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COEFFICIENT QUANTIZATION EFFECTS ON NEW FILTERS BASED ON CHEBYSHEV FOURTH-KIND POLYNOMIALS

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Abstract. *The aim of this paper is to construct non-recursive filters, extensively used type of digital filters in digital signal processing applications, based on Chebyshev orthogonal polynomials. The paper proposes the use of the fourth-kind Chebyshev polynomials as functions in generating new filters. In this kind, low-pass filters with linear phase responses are obtained. Comprenhansive study of the frequency response characteristics of the generated filter functions is presented. The effects of coefficient quantization as one type of quantization that influences a filter characteristic are investigated here also. The quantized-coefficient errors are considered based on the number of bits and the implementation algorithms.*

Key words: *Chebyshev recursion, orthogonal polynomials, non-recursive filters, linear phase characteristic, coefficient quantization, implementation structure*

1. INTRODUCTION

Digital filters are used in a wide variety of digital signal processing (DSP). Most of the time, the final goal of using a filter is to achieve a kind of frequency selectivity on the spectrum of the input signal. Till now, researchers have suggested different methods for improving the digital filter efficiency as presented in [1]-[8]. Different methods presented in [4]-[8] share the same idea, namely, using CIC (Cascade-Integrator-Comb) sections with some added solutions to improve filter attenuation. Different methods are proposed to compensate for the passband droop as it is described in [9]-[17]. The objective of the paper [11] has been to categorize and describe the most important methods for compensator designs, proposed till than, and to propose some future direction for the compensator designs. In [12]-[17], new compensator designs are desribed and applied to different filters.

Chebyshev polynomials have been the focus of many studies and have drawn a wide attention due to their frequent appearance in various applications in polynomial approximation, integral and differential equations, etc. There are four kinds of these polynomials as projected by Mason and Handscomb [18] which leads to an extended

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range of new filter functions. The Chebyshev polynomials of the first-kind [19]-[21] are used to generate FIR (Finite Impulse Response) filters in [3] and integer coefficients of Coleman filter in [20]. Filter functions recently presented in [22] and [23] prove the use of the first- and second-kind Chebyshev polynomials in filter design, respectively. The principle idea behind obtaining these filter functions is based on Coleman filter described in [20].

Usually, Chebyshev polynomials of the fourth-kind are known less. The aim of this paper is to construct non-recursive filters, extensively used type of digital filters in DSP applications, based on these polynomials. The presented design algorithm leads to filter functions having a linear phase characteristic which is very important. However, if the phase characteristic is nonlinear, the phase delay of the individual frequency components of the signal is not equal, and a change in signal shape occurs.

In DSP, quantization has a great influence. Coefficient quantization represents a type of quantization that influences a frequency response of digital filter. Hardware implementation of digital filters requires quantization of filter coefficients to the word size of registers. The goal is to obtained quantized frequency characteristic that has more similarity to frequency characteristic of infinite precision filter.

The contribution of this study is twofold. Firstly, it has been mathematically proved that it is possible to obtain novel filter functions by considering Chebyshev orthogonal polynomials. Some characteristics of them are shown here to demonstrate their usefulness in communication systems. To the best of authors' knowledge, this is the first study that investigates the use of the fourth-kind Chebyshev polynomials as functions in generating new filters. As the second contributation, after designing a digital filter, the effects of the coefficient quantization on its frequency characteristic have to be examined and studied. If the quantized filter does not meet the target specifications, the designer needs to redesign the filter. In particular, it will be study how magnitude response of the filter is affected by coefficient quantization. Also, different implementation structures will be considered in order to suggest the best solution.

This paper is unscrewed into several sections. The design procedure of the novel filters and the verification of the designed filters are discussed in Sections 2 to 4. Then, Section 5 elaborates the coefficient quantization's effects on the filter frequency characteristic depending of the number of bits and the structure implementation algorithm. Finally, the summary is drawn in Section 6.

2. Coleman Filter: A Brief Overview

The recursion for generation of the Chebyshev polynomials of the first-kind [19]-[21] denoted as $T_N(x)$ is

$$T_N(x) \triangleq \begin{cases} 1 & , \ N=0 \\ x & , \ N=1 \\ 2x \cdot T_{N-1}(x) - T_{N-2}(x), & N>1 \end{cases} \quad (1)$$

The Chebyshev polynomials of the first-kind are used to generate Coleman filter form given in [20]. The frequency response is calculated as $G(f) = 2^{-20} \cdot T_7(F(f))$, where the function $F(f) = 2 + 2 \cdot \cos(2\pi f)$ is used and $T_7(x)$ represents the Chebyshev polynomial of the first-kind of degree seven. There, the exact functions with Chebyshev polynomials of degree N are obtained by following relation

$$G_N(f) = G(0) \cdot T_N(F(f)), \tag{2}$$

where the constant is calculated as $G(0) = T_N(F(0))$.

3. The New Generated Filter Forms

Design of a digital filter involves the following steps: (1) Filter specification, (2) Filter coefficient calculation, (3) Realization, (4) Analysis of finite word length effect and (5) Implementation. Some of the steps will be described in the following sections.

