d)'zigzag' distribution $\mathbf{P}_i$ = [0.18, 0.07, 0.18, 0.07], where $\mathbf{P}_i = [P_1, P_2, P_3, P_4]$.

For all presented distributions of the probabilities of constellation points and for different values of the SNR, the symbol error probabilities for ML and MAP detectors are calculated by using (17) and (19) and listed in the Table 1.

In order to better distinct the difference in the performances of the ML and MAP detectors, we introduce the relative difference between the symbol error probabilities δ:

$$\delta[\%] = \frac{|\Delta P_e|}{P_e^{MAP}} \times 100 = \frac{|P_e^{ML} - P_e^{MAP}|}{P_e^{MAP}} \times 100 \;, \tag{21}$$

where P_e^{MAP} and P_e^{ML} are the error probabilities for MAP and ML detector, respectively. In Fig. 3 the δ in function of the SNR is graphically presented.

By analyzing the results presented in Fig. 3 and Table 1, we can derive several conclusions. In every case the error probability per symbol of MAP detector is lower than the corresponding one of ML detector (the known fact). For PAM constellation with equiprobable symbols (see the first case) we confirm that there is no difference between the symbol error probabilities of ML and MAP detectors. Here it should be also noticed that the first case defines the conventional PAM constellation [1]. For the second case, that is, when the distribution of constellation probabilities is decreasing, we observe that the relative difference between the symbol error probabilities of ML and MAP detectors amounts about 4.6 % for SNR = 0 dB and 2.5 % for SNR = 20 dB. Besides that, the symbol error probability for both types of detectors is for three orders lower than the corresponding one in the first case. In the third case, when the distribution of probabilities is increasing, relative difference in symbol error probabilities is somewhat greater and ranges from 7.5 % to 3.2 % for the signal – to – noise ratio per bit ranging from 0 dB to 20dB. However, for this kind of probability distribution one should observe that the error probability for both detectors is for three orders higher than the error probability of a conventional PAM. Finally, the fourth case with the 'zigzag' distribution of the constellation points probabilities the relative difference in symbol error probabilities has a significantly greater value than it is in other cases. This difference reaches 19.1% for SNR = 0dB and 9.2% for SNR= 20dB. Thereby, the symbol error probability for both detectors is of the same order as the error probability of the conventional PAM constellation. From Fig. 3 we can also observe that curves in all cases approach the curve corresponding to the case where there is no difference between MAP and ML detection. This actually means that with the increase of SNR the relative difference in the symbol error probability decreases.

Table 1 Symbol Error Probability for ML and MAP detectors in function of SNR: a) Case 1 and Case 2, b) Case3 and Case 4

a)

P_e(SNR)	CASE 1		CASE 2	
SNR[dB]	ML	MAP	ML	MAP
0	0.519	0.519	0.461	0.441
5	0.299	0.299	0.201	0.194
10	0.080	0.080	0.025	0.024
15	0.002	0.002	6.91×10^{-5}	6.73×10^{-5}
20	7.90×10^{-9}	7.90×10^{-9}	1.61×10^{-12}	1.57×10^{-12}

b)

P_e(SNR)	CASE 3		CASE 4	
SNR[dB]	ML	MAP	ML	MAP
0	0.517	0.481	0.519	0.436
5	0.337	0.320	0.277	0.243
10	0.124	0.119	0.059	0.054
15	9.36×10^{-3}	9.05×10^{-3}	9.17×10^{-4}	8.35×10^{-4}
20	6.03×10^{-6}	5.84×10^{-6}	4.34×10^{-9}	3.97×10^{-9}

