FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 3 (2022), 585–593
https://doi.org/10.22190/FUMI211013040A
Original Scientific Paper

STATISTICAL UNBOUNDED ORDER CONVERGENCE IN RIESZ SPACES

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Abstract. Some kinds of statistical unbounded convergence were studied and investigated with respect to the solid topology and order convergence. In this paper, we study the concept of statistical unbounded order convergence in Riesz spaces by a topology-free technique with the order convergence on Riesz spaces. Moreover, we give some relations with other kinds of statistical convergences.
Key words: statistical *uo*-convergence, order convergence, statistically order convergence, Riesz spaces

1. Introduction and Preliminaries

Statistical convergence is a generalization of the ordinary convergence of a real sequence. It was introduced by Steinhaus in [17]. It is enough to mention the theory of statistical convergence (cf. [8, 9, 14]). On the other hand, Riesz space (or, vector lattice) is another concept of functional analysis that was introduced by F. Riesz [15]. Then many others have developed the subject. An ordered vector space has many applications in measure theory, Banach space, operator theory, and applications in economics (cf. [1, 2, 4, 10, 11, 21]). Recently, the concept of statistical unbounded convergence on Banach lattice has been studied by some authors (cf. [19, 20]). The main aim of the present paper is to extend the concepts of statistical unbounded order convergence in Riesz spaces.

Received October 13, 2021, accepted: May 25, 2022
Communicated by Aleksandar Nastić
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2010 *Mathematics Subject Classification.* Primary 40A05; Secondary 46B42, 46J40, 47B65

For the definition of statistical convergence, the important point is the natural density of subsets of natural numbers. Recall that the *density of* a subset K of $\mathbb{N}$ is the limit $\lim_{n\to\infty}\frac{1}{n}\left|\{k\le n : k\in K\}\right|$ whenever this unique limit exists. Also, it is mostly abbreviated by $\delta(K)$, where $\left|\{k\le n : k\in K\}\right|$ is the cardinality of K and it does not exceed n. A sequence (x_n) of real numbers is called *statistical convergent* to a real number x if, for every $\varepsilon>0$, we have

$$\lim_{n\to\infty}\frac{1}{n}\big|\{k : n\ge k,\ |x_n-x|>\varepsilon\}\big|=0.$$

A lot of generalizations and applications of statistical convergence have been investigated by several authors (cf. [3, 5, 6, 7, 12, 18, 19, 20]). Throughout this paper, the vertical bar of sets will stand for the cardinality of the given sets.

A real-valued vector space E with an order relation is called an *ordered vector space* if, for each $x,y\in E$ with $x\le y$, we have $x+z\le y+z$ and $\alpha x\le\alpha y$ for all $z\in E$ and $\alpha\in\mathbb{R}_+$. A vector subspace F of an ordered vector space E is *majorizing* E if, for every $x\in E$ there exists some $y\in F$ with such that $y\le x$. An ordered vector space E is called *Riesz space* or *vector lattice* if, for any two vectors $x,y\in E$, the infimum and the supremum

$$x\wedge y=\inf\{x,y\}\quad\text{and}\quad x\vee y=\sup\{x,y\}$$

exist in E, respectively. A vector lattice is called *σ-order complete* if every nonempty bounded above countable subset has a supremum (or, equivalently, whenever every nonempty bounded below countable subset has an infimum). A subvector lattice F of a vector lattice E is called *order dense* in E if, for every $0<x\in E$, there exists some $y\in F$ such that $0<y\le x$. For an element x in a vector lattice E, *the positive part*, *the negative part*, and *module* of x are respectively

$$x^+:=x\vee 0,\quad x^-:=(-x)\vee 0\ \ \textit{and}\ \ |x|:=x\vee(-x).$$

Thus, in the present paper, the vertical bar $|\cdot|$ of elements in vector lattices will stand for the module of the given elements. Take a nonempty subset A of a vector lattice E. Then its disjoint complement is denoted by $A^d:=\{x\in E : x\perp y \text{ for all } y\in A\}$. If $|x|\wedge|y|=0$ holds for any two elements x and y then they are called *disjoint* (or, symbols $x\perp y$). A characterization of statistical convergence on vector lattices was introduced by Ercan in [7], and also, Şençim and Pehlivan studied the statistical order convergence on Riesz spaces; see [18]. Moreover, statistical convergence was introduced and studied by Aydın in [3] concerning unbounded order convergence on locally solid Riesz spaces. Also, Wang et al. studied unbounded order convergence on locally on Banach lattice in [19, 20].