3.1. Filter Specifications

The filter specifications are determined by the applications. Once the specifications are defined, various concepts and mathematics can be used to come up with a filter description that approximates the given set of specifications.

In the case of new suggested design, for given specifications like passband cut-off frequency, stopband cut-off frequency, maximum and minimum attenuations, the sampling frequency as well as the filter order can be calculated by using exact formulas for design of Chebyshev filters. The actual attenuation of the filter depends on the filter order (the higher the order, the higher the attenuation). Here, in case of lowpass filter, the group delay is equal to the filter order.

Meeting the specifications is not guaranteed a-priori, trial and error is often required. If the resulting filter does not meet the specifications, one can adjust the filter parameters, i.e. changing the filter order could help to resolve the problem. In order to obtain more attenuation, some optimization can be done, like increasing filter order or free parameter v. This design approach is superior in that by varying one parameter v a much better design can be obtained for the same filter order.

3.2. Definition of Fourth-kind Chebyshev Polynomials

A brief overview of basic definition of Chebyshev polynomials of the fourth-kind [9], [21] is given here. Usually, the polynomials are defined according to the trigonometric formula. The N^{th} Chebyshev polynomials of the fourth-kind at the point x are denoted as $W_N(x)$. A simplified definition of the polynomials on the interval $x \in [-1,1]$ by using recurrence relations is as follows

$$W_N(x) = 2 \cdot x \cdot W_{N-1}(x) - W_{N-2}(x), \tag{3}$$

where $N = 2,3,...$ represents polynomial degree and the initial conditions are $W_0(x) = 1$ and $W_1(x) = 2 \cdot x + 1$.

3.3. Design Procedure: Filter Coefficient Calculation

The Chebyshev polynomials of the fourth-kind are used here to generate different low-pass filter functions. In this case, the new filter functions are generated by equation

$$G_{N,new}(f) = G_{new}(0) \cdot W_N(F_{new}(f)) \,, \tag{4}$$

where the normalized constant $G_{new}(0) = W_N(F_{new}(f))$ is calculated for $f = 0$, and applied cosine function

$$F_{new}(f) = v + 2 \cdot \cos(2\pi f) \,, \tag{5}$$

with parameter $1 \le v \le 2$.

Equations (4) and (5) allow one to predict how the filter will respond to varying frequency and constant parameter v.

In order to analyze filter characteristics, a few filter examples are designed by using MATLAB environment. The function proposed by J. Coleman [20], Eq. (2), and functions of new filters, Eq. (4), for filter order N, are arranged here in rectangular form as the following unique function of frequency ω

$$G_N(\omega) = [\, a(0) + a(1) \cdot cos(\omega) + ... + a(N) \cdot \cos(N\omega)] \,/\, G(0) \,, \tag{6}$$

which is normalized with the constant $G(0)$ or $G_{new}(0)$.

Assume that the transfer function $H(z)$ of designed filter can be presented as

$$H(z) = h(N) + \sum_{k=0}^{N-1} h(k) \cdot [z^{N-k} + z^{-(N-k)}] \,. \tag{7}$$

The filter coefficients show the symmetry around the center value.

4. Design Examples

4.1. Magnitude and phase responses

Normalized curves of designed low-pass filters with functions $F(f)$, $v = 2$ and $F_{new}(f)$, $v = 1$ or 2, versus normalized frequency $f = \omega / \pi$, are summarized in Figs. 1 and 2, for filter orders $N = 5$ and $N = 6$, respectively. Examples of odd and even orders are generated. The filter characteristics generated by Eq. (4), for the function $F_{new}(f)$ and $v = 1$, and different filter order N show higher selectivity in the transient area in comparison with both designed functions $F(f)$ and $F_{new}(f)$ for $v = 2$. The worst-case suppression in the stop-bands for new filters with $F_{new}(f)$ is between 63.81 dB and 76.41 dB, for $N = 5$ as shown in Fig. 1. The worst-case suppression in the stop-bands for new filters with $F_{new}(f)$, shown in Fig. 2, for the case of $N = 6$ are between 75.52 dB and 90.78 dB.

