FACTA UNIVERSITATIS
Series: **Electronics and Energetics** Vol. 35, Nº 3, September 2022, pp. 313-331
https://doi.org/10.2298/FUEE2203313D

Original scientific paper

A NEW APPROACH FOR DIRECT DISCRETIZATION OF FRACTIONAL ORDER OPERATOR IN DELTA DOMAIN

Sujay Kumar Dolai[1], Arindam Mondal[2], Prasanta Sarkar[3]

[1]Dream Institute of Technology, Faculty of Electrical Engineering, India
[2]Pailan College of Management & Technology,
Faculty of Electrical and Electronics Engineering, India
[3]NITTTR Kolkata, Faculty of Electrical Engineering, India

Abstract. *The fractional order system (FOS) comprises fractional order operator. In order to obtain the discretized version of the fractional order system, the first step is to discretize the fractional order operator, commonly expressed as* $s^{\pm\mu}$, $0 < \mu < 1$. *The fractional order operator can be used as fractional order differentiator or integrator, depending upon the values of* μ. *In general, there are two approaches for discretization of fractional order operator, one is indirect method of discretization and another is direct method of discretization. The direct discretization method capitalizes the method of formation of generating function where fractional order operator* $s^{\pm\mu}$ *is expressed as a function of Z in the shift operator parameterization and continued fraction expansion (CFE) method is then utilized to get the corresponding discrete domain rational transfer function. There is an inherent problem with this discretization method using shift operator parameterization (discrete Z-domain). At fast sampling time, the discretized version of the continuous time operator or system should resemble that of the continuous time counterpart if the sampling theorem is satisfied. At very high sampling rate, the shift operator parameterized system fails to provide meaningful information due to its numerical ill conditioning. To overcome this problem, Delta operator parameterization for discretization is considered in this paper, where at fast sampling rate, the continuous time results can be obtained from the discrete time experiments and therefore a unified framework can be developed to get the discrete time results and continuous time results hand to hand. In this paper a new generating function is proposed to discretize the fractional order operator using the Gauss-Legendre 2-point quadrature rule. Additionally, the function has been expanded using the CFE in order to obtain rational approximation of the fractional order operator. The detailed mathematical formulations along with the simulation results in MATLAB, with different fractional order systems are considered, in order to prove the newness of this formulation for discretization of the FOS in complex Delta domain.*

Key words: *continuous fraction expansion, direct discretization, delta operator, fractional order operator, fractional order system*

Received September 30, 2021; revised January 26, 2022; accepted July 4, 2022
Corresponding author: Arindam Mondal
Pailan College of Management & Technology, Faculty of Electrical and Electronics Engineering, India
E-mail: arininstru@gmail.com

© 2022 by University of Niš, Serbia | Creative Commons License: CC BY-NC-ND

1. INTRODUCTION

Around 300 years ago the concept of fractional calculus [1-2] came into existence. It has been an untouched and undiscovered part of engineering until the conceptual furtherance of fractional calculus eventuated in the mid nineteenth century. With time this part attracted the researchers towards its diversified properties that can be implemented in various field of engineering, as well as various part of science [3-7]. The postulation of fractional order calculus has an immense perspective to change the technique we see, manipulate and design the nature that is around us. The fundamental unit of the non-integer order system is the operator ($s^{\pm\mu}$), which can also be coined as fractional order differentiator or integrator [8-9] for variation of μ by making it either positive or negative. The important part of digital realization of fractional order system is the discretization of this operator. In order to implement the FOS in real time, the rationalization is the only procedure, either in continuous time or in discrete time. There are various methods for continuous time approximation of fractional order operators [10-12]. Once it is converted to continuous time rational transfer function [13], there are methods of discretization to get the discretized version of the FOS [14-18]. This is known as indirect method of discretization of FOS. There is a second method known as direct discretization method, where the rational transfer function in Z-domain is directly obtained via different generating functions, as proposed by Euler, Tustin, Al-Alauoi. In the subsequent step, the generating function is expanded using methods such as continued fraction expansion (CFE) [19].

There has been an increased demand in digital system implementation. In order to implement the FOS digitally, the sampling rate must be increased to at least 10 times the original system bandwidth. The increased sampling rate makes the poles closer to each other in Z- domain transfer function and gets focused near the point (1,0) in the discrete Z- plane. This will result in an unstable system due to finite word length effect [15]. The conventional or shift operator representation of discrete time system fails to furnish the significant portrayal of the conventional continuous-time system at fast sampling rate. To circumvent this problem delta operator parameterization is introduced [20] where, at very high sampling frequency the continuous time results and discrete time results are obtained at the same time. The superiority of the delta operator parameterization along with its various applications are found in [21-29].

In this paper, a method is proposed by which the fractional order operator is directly discretized [30-31] in delta domain. Initially, a generating function is proposed in delta domain by using one of the useful numerical computational tools known as Gauss-Legendre 2-point quadrature rule [32]. The classical CFE method is adopted to expand this generating function to get the rational approximation of the fractional operator in discrete delta domain.

The significant contributions are made in this paper as given below: earlier research work so far done on the discretization of the fractional order system through the discretization of the fractional order operator in shift operator parameterization. In this work the FOS has been directly discretized using delta operator parameterization so that at a very fast sampling frequency, the discrete time results resemble that of the continuous time counterpart. One more important contribution of this work is that here Gauss-Legendre 2-point quadrature rule is used for the close form approximation of the $\log(1 + x)$ function to minimize the approximation error. The comparison with the other standard methods are done to prove the efficacy of this proposed method.

The paper has been well organized in the following sections as indicated: in Section 2, fractional order operator and systems are discussed; Section 3 enlightens the direct discretization method of FO operator in delta domain; simulation and result analysis are discussed with different examples in Section 4; and in Section 5 the conclusion is drawn.

2. FRACTIONAL ORDER SYSTEM AND ITS DISCRETIZATION (DIRECT METHOD) USING TRADITIONAL METHODS

2.1. Fractional order operator and fractional order system

Fractional order system literally means the order of the system is no longer integer that is non-integer order. A system of fractional order is represented as fractional order differential equation and Laplace transform of the system can be performed to get the transfer function.

A non-integer order system can be portrayed by the following equation [30].

$$a_n D^{\alpha n} y(t) + a_{n-1} D^{\alpha n-1} y(t) + ... + a_0 D^{\alpha 0} y(t) = b_m D^{\beta m} u(t) + b_{m-1} D^{\beta m-1} u(t) + ... + b_0 D^{\beta 0} u(t)$$

Where,

$$mD_\tau^\psi = \begin{cases} \dfrac{d^\psi}{d\tau^\psi} & (\psi > 0) \\ 1 & (\psi = 0) \\ \int_m^\tau (d\tau)^\psi & (\psi < 0) \end{cases} \tag{1}$$

is known as integro-diffrerentiator operator.

The Laplace transform of the Eq. (1) under consideration of zero initial condition, the transfer function that we get is:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m D^{\beta_m} + b_{m-1} D^{\beta_{m-1}} + ... + b_0 D^{\beta_0}}{a_n D^{\alpha_n} + a_{n-1} D^{\alpha_{n-1}} + ... + a_0 D^{\alpha_0}} \tag{2}$$

where, $L[y(t)] = Y(s), L[u(t)] = U(s)$.

If the fractional differential equation as given in Eq. (2) may be coined as commensurate order which further gets reduced to the following form.

$$\sum_{k=0}^{n} a_k D^{ka} y(t) = \sum_{k=0}^{m} \beta_k D^{ka} u(t) \tag{3}$$

where, $a_k, \beta_k = ka, \in \Re_+$

There are two popular definitions, such as Grünwald-Letnikov (GL) and Riemann-Liouville (RL) definitions, to express this operator mD_τ^ψ . Here, RL definition is considered.

The RL definition is

$$mD_\tau^\psi \phi(t) = \frac{1}{\Gamma(k-\mu)} \frac{d^k}{dt^k} \int \frac{\phi(p)}{(t-p)^{\mu-k+1}} dp \tag{4}$$

Where m and τ are the bounds of operation and Γ is used to represent the Euler's gamma function.