We turn our attention to unbounded order convergence. The *uo*-convergence was introduced in [13] under the name of individual convergence. We refer the reader for an exposition on *uo*-convergence to [10]. The order convergence and *uo*-convergence is crucial for this paper, and so, we continue with their definitions.

Definition 1.1. Let (x_n) be a sequence in a vector lattice E. Then it is called

(1) *order convergent* to $x \in E$ if there exists another sequence $y_n \downarrow 0$ (i.e., $\inf y_n = 0$ and $y_n \downarrow$) such that $|x_n - x| \leqslant y_n$ holds for all $n \in \mathbb{N}$, and abbreviated as $x_n \xrightarrow{o} x$,

(2) *unbounded order convergent* to $x \in E$ if $|x_n - x| \wedge u \xrightarrow{o} 0$ for every $u \in E_+$, and so, we write $x_n \xrightarrow{uo} x$.

It is clear that the order convergence implies the *uo*-convergence by virtue of $|x_n - x| \wedge u \leq |x_n - x|$ for any elements in vector lattices. The converse need not be true in general. To see this, we give the following example.

Example 1.1. Consider the sequence (e_n) of the standard unit vectors in c_0. Then we have $e_n \xrightarrow{uo} 0$ because *uo*-convergence in c_0 is equivalent to the coordinate-wise convergence; see [10, p.22]. But, it is not convergent in the order convergence because (e_n) is not order bounded in c_0.

It is clear that *uo*-convergence is equivalent to order convergence for order bounded sequences.

2. The statistical *uo*-convergence

We begin the section with the notion of statistical monotonicity. It was introduced in [16] for real sequences. We take the following notions from [18].

Definition 2.1. Let E be a vector lattice and (x_n) be a sequence in E. Then (x_n) is called

(a) *statistical monotone convergent* to $x \in E$ if there exists a subset $J = \{j_1 < j_2, \cdots\}$ in $\mathbb{N}$ such that $\delta(J) = 1$ and $(q_{j_n})_j \downarrow x$, and we abbreviate it as $q_n \downarrow^{st} x$,

(b) *statistical order converges* to $x \in E$ if there are a sequence $q_n \downarrow^{\text{st}} 0$ and a subset $\delta(J) = 1$ of $\mathbb{N}$ such that $|x_n - x| \leq q_n$ for all $n \in J$, and we write $x_n \xrightarrow{\text{st-o}} x$.

One can observe that every order convergent monotone sequence in vector lattices is statistical monotone convergent to its order limit. Motivated from above definitions, we give the following main notion of this paper, which is also introduced by Wang et al., in [20].

Definition 2.2. Let E be a vector lattice and (x_n) be a sequence in E. Then (x_n) is called *statistical unbounded order convergent* (or, *statistical uo-convergent*, shortly) to $x \in E$ if, for every $u \in E_+$, there exists a sequence $q_n \downarrow^{st} 0$ and a subset J of the natural numbers with $\delta(J) = 1$ such that

$$|x_{j_n} - x| \wedge u \leqslant q_{j_n}$$

for all $j_n \in J$. We abbreviate it as $x_n \xrightarrow{\text{st-uo}} x$.

It is clear that that the statistical *uo*-convergence can be redefined as follows: if, for each $u \in E_+$, there exists a sequence $q_n \downarrow^{\mathrm{st}} 0$ such that $\delta\big(\{n \in \mathbb{N} : |x_n - x| \wedge u \nleq q_n\}\big) = 0$ then $x_n \xrightarrow{\text{st-uo}} x$.

Example 2.1. Let's consider the vector lattice $E := \mathbb{R}$ the set of all real numbers. Take a sequence (x_n) in E denoted by n^3 whenever $n = k^3$ for some $k \in \mathbb{N}$, and otherwise denoted by $1/(1 + 2n)$. Now, we choose a sequence (q_n) such that

$$q_n := \begin{cases} n, & n = k^3 \\ \frac{1}{n}, & \text{otherwise} \end{cases}.$$

It is clear that $q_n \downarrow^{\mathrm{st}} 0$. Also, when we consider the set J as $\{1, n \in \mathbb{N} : n \neq k^3 \text{ for some } k \in \mathbb{N}\}$, it can be seen that $\delta(J) = 1$ and $|x_{j_n}| \wedge u \leq q_{j_n}$ for all $u \in E_+$ and for every $j_n \in J$. So, we obtain $x_n \xrightarrow{\text{st-uo}} 0$.