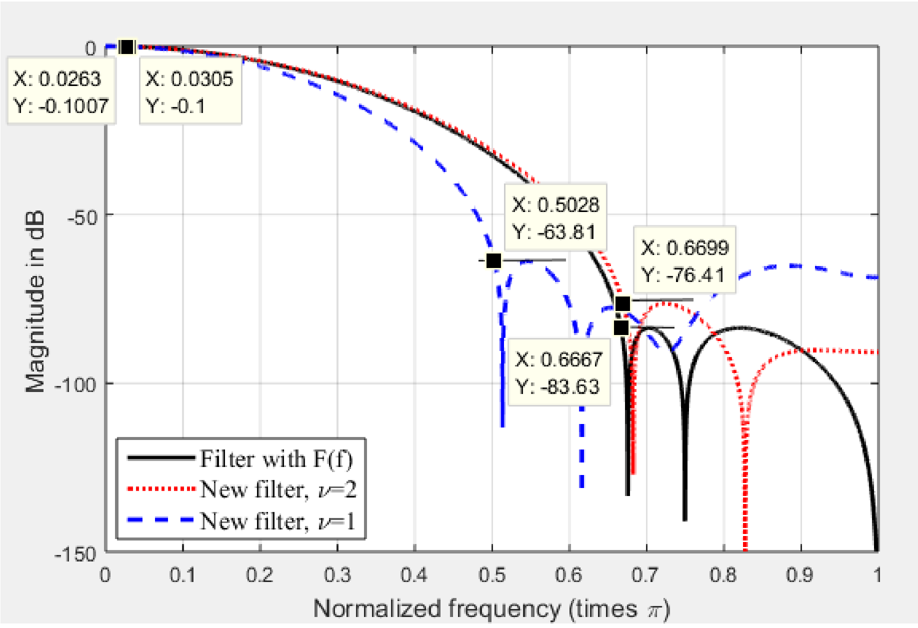

Fig. 1 Normalized filter curves in dBs for case $N = 5$

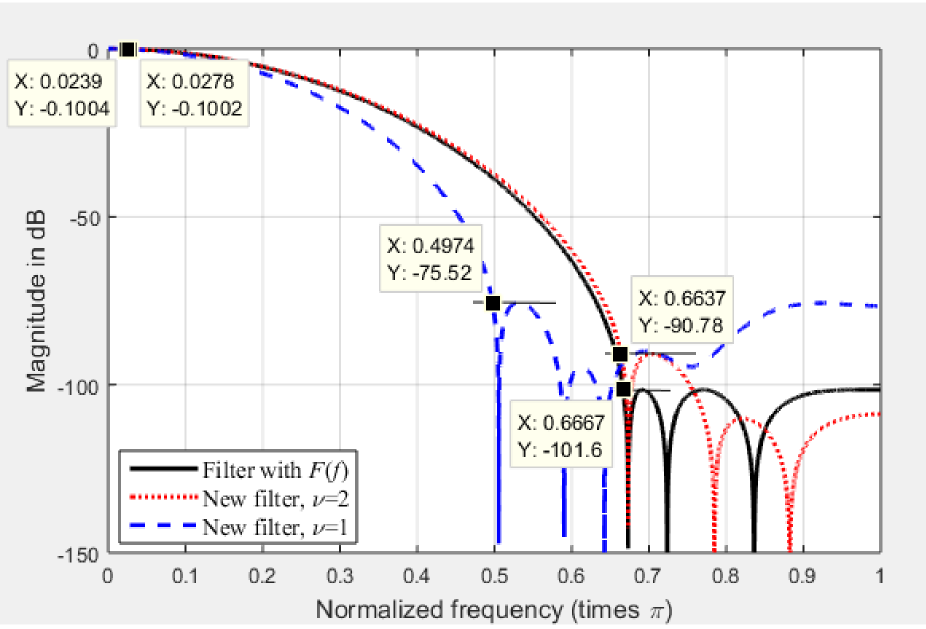

Fig. 2 Normalized filter curves in dBs for case $N = 6$

Table 1 gives some filter characteristics, such as pass-band cut-off frequency f_{cp} at $\alpha_{\max} = 0.1\ dB$, and minimum attenuation $\alpha_{\max}$ at stop-band cut-off frequency f_{cn}. The listed values of stop-band cut-off frequencies indicate higher selectivity of filter functions with parameter $v = 1$. A low-pass filter seeks to eliminate all frequencies in the stop-band, that is, all frequencies above cut-off frequency are desired to be filtered out. In these cases, high suppression level in stop-band is evidently desirable. The main key features of the designed filters are good in band and out band performance.

Table 1 Characteristics of designed FIR filters

Parameter N	5		6	
Parameter v	1	2	1	2
α_{max} @ f_{cp}	0.0263	0.0305	0.0239	0.0278
α_{max} @ f_{cn}	63.81@0.50	76.41@0.67	75.52@0.50	90.78@0.66

The generated filter characteristics by Eq. (5) for the function $F_{new}(f)$, filter order $N = 5$ and different values of parameter $1 \le v \le 2$ are depicted in Fig. 3. The magnitude characteristics of designed filters have a passband droop in the passband that is dependent upon the parameter v. Depending on chosen normalized frequency, passband droop can vary about 0.0038 dB at normalized frequency 0.01, 0.0152 dB at frequency 0.02, or 0.0581 at 0.025, for different parameter v values. The passband droop is higher in case when the parameter v has smaller value (the width of passband is smaller in this case as shown in Table 1).