For the analysis purpose, the fractional order differentiator is considered in this section. The fractional order system (differentiator) is realized in complex S-domain for the ease, which can be acquired by taking the Laplace transform of the Eq. (4), thus the Laplace transform of the equation is

$$L\{mD_{\tau}^{\psi}\phi(t)\} = s^{\psi}\varphi(s),\ for\ 0<\psi<1 \quad (5)$$

3. DELTA DOMAIN DISCRETIZATION METHOD OF FRACTIONAL ORDER OPERATOR

In contrast to get better finite-word-length effect under fast sampling, forward shift operator is going to be replaced by the delta operator [20]. The forward difference operator of delta operator is defined as

$$\delta = \frac{q-1}{\Delta} \quad (6)$$

Where q is the forward shift operator and Δ is termed as sampling time or internal. Employing a differentiable signal $x(t)$, at high sampling time ($\Delta \to 0$) the delta (δ) operator gravitates with continuous-time derivative operator as shown in Eq. (7).

$$\lim_{\Delta\to 0} \delta x(t) = \lim_{\Delta\to 0} \frac{x(t+\Delta)-x(t)}{\Delta} = \frac{dx(t)}{dt} \quad (7)$$

The variable corresponding to z in the shift operator parameterization is denoted by γ in complex delta domain and relationship between the two complex variables are given in Eq. (8)[20].

$$\gamma = \frac{z-1}{\Delta} = \frac{e^{s\Delta}-1}{\Delta} \quad (8)$$

At high sampling time limits ($\Delta \to 0$) the delta discrete-time frequency variable (γ) coincides with the continuous-time frequency variable (s) as follows and it is the philosophy which is capitalized in this work.

$$\lim_{\Delta\to 0}\gamma = \lim_{\Delta\to 0}\frac{e^{s\Delta}-1}{\Delta} = \lim_{\Delta\to 0}\frac{1+s\Delta+\frac{s^2\Delta^2}{2!}+....-1}{\Delta} = s \quad (9)$$

To obtain the mapping between s and γ, we need to replace $z = e^{s\Delta}$ in Eq. (8) as shown above. After taking logarithm on both sides the relationship between the two domains can be established by Eq. (10).

$$s = \frac{1}{\Delta}\ln(1+\gamma\Delta) \quad (10)$$

Now, $\ln(1+\gamma\Delta)$ function is approximated in a closed form and the CFE expansion is made possible. Upon applying different Trapezoidal quadrature rule [32], the close form approximation of $\ln(1+x)$ is obtained through 2P-GILOG approximation as follows:

$$\ln(1+x) \approx \frac{6x+3x^2}{6+6x+x^2} \quad (11)$$

This Approximation is known as 2P-GILOG.

Now replacing x by $\gamma\Delta$ in Eq. (11), the expression becomes,

$$\ln(1+\Delta\gamma) \approx \frac{6\gamma\Delta + 3(\gamma\Delta)^2}{6+6(\gamma\Delta)+(\gamma\Delta)^2} \tag{12}$$

The Eq. (10) is re-established by using Eq. (12) and Eq. (13) and is obtained as follows:

$$s = \left\{\frac{1}{\Delta}\ln(1+\gamma\Delta)\right\} \approx \left\{\frac{6\gamma + 3\Delta\gamma^2}{6+6\gamma\Delta+\Delta^2\gamma^2}\right\} \tag{13}$$

At fast sampling limit ($\Delta \to 0$) the discrete-time frequency variable (γ) in delta domain coincides with the continuous-time frequency variable (s) as can be found out from Eq.(13) Therefore, at fast sampling limit, the complex variable in continuous domain is approximated as the complex variable in discrete delta domain.

A FO differentiator is framed as:

$$G(s) = s^r \, (0 < r < 1) \tag{14}$$

CFE2P-GILOG method is used for discretization of s^r directly in delta domain.

The fractional order operator discretization is accomplished in two stages. Initially, the required generating function is selected and that is going to define the approximate mapping between delta discrete-time variable (γ) and continuous-time variable (s). In the next stage, to obtain the discrete time approximation of s^r in the form of transfer function in delta domain, the selected generating function is expanded. In this work, Eq. (13) is chosen as the generating function and CFE method is used to expand it to get respective integer order approximation of s^r in delta domain.

$$s^r \approx G_\delta(\gamma) \approx CFE\left\{\left(\frac{6\gamma\Delta + 3(\gamma\Delta)^2}{6+6(\gamma\Delta)+(\gamma\Delta)^2}\right)^r\right\} \tag{15}$$

The mathematical expression for CFE approximation is as follow:

$$(1+p)^r = 1 + \cfrac{rp}{1 + \cfrac{(1-r)p}{2 + \cfrac{(1+r)p}{3 + \cfrac{(2-r)p}{2 + \cfrac{(2+r)p}{5 + \cfrac{(3-r)p}{2+\ldots}}}}}} \tag{16}$$

$\frac{6\gamma + 3\Delta\gamma^2}{6+6\Delta\gamma+\Delta^2\gamma^2} - 1$ is substituted in place of p in the Eq. (16) to get the equivalent form of Eq. (15). Now executing CFE approximation of $\left\{\left(\frac{6\gamma + 3\Delta\gamma^2}{6+6\Delta\gamma+\Delta^2\gamma^2}\right)^r\right\}$ for third order, and fifth order in delta domain are obtained as given in Eq. (17) and Eq. (18) respectively.

$$s^r \approx G_{\delta 3}(\gamma) = CFE\left\{\left(\frac{6\gamma + 3\gamma\Delta^2}{6+6\Delta\gamma+\Delta^2\gamma^2}\right)^r\right\} = \frac{a^0\gamma^{-3} + a^1\gamma^{-3} + a^2\gamma^{-3} + a^3}{b^0\gamma^{-3} + b^0\gamma^{-2} + b^2\gamma^{-1} + b^3} \tag{17}$$

$$s^r \approx G_{\delta 5}(\gamma) = CFE\left\{\left(\frac{6\gamma + 3\Delta\gamma^2}{6 + 6\gamma\Delta + \Delta^2\gamma^2}\right)^r\right\} = \frac{a_0\gamma^{-5} + a_1\gamma^{-4} + a_2\gamma^{-3} + a_3\gamma^{-2} + a_4\gamma^{-1} + a_5}{b_0\gamma^{-5} + b_1\gamma^{-4} + b_2\gamma^{-3} + b_3\gamma^{-2} + b_4\gamma^{-1} + b_5} \quad (18)$$

Table 1 Numerator coefficients for fifth order approximation in Delta Domain

$$D_{num5} = ((r+1)(r+2)(1073741824r^{15} + 16911433728r^{14} + 13354663936r^{13} - 1002254106624r^{12}$$
$$- 3869945888768r^{11} + 20886278111232r^{10} + 129327405203456r^{9} - 138817498447872r^{8}$$
$$- 1829934470742016r^{7} - 712034173267968r^{6} + 12608455533286400r^{5} + 13516106683236096r^{4}$$
$$- 41479456532696640r^{3} - 59408759887249392r^{2} + 55098059015583104r$$
$$+ 92016444345172880)))$$