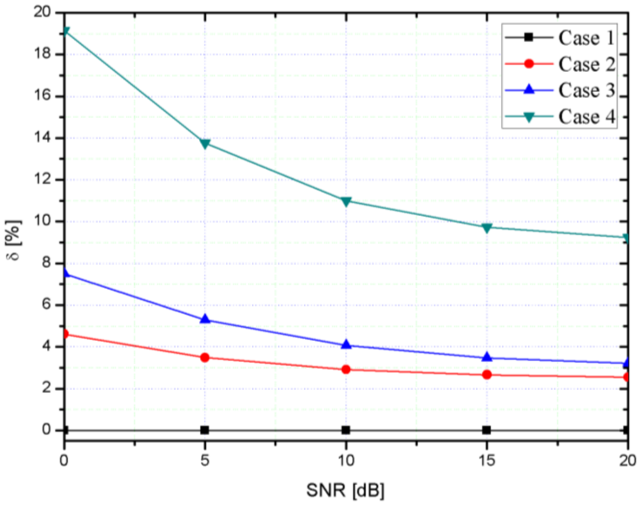

Fig. 3 Relative difference in the symbol error probabilities of the ML and MAP detectors.

As we have already noticed the PAM constellation with the increasing distribution of the constellation point probabilities does not represent a good constellation because its symbol error probability is significantly higher than that for the conventional PAM constellation. This points out that the nonuniform distribution of the constellation points probabilities does not necessarily improve the symbol error probability. Because of that, determination of constellation points probabilities with the aim to reduce the symbol error probability is an important task. In this paper we propose that the constellation points probabilities follows the Gaussian distribution. Namely, we assume that the constellation point probability P_i is equal with the probability that Gaussian variable with zero mean value and unit variance belongs to segment $[a_i - d/2, a_i + d/2]$:

$$P_i = \frac{1}{2}\left[\mathrm{erfc}\left(\frac{(i-1)d}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{id}{\sqrt{2}}\right)\right], \quad i = 1, \ldots, \frac{M}{2}. \tag{22}$$

For M = 16 and PAM constellation with equidistant constellation points whose probabilities are defined with (22) we calculate symbol error probability by utilizing (17) and tabulate the obtained results in Table 2. In Table 2 we also tabulate the symbol error probability of the conventional PAM constellation. One can evident that the PAM constellation we propose achieves significantly lower symbol error probability than it is in the case of conventional PAM constellation, whereby the achieved gain grows with the SNR. Thus, for SNR = 20 dB it is reached that the symbol error probability of our PAM constellation is for 2.57×10^{6} times lower than that of its conventional counterpart. Furthermore, the considered PAM constellation outperform the best PAM constellation in [17] whose symbol error probability amounts 4.57×10^{-6} for SNR = 20 dB. Unlike the PAM constellation defined with (22), the PAM constellation in [17] is characterized with only two different constellation points probabilities, so that it represents the PAM constellation of smaller complexity than it is the PAM constellation proposed in this paper.

Table 2 Symbol Error Probability for 16-PAM Constellation Based on (22)

SNR[dB]	P_e	$P_e^{\text{conventional PAM}}$
0	0.515	0.712
5	0.265	0.549
10	0.051	0.311
15	5.36×10^{-4}	0.079
20	7.86×10^{-10}	2.02×10^{-3}

4. Conclusion

In this paper, for PAM constellation with equidistant and non-equiprobable symbols the comparative analysis of the MAP and ML detectors has been performed. One of the results of analysis is formula for the symbol error probability of MAP detector. It has been also noticed that in all observed cases, except for the first one, with the increase of the signal-to-noise ratio the relative difference in the symbol error probabilities of the ML and MAP detectors decreases so that it can be expected that it becomes negligible for higher SNR values. This means that in the channels with higher values of signal-to-noise ratio the detector of smaller complexity - the ML detector can be used. The another finding is that when the probability distribution has ‘zigzag’ form, the choice of the detector is of great importance since it significantly influences the symbol error probability. In such situations, the MAP detector is preferable solution. It has been observed that the increasing distribution of the constellation points probabilities negatively effects the symbol error probability, while the decreasing distribution decreases the symbol error probability. Finally, this observation has helped us to define the method for determining constellation points probabilities. The proposed method has led to the symbol error probability of PAM constellation being significantly decreased in comparison to that for the conventional PAM constellation.

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