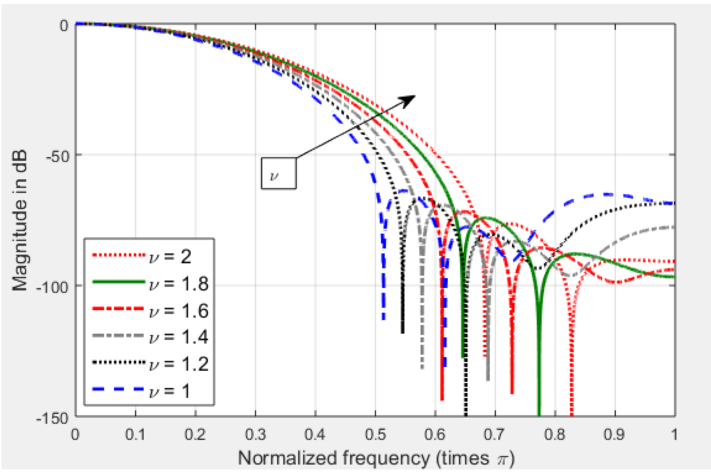

Fig. 3 Normalized new filter curves in dBs for case $N = 5$ and $1 \le v \le 2$.

In order to achieve correct performance, the filter should have a flat passband. The passband droop can be further improved by applying an additional filter, called compensating filter or compensator. This idea for improving the passband is given and applied on some other filters (comb, CIC and new CIC filters) like it is shown in [4] and [9]-[17]. Investigation can be done in order to elaborate which suggested method is the most convenient for passband droop compensation in these suggested filters.

The comparison of filter coefficients, $a(i)$, $i = 0,1,2,..., N$, from Eq. (6) and $N = 5$is done in tabular form, Table 2. There are listed coefficients of the filters based on first-, second- and fourth-kind Chebyshev polynomials and different value of parameter $v = 1$ or 2. It is obvious in all cases that the filter coefficients calculated for case of v =1 have smaller values then the one calculated for v =2.

Table 2 Coefficients of designed filter functions $G_N(\omega)$: A comparison

Literature	[20]	[23]	[22]		New filter functions	
$N = 5$	First-kind ChPoly* $v = 2$	First-kind ChPoly $v = 1$	Second-kind ChPoly $v = 2$	Second-kind ChPoly $v = 1$	Fourth-kind ChPoly $v = 2$	Fourth-kind ChPoly $v = 1$
$G(0)$	15124	3363	30744	6930	34633	8107
$a(0)$	3642	681	7436	1414	8477	1679
$a(1)$	6130	1210	12492	2508	14180	2964
$a(2)$	3600	840	7296	1728	8168	2024
$a(3)$	1400	440	2816	896	3072	1024
$a(4)$	320	160	640	320	672	352
$a(5)$	32	32	64	64	64	64

*ChPoly – Chebyshev polynomials

In comparison with traditional FIR filters, the designed filters have higher suppression level in stopband for the same filter order. The difference is more than 30 dB in favor of new designed filters.

Figure 4 shows the linear phase response of the designed new filter. The amplitudes of the individual frequency components will not change by passing the signal through such a filter. The group delay of the designed filters is equal to the filter order N.

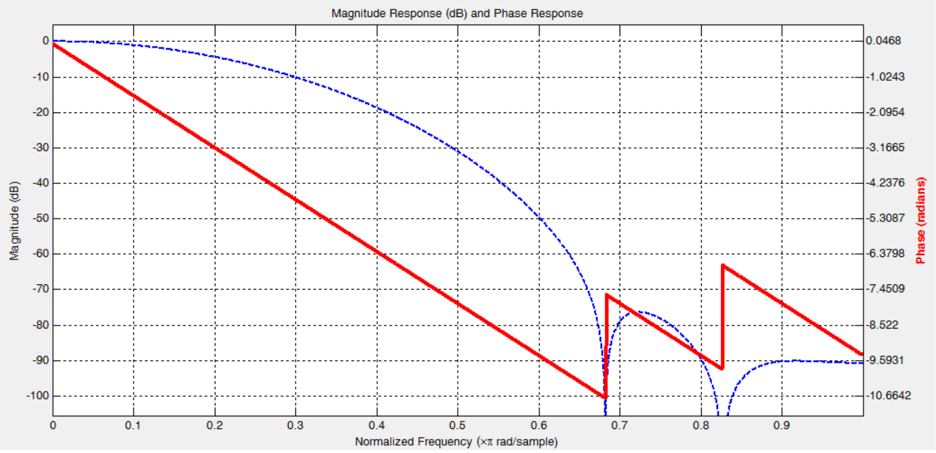

Fig. 4 Magnitude and phase responses for case $N = 5$ and $v = 2$

4.2. Pole-zero plots

Figure 5 graphically displays the locations of an unquantized system's poles and zeros. This is a two-dimensional plot of the z-plane that shows the unit circle, the real and imaginary axes, and the position of the system's zeros. A location having multiple poles is marked with a number next to that location to indicate how many poles exist there.