Coefficients	Numerator
a_0	$(-((3/\Delta)^r\,(1073741824r^{17} - 20132659200r^{16} + 66236448768r^{15} + 928367247360r^{14}$ $- 6849998880768r^{13} - 7271932231680r^{12} + 184246347759616r^{11} - 290937273384960r^{10}$ $1987732155678720r^{9} + 6479472582389760r8 + 6812484071998464r^{7}$ $49917404936559360r^{6} + 24285774583584448r^{5} + 156814916118867136r^{4}$ $206087133711558336r^{3} - 138493101617423408r^{2} + 386245451066684864r$ $- 184032888690345760))\,/\,D_{num5}$
a_1	$(-((3/\ \ )r\,(140928614400\,r^{15}\quad 8053063680\,r^{16}\quad 320612597760\,r^{14}\quad 7395229040640\,r^{13}$ $42899897057280\,r^{12}\quad 102849152286720\,r^{11}\quad 1256995122708480\,r^{10}$ $791103107235840\,r^{9}\quad 15398862690017280\,r^{8}\quad 30803473842032640\,r^{7}$ $81700461363356160\,r^{6}\quad 272889395307594240\,r^{5}\quad 112347728802349920\,r^{4}$ $1002215818766418432\,r^{3}\quad 425460690581907136\,r^{2}\quad 1423614499033773056\,r$ $1274294568187684864))\,/\,D_{num5}$
a_2	$(-((3/\Delta)^r\,(28185722880\Delta^2r^{15} - 422785843200\Delta^2r^{14} + 65179484160\Delta^2r^{13} + 26212722278400\Delta^2r^{12}$ $-87432002273280\Delta^2r^{11} - 570237360537600\Delta^2r^{10} + 3060794314260480\Delta^2r^{9}$ $+4379676740812800\Delta^2r^{8} - 43583729456762880\Delta^2r^{7} + 7143762905395200\Delta^2r^{6}$ $+300546723808853760\Delta^2r^{5} - 267563840484326400\Delta^2r^{4} - 992987073349200768\Delta^2r^{3}$ $+1262317112875803648\Delta^2r^{2} + 1327256882051046912\Delta^2r - 2019580911193378048\Delta^2))\,/\,D_{num5}$
a_3	$(-((3/\Delta)^r\,(634178764800\Delta^3r^{13} - 56371445760\Delta^3r^{14} + 2247811399680\Delta^3r^{12} - 44392513536000\Delta^3r^{11} +$ $12672785448960\Delta^3r^{10} + 1194755633971200\Delta^3r^{9} - 1827599513026560\Delta^3r^{8} - 15451885751500800\Delta^3r^{7}$ $+32314828390440960\Delta^3r^{6} + 102031042011494400\Delta^3r^{5} - 243411076800568320\Delta^3r^{4}$ $-330417309695155200\Delta^3r^{3} + 842112615992487808\Delta^3r^{2}$ $+436883676571452608\Delta^3r - 1161233166549210368\Delta^3))\,/\,D_{num5}$
a_4	$(-((3/\Delta)^r\,(63417876480\Delta^4r^{13} - 396361728000\Delta^4r^{12} - 4510596464640\Delta^4r^{11} + 27834502348800\Delta^4r^{10}$ $+122752979435520\Delta^4r^{9} - 750819041280000\Delta^4r^{8} - 1602561749483520\Delta^4r^{7} + 9729145062604800\Delta^4r^{6}$ $+10683837557406720\Delta^4r^{5} - 64370317952505600\Delta^4r^{4} - 34908135243891840\Delta^4r^{3}$ $+208843823902442400\Delta^4r^{2} + 46559875383003776\Delta^4r - 276641682514481024\Delta^4))\,/\,D_{num5}$
a_5	$((3/\Delta)^r\,(31708938240\Delta^5r^{12} - 2255298232320\Delta^5r^{10} + 61376489717760\Delta^5r^{8}$ $-801280874741760\Delta^5r^{6} + 5341918778703360\Delta^5r^{4} - 17454067621945920\Delta^5r^{2}$ $+23279937691501888\Delta^5))\,/\,D_{num5}$

Table 2 Denominator coefficients of fifth order approximation in Delta domain

$$Dn_5 = ((r+1)(r+2)(55098059015583104r - 59408759887249392r^2 - 41479456532696640r^3 +13516106683236096r^4 +12608455533286400r^5 - 712034173267968r^6 - 1829934470742016r^7 -138817498447872r^8 - 712034173267968r^6 - 1829934470742016r^7 - 138817498447872r^8 +129327405203456r^9 + 20886278111232r^{10} - 3869945888768r^{11} - 1002254106624r^{12} +13354663936r^{13} + 16911433728r^{14} + 1073741824r^{15} + 92016444345172880))$$

Coefficients	Denominator
b_0	$((386245451066684864r + 138493101617423408r^2 - 206087133711558336r^3 - 156814916118867136r^4 + 242857745835844448r^5 + 49917404936559360r^6 + 6812484071998464r^7 - 6479472582389760r^8 - 1987732155678720r^9 + 290937273384960r^{10} + 184246347759616r^{11} + 7271932231680r^{12} - 6849998880768r^{13} - 928367247360r^{14} + 66236448768r^{15} + 20132659200r^{16} + 1073741824r^{17} + 184032888690345760)) / Dn_5$
b_1	$((1274294568187684864\Delta + 1423614499033773056\Delta r - 425460690581907136\Delta r^2 - 1002215818766418432\Delta r^3 - 112347728802349920\Delta r^4 + 272889395307594240\Delta r^5 + 81700461363356160\Delta r^6 - 30803473842032640\Delta r^7 - 15398862690017280\Delta r^8 + 791103107235840\Delta r^9 + 1256995122708480\Delta r^{10} + 102849152286720\Delta r^{11} - 42899897057280\Delta r^{12} - 7395229040640\Delta r^{13} + 320612597760\Delta r^{14} + 140928614400\Delta r^{15} + 8053063680\Delta r^{16})) / Dn_5$
b_2	$((1327256882051046912\Delta^2 r + 2019580911193378048\Delta^2 - 1262317112875803648\Delta^2 r^2 - 992987073349200768\Delta^2 r^3 + 267563840484326400\Delta^2 r^4 + 300546723808853760\Delta^2 r^5 - 7143762905395200\Delta^2 r^6 - 43583729456762880\Delta^2 r^7 - 4379676740812800\Delta^2 r^8 + 3060794314260480\Delta^2 r^9 + 570237360537600\Delta^2 r^{10} - 87432002273280\Delta^2 r^{11} - 26212722278400\Delta^2 r^{12} + 65179484160\Delta^2 r^{13} + 422785843200\Delta^2 r^{14} + 28185722880\Delta^2 r^{15})) / Dn_5$
b_3	$((436883676571452608\Delta^3 r + 1161233166549210368\Delta^3 - 842112615992487808\Delta^3 r^2 - 330417309695155200\Delta^3 r^3 + 243411076800568320\Delta^3 r^4 + 102031042011494400\Delta^3 r^5 - 32314828390440960\Delta^3 r^6 - 15451885751500800\Delta^3 r^7 + 1827599513026560\Delta^3 r^8 + 1194755633971200\Delta^3 r^9 - 12672785448960\Delta^3 r^{10} - 44392513536000\Delta^3 r^{11} - 2247811399680\Delta^3 r^{12} + 634178764800\Delta^3 r^{13} + 56371445760\Delta^3 r^{14})) / Dn_5$

b_4	$((46559875383003776\Delta^4 r + 276641682514481024\Delta^4$ $-2088438239024424 00\Delta^4 r^2 - 34908135243891840\Delta^4 r^3$ $+64370317952505600\Delta^4 r^4 + 10683837557406720\Delta^4 r^5$ $-9729145062604800\Delta^4 r^6 - 1602561749483520\Delta^4 r^7$ $+750819041280000\Delta^4 r^8 + 122752979435520\Delta^4 r^9$ $-27834502348800\Delta^4 r^{10} - 4510596464640\Delta^4 r^{11}$ $+396361728000\Delta^4 r^{12} + 63417876480\Delta^4 r^{13})) / Dn_5$
b_5	$((23279937691501888\Delta^5 - 17454067621945920\Delta^5 r^2$ $+5341918778703360\Delta^5 r^4 - 801280874741760\Delta^5 r^6$ $+61376489717760\Delta^5 r^8 - 2255298232320\Delta^5 r^{10}$ $+31708938240\Delta^5 r^{12}))\gamma^5) / Dn_5$

4. Simulation and Result Analysis

To prove the effectiveness of the portrayed approach, three examples are taken.