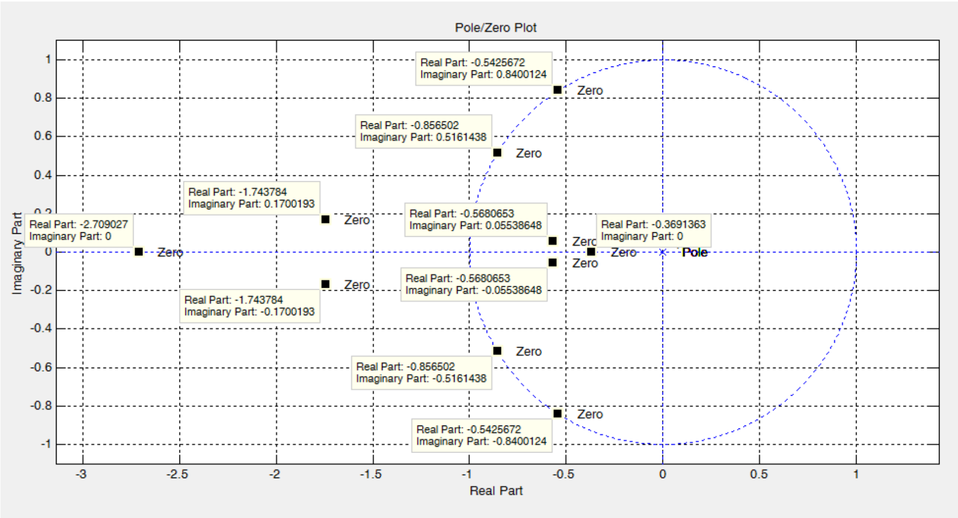

Fig. 5 Pole-zeros plot of filter designed for $N = 5$ and $v = 2$

The zeros of a linear-phase filter are of three different arrangements which include: a) a pair of complex-conjugate zeros which are not on the unit circle along with their reciprocals, e.g. z_2, z_2^*, $1 / z_2^*$ and $1/ z_2$; b) a pair of complex-conjugate zeros on the unit circle such as z_3, z_3^*, z_4 and z_4^*, and finally, c) a real zero which is not on the unit circle along with its reciprocal such as z_1 and $1 / z_1$. The filter with the transmission function having zeros in the left half-plane has a smaller phase, which means smaller signal delay. The zeros do not need to be inside the unit circle to maintain the stability.

As shown in this figure, some zeros are moved out of the unit circle. Recall that zeros near the unit circle can be expected to have a strong influence on the magnitude frequency response of the filter. This example shows that after designing a filter, the effect of the coefficient quantization have to be examined because they have direct influence on filter zeros. If the quantized filter does not meet the target specifications, the designer needs to redesign the filter.

5. Realization and Analysis of Finite Word Length Effect

A design of digital filters involves finding the coefficients of the filter which is described in Section 3. For computing the coefficients of digital filter, infinite precision arithmetic is used (the number of digits is only limited by available memory of the system).

As it is known, a common filter design tool like MATLAB, can run the design algorithm and return the filter coefficients.

After designing a digital filter, the effects of the coefficient quantization on its frequency response have to be examined and studied. If the quantized filter does not meet the target specifications, the designer needs to redesign the filter.

In DSP system, the number of bits in designing a filter is limited by word length of the register used to store them (registers have a fixed number of bits). Quantization methods like rounding and truncating are used to quantize the filter coefficients to the word size of the register [24], [25].

The designed coefficients have to be converted or quantized to a fixed-point representation [24]. A signed fixed-point representation with $B + 1$ total bits including sign is used here. Notice that filter coefficients of the novel designed FIR filters are smaller than 1. Because of that, $B + 1$-bit format which reserve 1 bit for sign, 1 bit for integer part and $B - 1$ bits for the fractional part has been chosen for coefficient representation.

Since a digital filter uses a finite number of bits to represent signals and filter coefficients, it is needed to find structure which can somehow retain the target filter specifications even after quantizing the coefficients. Below is given analysis of the direct-form structure and cascade-form structure, respectively. The main difference between the aforementioned realization structures is their sensitivity to using a finite length of bits.

5.1. Direct-form Structure

The direct-form structure is directly obtained from the difference equation. The coefficients of the new filter according to Eq. (7), for $N = 6$ and parameter $v = 2$ are listed in Table 3, and for parameter $v = 1$ in Table 4. The filter coefficients obtained by using different number of bits are presented in these tables.

Table 3 The unquantized and quantized filter coefficients, $B+1$-bit format, for $N = 6$ and $v = 2$

k	$h(k)$ - unquantized	$h_{new}(k)$ - quantized B=7	$h_{new}(k)$ - quantized B=15	$h_{new}(k)$ - quantized B=31
0	0.000234721982814	0	0.000244140625000	0.000234722159803
1	0.002934024785174	0	0.002929687500000	0.002934024669230
2	0.016371858301273	0.015625000000000	0.016357421875000	0.016371858306229
3	0.054455500012836	0.046875000000000	0.054443359375000	0.054455500096083
4	0.121409945610516	0.125000000000000	0.121398925781250	0.121409945189953
5	0.192611392084735	0.187500000000000	0.192626953125000	0.192611391656101
6	0.223965114445304	0.218750000000000	0.223937988281250	0.223965113982558