Example 1:

A 1/4th order differentiator is considered in this example [25] with transfer function as shown below:

$$G(s) = s^r = s^{0.25} \tag{19}$$

The direct discretization of 1/4th order differentiator in delta domain is expressed as follows:

$$s^{0.25} \approx G_{2P-GILODEL}(\gamma) \approx CFE\left(\left(\frac{6\gamma + 3\Delta\gamma^2}{6 + 6\Delta\gamma + \Delta^2\gamma^2}\right)^{0.25}\right)_{\Delta=0.01} \tag{20}$$

The third and fifth order approximation of $s^{0.25}$ in delta domain after continued fraction expansion of $\left(\left(\frac{6\gamma + 3\Delta\gamma^2}{6 + 6\Delta\gamma + \Delta^2\gamma^2}\right)^{0.25}\right)$ results in Eq. (21) and Eq. (22) respectively. The sampling time is considered to be $\Delta = 0.01s$

$$\begin{aligned} s^{0.25} &\approx G_{2P-GILODEL3}(\gamma)\Big|_{\Delta=0.01} = CFE\left(\left(\frac{6\gamma + 3\Delta\gamma^2}{6 + 6\Delta\gamma + \Delta^2\gamma^2}\right)^{0.25}\right)_{\Delta=0.01} \\ &= \frac{5.48^{-7}\gamma^3 + 0.0003722\gamma^2 + 0.06519\gamma + 1.8}{1.317^{-7}\gamma^3 + 0.0001026\gamma^2 + 0.02198\gamma + 1} \end{aligned} \tag{21}$$

$$\begin{aligned} s^{0.25} &\approx G_{2P-GILODEL5}(\gamma)\Big|_{\Delta=0.01} = CFE\left(\left(\frac{6\gamma + 3\Delta\gamma^2}{6 + 6\Delta\gamma + \Delta^2\gamma^2}\right)^{0.25}\right)_{\Delta=0.01} \\ &= \frac{3.238^{-11}\gamma^5 + 3.685^{-8}\gamma^4 + 1.466^{-5}\gamma^3 + 0.002369\gamma^2 + 0.1322\gamma + 1.439}{7.781^{-12}\gamma^5 + 9.632^{-9}\gamma^4 + 4.252^{-6}\gamma^3 + 0.0007911\gamma^2 + 0.05562\gamma + 1} \end{aligned} \tag{22}$$

For $G_{2P-GILOGDel5}(\gamma)$ the denominator and numerator coefficient are calculated using Table 1 and Table 2 taking $r = 0.25$ and $\Delta = 0.01$. The frequency responses of delta domain transfer functions, $G_{2P-GILOGDel3}(\gamma)$ and $G_{2P-GILOGDel5}$ are shown in Fig. 1. The magnitude and phase error of the third order and fifth order approximate transfer function with respect to the original 1/4th order differentiator are demonstrated in Fig. 2. It can be seen through the graph that as the order of approximation goes higher, the precision of approximation gets better.

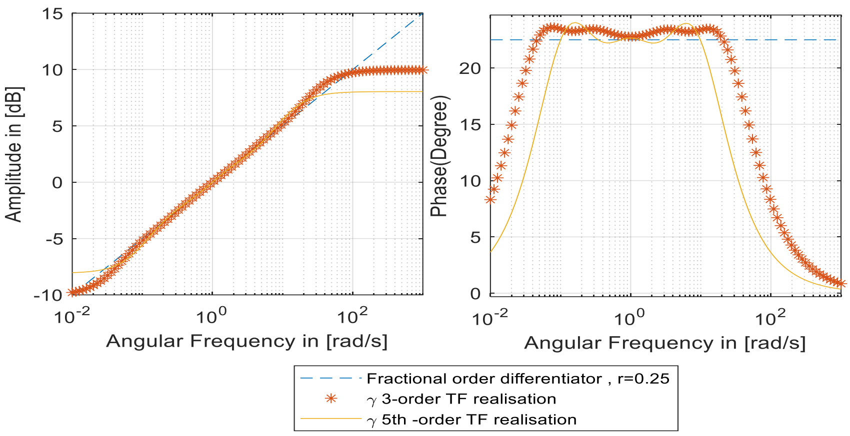

Fig. 1 Fifth order and third order approximation of $s^{0.25}$ in delta domain using proposed method

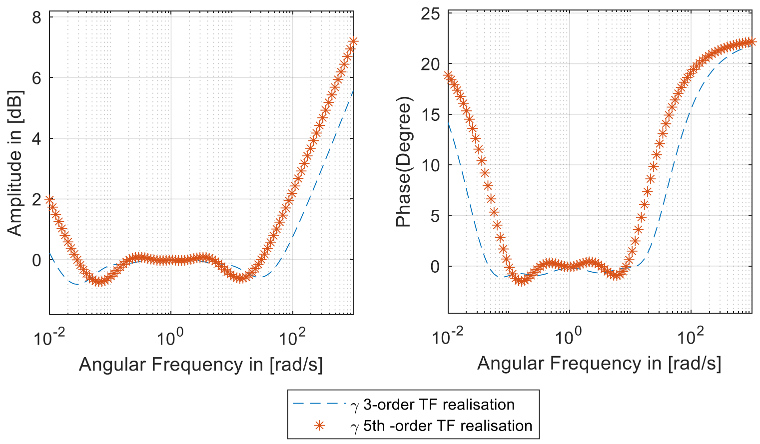

Fig. 2 Error comparison between fifth order and third order approximation of $s^{0.25}$ in delta domain using proposed method

While taking the whole range of frequency into consideration, the magnitude is more accurate as compared to the phase response. The approximation is compared on the basis of the maximum absolute magnitude and phase error as shown in Table 3. As we can see that the approximation results for the fifth order are more prominent than those of the third order, therefore fifth order CFE approximation has been chosen to develop the frequency responses for the different systems considered in this paper. At a sampling time of $\Delta = 0.01s$, the fifth order discrete realization of 1/4th order differentiator is considered based upon the four methods described in this paper namely CFE of Al-Alaoui (CFEAL), CFE of Tustin (CFETO), CFEDO and CFE of 2P GILOG in Delta domain (CFE2P-GILOGDel) and following results are obtained.

$$G_{Al5}(z)\big|_{\Delta=0.01} = \frac{(1409\text{-}3221z^{-1}+2435z^{-2}\text{-}639.5z^{-3}+6.82\ z^{-4}+5.449\ z^{-5})}{(430.9\text{-}861.9\ z^{-1}+533.6\ z^{-2}\text{-}90.88\ z^{-3}\text{-}7.06\ z^{-4}+z^{-5})} \quad (23)$$

$$G_{Tus5}(z)\big|_{\Delta=0.01} = \frac{(226.5 - 56.63\ z^{-1} - 245.4\ z^{-2} + 43.65\ z^{-3} + 51.03\ z^{-4} - 3.761\ z^{-5})}{(60.24 + 15.06\ z^{-1} - 65.25\ z^{-2} - 11.61\ z^{-3} + 13.57\ z^{-4} + z^{-5})} \quad (24)$$

$$G_{2P-GILOGDel5}(\gamma)\big|_{\Delta=0.01} = \frac{3.238\gamma^{11} + 3.685^{8}\gamma^{4} + 1.466^{5}\gamma^{3} + 0.002369\gamma^{2} + 0.1322\gamma + 1.439}{7.781^{12}\gamma^{5} + 9.632^{9}\gamma^{4} + 4.252^{6}\gamma^{3} + 0.000791\gamma^{2} + 0.05562\gamma + 1} \quad (25)$$

$$G_{CFEDO5}(\gamma)\big|_{\Delta=0.01} \frac{7157\gamma^{5} + 1.282^{5}\gamma^{4} + 4.186^{5}\gamma^{3} + 3.512^{5}\gamma^{2} + 7.025^{4}\gamma + 2057}{2417\gamma^{5} + 7.373^{4}\gamma^{4} + 3.56^{5}\gamma^{3} + 4.158^{5}\gamma^{2} + 1.242^{5}\gamma + 6526} \quad (26)$$

Table 3 Absolute maximum phase error and magnitude error for discretization of 0.25th -order differentiator using CFE2P-GILOGDel

Approximation order	Maximum magnitude error (dB)	Maximum phase error (degree)
Fifth	0.92	7.7415
Third	1.27	30.5

Example 2: A fractional order system [25] is considered:

$$G_1(s) = 0.191 + \frac{2.813}{s^{0.97}} \quad (27)$$

For the discretization of the above system, sampling time considered is $\Delta = 0.0001s$. The discretization of this continuous time transfer function results in four rational approximation T.F. as given by Eq. (28), Eq. (29), Eq. (30) and Eq. (31), by using four methods CFEAL, CFETO, CFEDO and CFE2P-GILOGDel, respectively,.

$$G_{Al5}(z)\big|_{\Delta=0.0001} = \frac{1.693^{7} - 4.564^{7}\ z^{-1} + 4.33^{7}\ z^{-2} - 1.665^{7}\ z^{-3} + 2.032^{6}\ z^{-4} + 2.54^{4}\ z^{-5}}{8.85^{7} - 2.387^{8}\ z^{-1} + 2.266^{8}\ z^{-2} - 8.719^{7}\ z^{-3} + 1.064^{7}\ z^{-4} + 1.33^{5}\ z^{-5}} \quad (28)$$

$$G_{Tus5}(z)\big|_{\Delta=0.0001} = \frac{(3.142^5\text{-}3.048^5\ z^{-1}\text{-}\ 2.177^5 z^{-2} + 2.052^5 z^{-3} + 1.822^4 z^{-4}\text{-}1.486^4\ z^{-5})}{(21.15\text{+}20.51 z^{-1}\text{-}14.65\ z^{-2}\text{-}13.81\ z^{-3}\text{+}1.226\ z^{-4}\text{+}z^{-5})} \quad (29)$$

$$G_{CFEDO5}(\gamma)\big|_{\Delta=0.0001} = \frac{\begin{array}{c}7157^{-18}\gamma^4 + 1.076^{-13}\gamma^4 + 3.584^{-9}\gamma^3 + 4.511^{-5}\gamma^2 + 0.1549\gamma \\ + 9.672\end{array}}{\begin{array}{c}6.698^{-18}\gamma^5 + 5.628^{-13}\gamma^4 + 1.874^8\gamma^3 + 0.0002356\gamma^2 + 0.8057\gamma \\ + 35.91\end{array}} \quad (30)$$

$$G_{2P-GILOGDel5}(\gamma)\big|_{\Delta=0.0001} = \frac{4238\gamma^5 + 3.512^4\gamma^4 + 7.721^5\gamma^3 + 1.341^6\gamma^2 + 6.34^5\gamma + 6.204}{2.207^4\gamma^5 + 2.25^5\gamma^4 + 4.603^5\gamma^3 + 2.431^5\gamma^2 + 2861\gamma + 24.51} \quad (31)$$

Example 3:

The FO system [14] is chosen and the transfer function is as follows:

$$G_2(s) = 428.68 + \frac{41.89}{s^{0.638}} \quad (32)$$

Here the sampling rate is made higher and that is considered as $\Delta = 0.00001s$. The discretization of this continuous time transfer function results in four rational approximation T.F., as given by Eq. (33), Eq. (34), Eq. (35) and Eq. (36), by using four methods CFEAL, CFETO, CFEDO and CFE2P-GILOGDel, respectively.

$$G_{Al5}(z)\big|_{\Delta=0.00001} = \frac{\begin{array}{c}8.257^8 - 2.07^9 z^{-1} + 1.782^9 z^{-2} - 5.872^8 z^{-3} \\ +4.794^7 z^{-4} + 3.057^6 z^{-5}\end{array}}{\begin{array}{c}1.926^6 - 4.829^6 z^{-1} + 4.156^6 z^{-2} - 1.37^6 z^{-3} \\ +1.118^5 z^{-4} + 7132 z^{-5}\end{array}} \quad (33)$$

$$G_{Tus5}(z)\big|_{\Delta=0.00001} = \frac{\begin{array}{l}2.732^7 - 1.743^7 z^{-1} - 2.541^7 z^{-2} + 1..277^7 z^{-3} \\ +4.282^6 z^{-4} - 1.033^6 z^{-5}\end{array}}{\begin{array}{l}6.372^4 - 4.065^4 z^{-1} + 5.927^4 z^{-2} + 2.978^4 z^{-3} \\ +9987 z^{-4} - 2410 z^{-5}\end{array}} \quad (34)$$

$$G_{CFEDO5}(\gamma)\big|_{\Delta=0.00001} = \frac{\begin{array}{c}5.662^6\gamma^5 + 7.637^7\gamma^4 + 2.036^8\gamma^3 + 1.436^8\gamma^2 \\ +2.053^7\gamma + 7.853^4\end{array}}{\begin{array}{c}1.315^4\gamma^5 + 1.751^5\gamma^4 + 4.964^5\gamma^3 + 2.913^5\gamma^2 \\ +3.077^4\gamma + 546.5\end{array}} \quad (35)$$

$$G_{2P-GILOGDel5}(\gamma)\big|_{\Delta=0.00001} = \frac{\begin{array}{c}5.507^{-21}\gamma^5 + 5.827^{-15}\gamma^4 + 2.116^{-9}\gamma^3 + \\ 0.0003013\gamma^2 + 13.34\gamma + 7.089^4\end{array}}{\begin{array}{c}1.285^{-23}\gamma^5 + 1.359^{-17}\gamma^4 + 4.936^{-12}\gamma^3 \\ +7.029^7\gamma^2 + 0.03112\gamma + 165.3\end{array}} \quad (36)$$

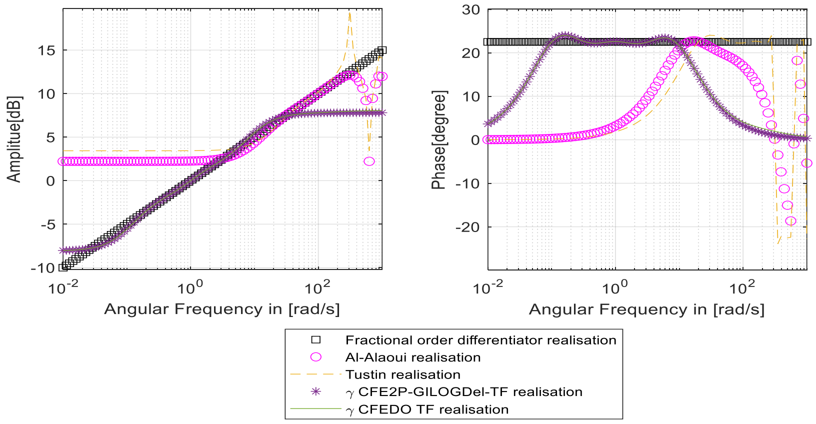

Fig. 3 Frequency response comparison after discretization of $G(s)$ using four methods at $r = 0.25$ and $\Delta = 0.01$

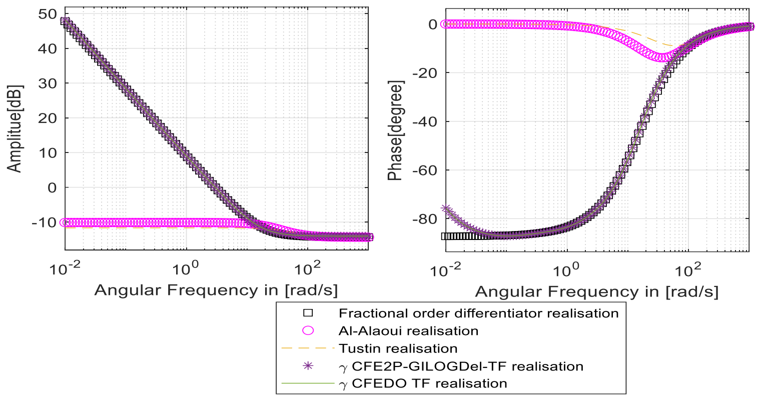

Fig. 4 Frequency response comparison after discretization of $G_1(s)$ using four methods $r = 0.97$ and $\Delta = 0.0001$