Table 4 The unquantized and quantized filter coefficients, $B+1$-bit format, for $N=6$ and $v=1$

k	$h(k)$ - unquantized	$h_{new}(k)$ - quantized B=7	$h_{new}(k)$ - quantized B=15	$h_{new}(k)$ - quantized B=31
0	0.001354526021715	0	0.001342773437500	0.001354525797069
1	0.008804419141146	0.015625000000000	0.008789062500000	0.008804419077933
2	0.030138203983153	0.031250000000000	0.030151367187500	0.030138203874230
3	0.070435353129167	0.078125000000000	0.070434570312500	0.070435353554785
4	0.123600499481471	0.125000000000000	0.123596191406250	0.123600499704480
5	0.170797265550594	0.171875000000000	0.170776367187500	0.170797265134752
6	0.189739465385511	0.187500000000000	0.189758300781250	0.189739465713501

Obviously, there are changes in the filter coefficients and they depend on used number of bits, so the filter frequency response will change correspondingly. As examples of affecting the magnitude response of a filter by coefficient quantization, new low-pass filters with different orders are designed and different number of bits are considered. Normalized curves are pictured in Figs. 6-9.

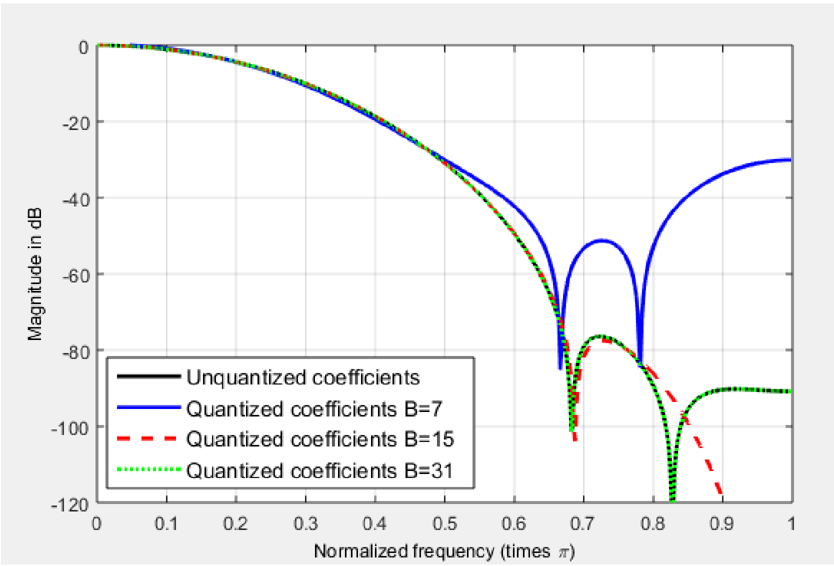

Fig. 6 New filter curves in dBs: $N=5$, $v=2$

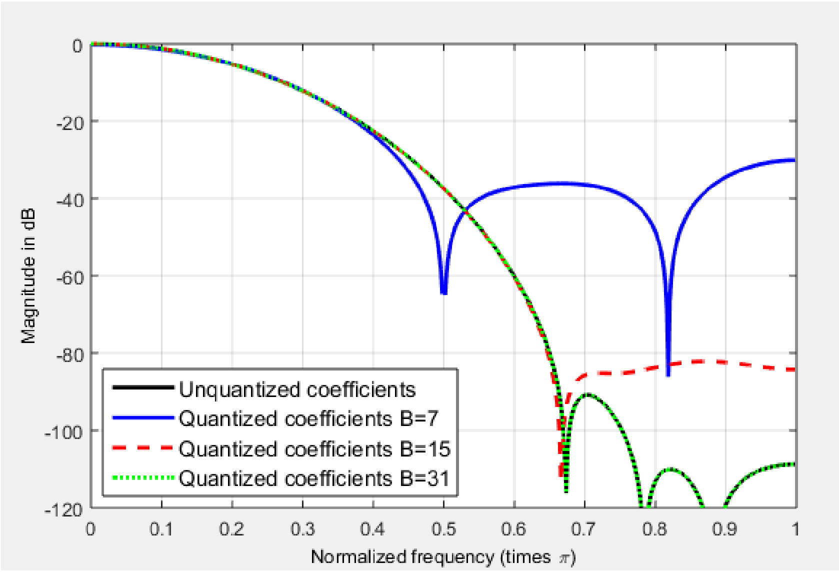

Fig. 7 New filter curves in dBs: $N = 6$, $v = 2$

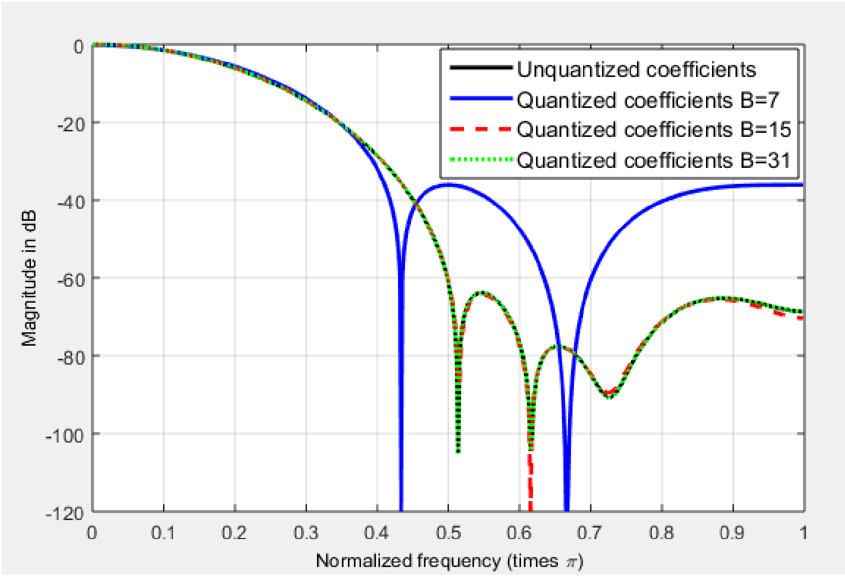

Fig. 8 New filter curves in dBs: $N = 5$, $v = 1$

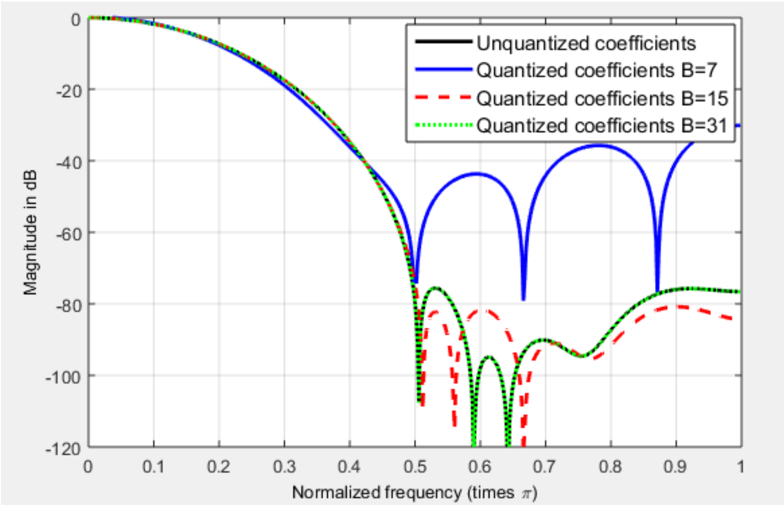

Fig. 9 New filter curves in dBs : $N=6$, $v=1$

It can be concluded that the quantization of filter coefficients creates a deviation in the frequency response of the filter as seen in Figs. 5-8. Note that if the wordlength B is not large enough, there will be undesirable effects. In summary, after coefficient quantization, a filter having a frequency response diverge from the frequency response of a filter with unquantized coefficients is obtained. The quantization effect is more visible and more significant in the stopband area.