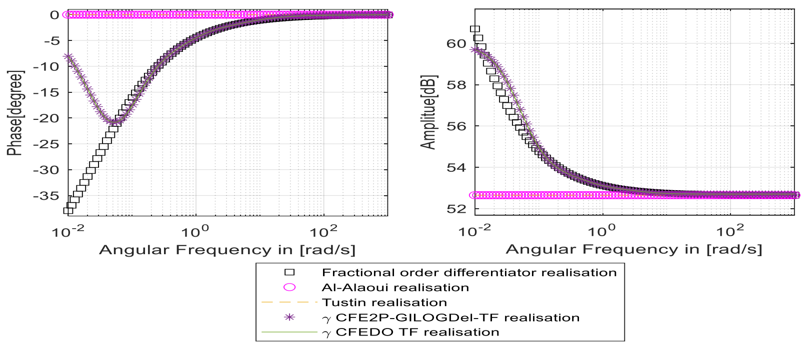

Fig. 5 Frequency response comparison after discretization of $G_2(s)$ using four methods at $r = 0.638$ and $\Delta = 0.00001$

Four different discretization methods are utilized to discretize three fractional order systems as shown in three examples. The frequency responses of all the systems (fractional order) along with the frequency responses of their corresponding discrete-time approximated systems are shown in Fig. 3, Fig. 4, and Fig. 5, respectively. In all the discretization methods magnitude approximation turns out to be superior over the phase approximation. From the Fig. 3, Fig. 4 and Fig. 5, it is evident that the proposed method, CFE2P-GILOGDel produces excellent frequency responses in the frequency range of (0.001 rad/s to 1000rad/sec). Therefore, through experimental analysis, the proposed method is more promising than the other three approaches for discretization with respect to approximation of original fractional order system. Moreover, the comparison of the outcomes with another method developed in the delta domain been made and superiority of the proposed method is established. The CFE2P-GILOGDel method at high sampling time ($\Delta = 0.00001$) provides frequency responses very much closer to the original fractional order system as can be seen from Fig. 5. This leads to a development of a unified approach towards the discretization of fractional order operator or system in complex delta domain means at high sampling rate the continuous time result and discrete time results can be obtained at the same time and is a sole reason for the development of discrete time systems' in delta operator parameterization.

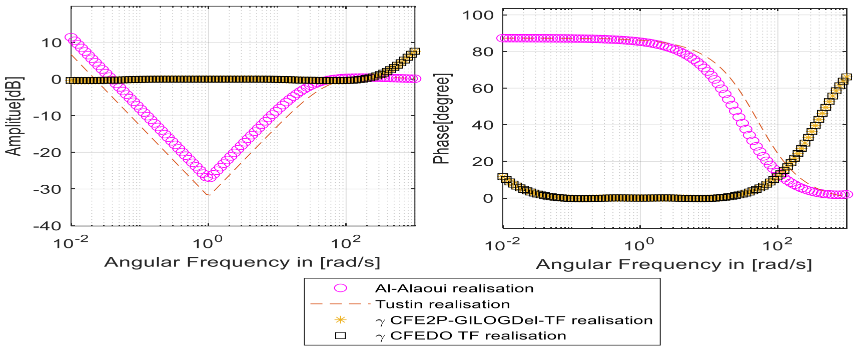

Fig. 6 Magnitude and phase error after discretization of $G(s)$ using four methods at $r = 0.25$ and $\Delta = 0.01$

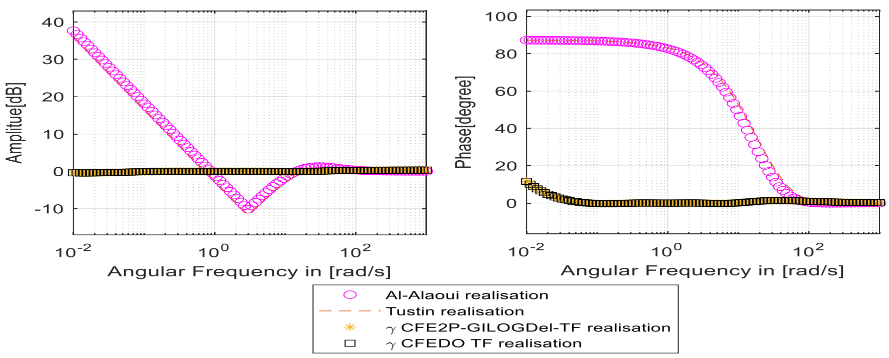

Fig. 7 Magnitude and phase error after discretization of $G_1(s)$ using four methods at $r = 0.97$ and $\Delta = 0.0001$

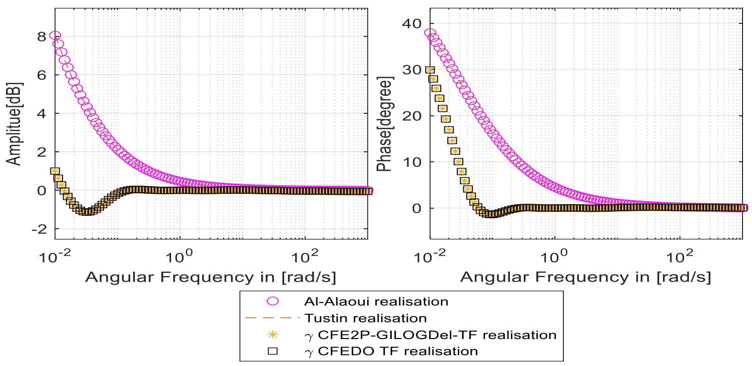

Fig. 8 Magnitude and phase error after discretization of $G_2(s)$ using four methods at $r = 0.638$ and $\Delta = 0.00001$

Table 4 Absolute maximum magnitude error and phase error for four discretization methods for different systems

FOS	Max. magnitude error (dB)				Max. phase error (degree)			
	CFE2PGILOGDel	CFEDO	Al-Alaoui	Tustin	**CFE2P-GILOGDel**	CFEDO	Al-Alaoui	Tustin
$G(s) = s^{0.25}$	**0.72**	1.06	1.11	1.2	**7.74**	18.3	44.79	44.88
$G_1(s) = 0.191 + \frac{2.813}{s^{0.97}}$	**1.66**	2.12	5.83	24.27	**44.46**	45.1	79.83	88.02
$G_2(s) = 428.68 + \frac{41.89}{s^{0.638}}$	**7.6**	7.94	28.76	35.78	**82.54**	82.8	103.52	112.44

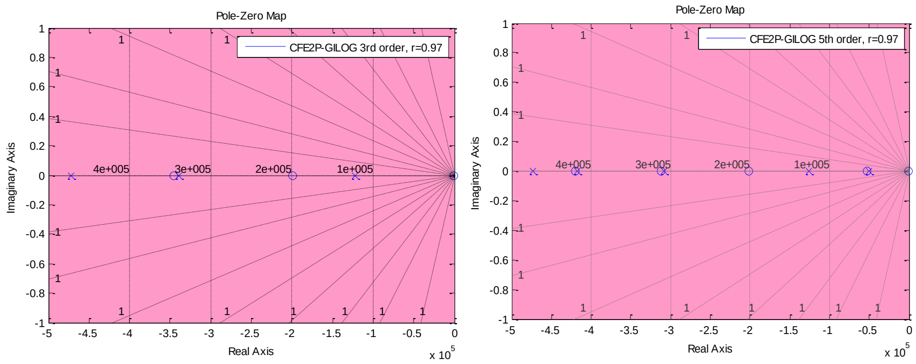

Fig. 9 Pole-zero plot for the third-order and fifth order approximation of $s^{0.97}$ using CFE2P-GILOGDel method

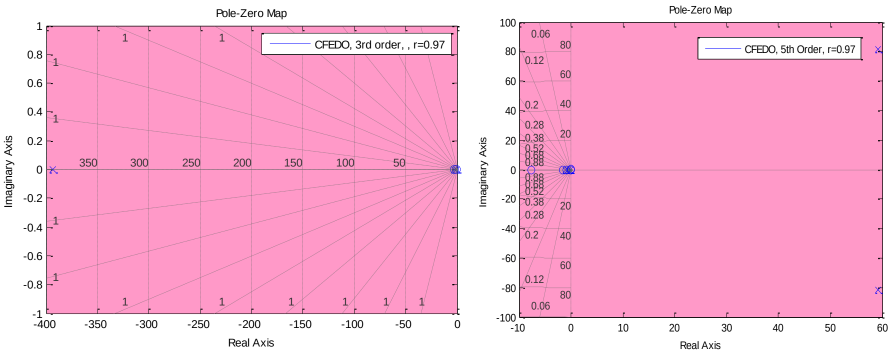

Fig. 10 Pole zero plot for the third-order and fifth order approximation of $s^{0.97}$ using CFE-DO method

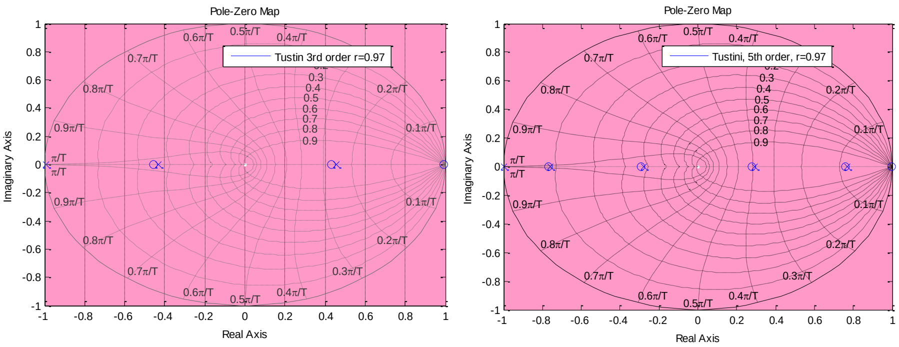

Fig. 11 Pole-zero plot for the third-order and fifth order approximation of $s^{0.97}$ using Tustin method

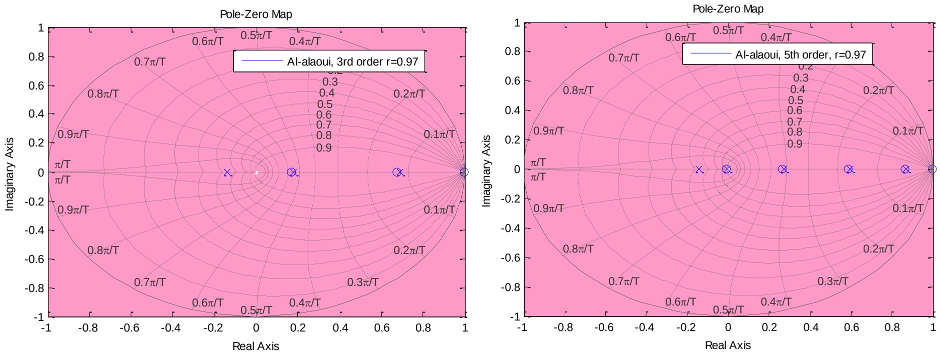

Fig. 12 Pole-zero plot for the third-order and fifth order approximation of $s^{0.97}$ using Al-Alaoui method

From the Table 4, it is clearly observed that when the sampling time is increased to a very high limiting value $\Delta = 0.00001s$, the maximum absolute magnitude error and phase error is much higher in case of discretization using Tustin and Al-Alaoui method in Z-domain in comparison to the discretization using Delta operator parameterization. The graphical representation can also be viewed from Fig. 8. Also, it can be seen that the proposed method is superior to the other methods in the literature. At the same time, a comparison has been made for the fifth order approximation of $s^{0.97}$ using another delta domain based approach, CFEDO method, where poles are in the right half of the plane Fig. 10, thus making the rational transfer function of the system unstable, whereas the method proposed in this paper shows that in both third order and fifth order the poles in the region itself are making the system stable. So, it is evident that the proposed method delivers preferable approximation amidst all four discretization methods and is a viable alternative in the literature of direct discretization of fractional order operator in delta domain.

The following analysis has been done to prove the novelty of the direct discretization of fractional order operator ($s^{\pm\mu}$, $0 < \mu < 1$) over the indirect discretization of the fractional order operator in delta domain. For the illustration purpose, a 1/4th order differentiator is considered for the discretization purpose. This operator is discretized using indirect discretization using Oustaloup approximation [33] method as an intermediate step.

Rational approximation of $s^{0.25}$ is obtained using [33] as given in Eq. (37).

$$\frac{3.162s^7 + 1899s^6 + 2.411e05s^5 + 7.763e06s^4 + 6.586e07s^3 + 1.472e08s^2 + 8.343e07s + 1e07}{s^7 + 834.3s^6 + 1.472e05s^5 + 6.586e06s^4 + 7.763e07s^3 + 2.411e08s^2 + 1.899e08s + 3.162e07} \quad (37)$$

Eq. (37) is discretized in delta domain to get the rational approximation of $s^{0.25}$.

$$\frac{3.162\gamma^7 + 1532\gamma^6 + 1.745e05\gamma^5 + 5.357e06\gamma^4 + 4.459e07\gamma^3 + 9.897e07\gamma^2 + 5.595e07\gamma + 6.7e06}{\gamma^7 + 667\gamma^6 + 1.056e05\gamma^5 + 4.52e06\gamma^4 + 5.243e07\gamma^3 + 1.619e08\gamma^2 + 1.273e08\gamma + 2.119e07} \quad (38)$$

The rational approximation of $s^{0.25}$ in delta domain using proposed direct discretization method is illustrated in Eq. (39)

$$\frac{2.55954\gamma^5 + 0.0235\gamma^4 + 0.000042\gamma^3 + 2.6066e(-08)\gamma^2 \;\; 6.5524e(-12)\gamma + 5.75821e(16)}{\gamma^5 + 0.00556\gamma^4 + 0.000007\gamma^3 + 4.25183e(-9)\gamma^2 + 9.63178e(-13)\gamma + 7.7805e(-17)} \quad (39)$$

A comparative analysis between the direct discretization and indirect discretization using delta operator based parameterization is graphically demonstrated in Fig. 13 and Fig. 14 respectively.

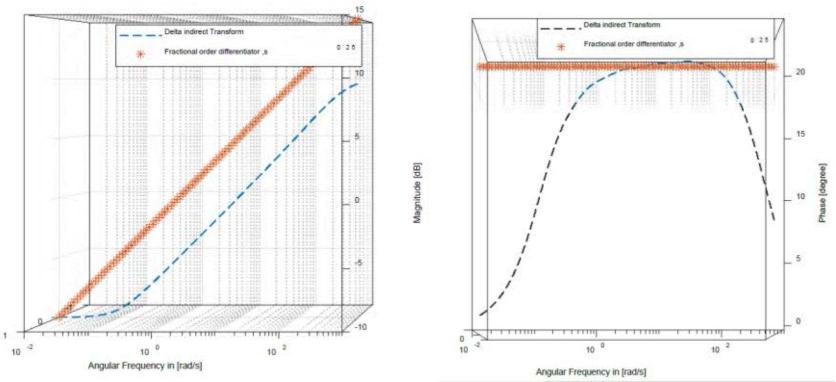

Fig. 13 Frequency Response using Indirect Discretization of $s^{0.25}$ at Δ=0.001s

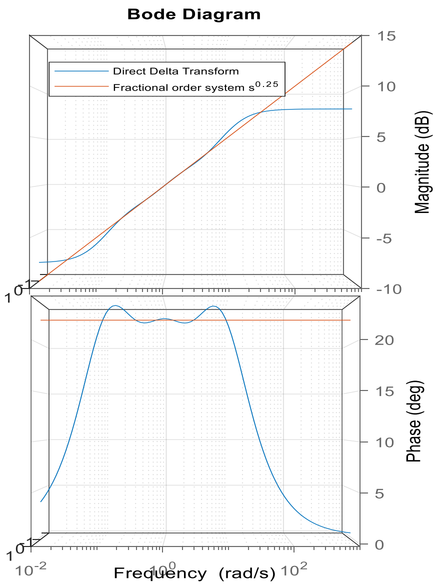

Fig. 14 Frequency response using direct discretization of $s^{0.25}$ at Δ=0.001s

From the above figure it is clear that using the direct discretization the magnitude and phase plot resembles that of the 1/4th order differentiator in continuous time domain, whereas there is a notable deviation of the magnitude and phase curve when indirect discretization is approached. Therefore, direct discretization of the fractional operator in delta domain is superior over indirect discretization.