Conclusion from this analysis and given characteristics: The new designed filters realizated by direct-form exhibit high sensitivity to the coefficient quantization.

5.2. Cascade-form Structure

In this part of the article, it will be shown that implementing a high-order filter as a cascade of second-order sections can significantly reduce the sensitivity to the coefficient quantization.

The cascade structure is obtained from the system function H(z). The idea is to decompose the target system function into a cascade of second-order FIR systems. In other words, we need to find second-order systems which satisfy

$$H(z)=\sum_{k=0}^{M-1} b_k \cdot z^{-k} = \prod_{k=1}^{[M/2]} (b_{0k}+b_{1k}\cdot z^{-1}+b_{2k}\cdot z^{-2}) = \prod_{k=1}^{[M/2]} H_k(z) \quad (8)$$

or

$$H(z)=\sum_{k=0}^{M-1} b_k \cdot z^{-k} = G\cdot \prod_{k=1}^{[M/2]} (b_{0k}+b_{1k}\cdot z^{-1}+b_{2k}\cdot z^{-2}) = G\cdot \prod_{k=1}^{[M/2]} H_k(z)\,, \quad (9)$$

where b_{0k}, b_{1k}, b_{2k} represent filter coefficients and G represents a gain factor.

A software implementation, such as a MATLAB code which use the *tf2sos* function, is applied to convert the transfer function into the cascade form insted of tedious mathematics. Function *tf2sos* converts forward and feedback path coefficients of the filter into numerator and denominator coefficients of the second-order sections. Then the coefficients of the second-order sections are quantized and the frequency response of the obtained structure with that of the unquantized system are compared.

To clarify converting a system function into the cascade form, as well as the effects of the chosen bit number for quantization, a few examples given below in Table 5 are reviewed.

Each row in the Table 5 gives the transfer function of one of the second-order sections. The first three numbers of each row represent the numerator of the corresponding second-order section and the second three numbers give its denominator. The *tf2sos* comand can also give a gain factor *g*, which can be included in the cascade-form structure. From the listed coefficient values it can be concluded that increasing coefficient word bit widths can be a viable option. Also, the case of using the *tf2sos* command which gives a gain factor is desirable one due to better coefficient resistance to quantization errors.

5.3. Performance Analysis Notes

The main difference between the aforementioned realization structures, direct- and cascade-form, is their sensitivity to using a finite length of bits. The realizations, such as direct forms, are very sensitive to quantization of the coefficients. However, structures with cascaded second-order sections show smaller sensitivity and are preferred. The analyzed examples show that implementing a high-order filter as a cascade of second-order sections can significantly reduce the sensitivity to the coefficient quantization.

Table 5 The second-order sections and their coefficients for $N = 5$ and $v = 2$

Function: `sos = tf2sos(b,a);`						
Unquantized coefficients	0.000923974244218	0.002844143378878	0.000923974244218	1.000000000000000	0	0
	1.000000000000000	3.487567927223607	3.069689062680776	1.000000000000000	0	0
	1.000000000000000	1.713003943968079	1.000000000000962	1.000000000000000	0	0
	1.000000000000000	1.085134395891595	0.999999999999881	1.000000000000000	0	0
	1.000000000000000	1.136130681644964	0.325765893411380	1.000000000000000	0	0
Quantized coefficients $B = 7$	0	0	0	1.000000000000000	0	0
	1.000000000000000	3.484375000000000	3.062500000000000	1.000000000000000	0	0
	1.000000000000000	1.718750000000000	1.000000000000000	1.000000000000000	0	0
	1.000000000000000	1.078125000000000	1.000000000000000	1.000000000000000	0	0
	1.000000000000000	1.140625000000000	0.328125000000000	1.000000000000000	0	0
	Note: *Impossible to get quantized transfer function, equal to zero!*					
Quantized coefficients $B = 15$	0.000915527343750	0.002868652343750	0.000915527343750	1.000000000000000	0	0
	1.000000000000000	3.487548828125000	3.069702148437500	1.000000000000000	0	0
	1.000000000000000	1.713012695312500	1.000000000000000	1.000000000000000	0	0
	1.000000000000000	1.085144042968750	1.000000000000000	1.000000000000000	0	0
	1.000000000000000	1.136108398437500	0.325744628906250	1.000000000000000	0	0
	Note: *The two graphs shows a very good agreement!*					
Function: `[sos,g] = tf2sos(b,a);`						
Unquantized coefficients	1.000000000000000	3.078163051271750	0.999999999999806	1.000000000000000	0	0
	1.000000000000000	3.487567927223607	3.069689062680776	1.000000000000000	0	0
	1.000000000000000	1.713003943968079	1.000000000000962	1.000000000000000	0	0
	1.000000000000000	1.085134395891595	0.999999999999881	1.000000000000000	0	0
	1.000000000000000	1.136130681644964	0.325765893411380	1.000000000000000	0	0
	g = 9.239742442179425e-04					
Quantized coefficients $B = 7$	1.000000000000000	3.078125000000000	1.000000000000000	1.000000000000000	0	0
	1.000000000000000	3.484375000000000	3.062500000000000	1.000000000000000	0	0
	1.000000000000000	1.718750000000000	1.000000000000000	1.000000000000000	0	0
	1.000000000000000	1.078125000000000	1.000000000000000	1.000000000000000	0	0
	1.000000000000000	1.140625000000000	0.328125000000000	1.000000000000000	0	0
	g = 9.239742442179425e-04					
	Note: *The two graphs shows a very good agreement!*					
Quantized coefficients $B = 15$	1.000000000000000	3.078186035156250	1.000000000000000	1.000000000000000	0	0
	1.000000000000000	3.487548828125000	3.069702148437500	1.000000000000000	0	0
	1.000000000000000	1.713012695312500	1.000000000000000	1.000000000000000	0	0
	1.000000000000000	1.085144042968750	1.000000000000000	1.000000000000000	0	0
	1.000000000000000	1.136108398437500	0.325744628906250	1.000000000000000	0	0
	g = 9.239742442179425e-04					
	Note: *The two graphs are barely distinguishable from each other!*					

6. CONCLUDING REMARKS

The presented approach for filter design which relies on the fourth-kind Chebyshev polynomials is computationally very simple. The Chebyshev polynomials can be used here to produce a set of low-pass non-recursive filter coefficients. Filter characteristics have been fully demonstrated through numerical examples to illustrate the efficiency and accuracy of the design approach.

When implementing a digital filter in the real world, a finite number of bits to represent each coefficient has to be used. The frequency response of the quantized filter might be quite different from that of the original design. Filter stability based on quantization effect on filter characteristics has been also analyzed on several examples. In practice, it is reasonable to do coefficient quantization of a single filter and measure the effect on its frequency response.

Implementing a higher-order filter requires a higher number of bits for coefficient representation, and a cascade of second-order sections used as implementation algorithm can significantly reduce the sensitivity to the coefficient quantization.

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