5. CONCLUSION

In this paper, a new direct discretization method for fractional order operator is proposed. The traditional discretization method for fractional order operator works in the discrete Z-domain and at a high sampling frequency, the resulting system fails to provide meaningful information. Instead, delta operator parameterized systems give continuous time results at high sampling frequency. In this work, an approximation mapping between the S-domain and delta domain is established through trapezoidal quadrature rule and traditional CFE, method is used to obtain rational transfer function corresponding to the fractional order operator in discrete delta domain.

Simulation results show that the proposed discretization method using delta operator is producing gratifying frequency response approximation of the original fractional order system in resemblance to other two discretization methods. At fast sampling rate, the discretized system produces almost the same frequency responses as those of continuous time counter-part. This successfully proves the efficiency of the suggested approach to be a viable alternative to that of the direct discretization methods of discretizing the fractional order operator or systems available in the concerned literature and leading to the development of a unified approach for direct discretization of FOS in delta domain.

REFERENCES

[1] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Application of Differentiation and Integration to Arbitrary Order*. Dover Books on Mathematics, 2006.

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York: Wiley, 1993.

[3] R. Caponetto, G. Dongola, L. Fortuna and I. Petras, *Fractional Order Systems, Modeling and Control Applications*, Singapore: World Scientific, 2010, pp. 43–54.

[4] H. H. Sun, B. Onaral and Y. Tsao, "Application of the positive reality principle to metal electrode linear polarization phenomena", *IEEE Trans. Biomed., Eng*. vol. 31, pp. 664–674, 1984.

[5] H. H. Sun, A. A. Abdelwahab and B. Onaral, "Linear approximation of transfer function with a pole of fractional order", *IEEE Trans. Autom. Control.*, vol. 29, pp. 441–444, 1984.

[6] S. B. Skaar, A. N. Michel and R. K. Miller, "Stability of viscoelastic control systems", *IEEE Trans. Auto. Control.*, vol. 33, no. 4, pp. 348–357, 1988.

[7] N. Engheta, "Fractional calculus and fractional paradigm in electromagnetic theory", In Proceedings of the VIIth International Conference on Mathematical Methods in Electromagnetic Theory, 1998, pp. 879–880.

[8] I. Podlubny, "Fractional-order systems and $PI^{\lambda}D^{\mu}$ controllers", *IEEE Trans. Automatic Contro.l*, vol. 44, pp. 208–214, 1999.

[9] N. Nakagava and K. Sorimachi,"Basic characteristics of a fractance device" *IEICE Trans. Fundamentals,* vol. E75-A, pp.1814–1819, 1992.

[10] B. M. Vinagre, I Podlubny, A Hernández and V. Feliu, "Some approximation of fractional-order operator used in control theory and application", *J. Fractional Calc. and Appl.*, vol. 3, pp. 231–248, 2000.

[11] D. Xue, C. Zhao and Y. Chen, "A modified approximation method of fractional order system", In Proceedings of the IEEE International Conference on Mechatronics and Automation (ICMA), Luoyang, 2006, pp. 1043–1048.

[12] M. Khanra, J. Pal and K. Biswas, "Rational approximation of fractional operator –a comparative study", In Proceedings of the IEEE international conference on power, Control and Embedded Systems (ICPCES), Allahabad, 2010, pp. 1–5.

[13] A. Oustaloup, *La Derivation Non Entiere: Theorie, Synthese et Applications*. Hermes, Paris, 1995.

[14] B. T. Krishna, "Studies on fractional-order differentiators and integrators: A survey", *Signal Processing*, vol. 91, pp. 386–426, 2011.

[15] G. Maione, "High-speed digital realizations of fractional operator in the delta-domain", *IEEE Trans. Automat. Control*, vol. 56, pp. 697–702, 2011.

[16] R. D. Keyser and C. I. Muresan, "Analysis of a new continuous-to-discrete-time operator for the approximation of fractional order system", In Proceedings of the IEEE International Conference on Systems, Man and Cybernetics (SMC), Budapest, 2016, pp. 003211–003216.

[17] B. T. Krishna, "Design of fractional-order differ integrators using reduced order s to z transforms", In Proceeding of the 10th IEEE International Conference on Industrial and Information Systems, Peradeniya, 2015, pp. 469–473.

[18] Y. Q. Chen and K. L. Moore, "Discretization schemes for fractional-order differentiators and Integrators", *IEEE Trans. Circuits System.*, vol. 49, no. 3, pp. 363–367, 2000.

[19] Y. Chen, I. Petráš and D. Xue, "Fractional Order Control-A Tutorial", In Proceeding of the American Control Conference, 2009, pp. 1397–1411.

[20] R. H. Middleton and G. C. Goodwin, *Digital Control and Estimation-A Unified Approach*. Englewood Cliffs, New Jersey: Prentice-Hall, 1990.

[21] R. H. Middleton and G. C. Goodwin, "Improved finite word length characteristics in digital control using delta operator", *IEEE Trans. Automatic Cont.*, vol. 31, pp. 1015–1021, 1986.

[22] J. Cortes-Romero, A. luviana-Juarez and H. Sira-Ramirez, "A Delta Operator Approach for the Discrete-Time Active Disturbance Rejection Control on Induction Motors", *Math. Probl. Eng.*, vol. 2013, pp. 1–9, 2013.

[23] Y. Zhao and D. Zhang, "H-∞ Fault Detection for Uncertain Delta Operator Systems with Packet Dropout and Limited Communication", In Proceedings of the American Control Conference, Seattle, 2017, pp. 4772–4777.

[24] J. Gao, S. Chai, M. Shuai, B. Zhang and L. Cui, "Detecting False Data Injection attack on cyber-physical system based on Delta Operator", In Proceeding of the 37th Chinese Control Conference, Wuhan, 2018, pp. 5961–5966.

[25] J. Swarnakar, P. Sarkar, M. Dey and L. Joyprakash Singh, "A unified approach for reduced order modelling of fractional order system in delta domain- a unified approach", In Proceeding of the IEEE region 10 Humanitarian Technology Conference (R-10 HTC), 2017, pp. 144–150.

[26] P. Sarkar, R. R. Shekh and A. Iqbal, "A unified approach for reduced order modelling of fractional order system in delta domain", In Proceeding of the IEEE International Automatic Control Conference (CACS), Taichung, 2016, pp. 257–262.

[27] S. Ganguli, G. Kaur and P. Sarkar, "Global heuristic methods for reduced-order modelling of fractional-order systems in the delta domain: a unified approach", *Ricerche di Matematica*, 2021.

[28] L. A. Quezada-Téllez, L. Franco-Pérez and G. Fernandez-Anaya, "Controlling Chaos for a Fractional-Order Discrete System", *IEEE Open J. Circuits Syst.*, vol. 1, pp. 263–269, 2020.

[29] O. Lamrabet, E. H. Tissir and F. E. Haoussi, "Controller design for delta operator time-delay systems subject to actuator saturation", In Proceedings of the International Conference on Intelligent Systems and Computer Vision (ISCV), 2020, pp. 1–6.

[30] I. Pan and S. Das, *Intelligent Fractional Order Systems and Control*, Studies in computational intelligence, Springer-Verlag, Berlin Heidelberg, 2013.

[31] B. M. Vinagre, Y. Q. Chen and I. Petrášc, "Two direct Tustin discretization methods for fractional-order differentiator/integrator", *J. Franklin Institute*, vol. 340, pp. 349–362, 2003.

[32] S. K. Khattri, "New close form approximation of ln(1+x)", *The Teaching of Mathematics*, vol. 12, pp. 7–14, 2009.

[33] J. Baranowski, W. Bauer, M. Zagorowska, T. Dziwinski and P. Piatek, "Time-domain oustaloup approximation"," In Proceedings of the 20th International Conference on Methods and Models in Automation and Robotics (MMAR), 2015, pp. 116–120.