Proposition 2.1. *Every uo-convergent sequence in a vector lattice is statistical uo-convergent to its uo-limit.*

Proof. Let (x_n) be *uo*-convergent to x and u be a fixed positive element in a vector lattice E. Then it follows from Definition 1.1 (2) that $|x_n - x| \wedge u \xrightarrow{o} 0$. Thus, by Definition 1.1 (1), there exists a sequence $y_n \downarrow 0$ in E such that $|x_n - x| \wedge u \leqslant y_n$ holds for all $n \in \mathbb{N}$. Since (y_n) is decreasing and order convergent to zero, we have $y_n \downarrow^{\mathrm{st}} 0$. Let's take the subset J as $\mathbb{N}$. Then we get the desired, $x_n \xrightarrow{\text{st-uo}} x$, result because $u \in E_+$ is arbitrary. □

Corollary 2.1. *The order convergence implies the statistical unbounded order convergent in vector lattices.*

Corollary 2.2. *The statistical order convergence implies statistical uo-convergence.*

3. Main Results

Theorem 3.1. *The lattice operations are continuous with the statistical uo-convergence.*

Proof. It is enough to show the continuity of the supremum operation. The other cases are analogies. Suppose that $x_n \xrightarrow{\text{st-uo}} x$ and $y_n \xrightarrow{\text{st-uo}} y$ hold in a vector lattice E. Fix $u \in E_+$. Then there exist sequences $q_n \downarrow^{st} 0$ and $p_n \downarrow^{st} 0$, and subsets J and K of the natural numbers with $\delta(J) = \delta(K) = 1$ such that $|x_{j_n} - x| \wedge u \leqslant q_{j_n}$ and $|x_{k_n} - y| \wedge u \leqslant p_{k_n}$ for all $j_n \in J$ and $k_n \in K$. Choose $M = J \cap K$. Thus, we have $\delta(M) = 1$, $|x_{n_m} - x| \wedge u \leqslant q_{n_m}$ and $|x_{n_m} - y| \wedge u \leqslant p_{n_m}$ for every $n_m \in M$. Thus, it follows from [2, Thm.1.2(2)] that

$$\begin{aligned} |x_{n_m} \vee y_{n_m} - x \vee y| \wedge u &\leq |x_{n_m} - x| \wedge u + |y_{n_m} - y| \wedge u \\ &\leq q_{n_m} + p_{n_m} \end{aligned}$$

for each $m \in \mathbb{N}$. Let consider a new sequence $r_n := q_n + p_n$ then we have $|x_{n_m} \vee y_{n_m} - x \vee y| \wedge u \leq r_{n_m}$ and $r_n \downarrow^{\mathrm{st}} 0$. Hence, we obtain $x_n \vee y_n \xrightarrow{\text{st-uo}} x \vee y$ in E. □

Corollary 3.1. *If* $x_n \xrightarrow{\text{st-uo}} x$ *holds in vector lattices then we have*

(i) $(x_n)^+ \xrightarrow{\text{st-uo}} x^+$,

(ii) $(x_n)^- \xrightarrow{\text{st-uo}} x^-$,

(iii) $|x_n| \xrightarrow{\text{st-uo}} |x|$.

It is not hard to see that a subsequence of a statistical *uo*-convergent sequence needs not be *uo*-convergent (cf. [18, Exam.4]).

The following basic results are motivated by their analogies from vector lattice theory.

Theorem 3.2. *Let* E *be a vector lattice. If* $x_n \xrightarrow{\text{st-uo}} x$ *and* $y_n \xrightarrow{\text{st-uo}} y$ *hold in* E *then we have the following results:*

(i) $x_n \xrightarrow{\text{st-uo}} x$ *iff* $(x_n - x) \xrightarrow{\text{st-uo}} 0$ *iff* $|x_n - x| \xrightarrow{\text{st-uo}} 0$;

(ii) The statistical uo-limit is linear;

(iii) The statistical uo-convergent has a unique limit;

(iv) The positive cone E_+ *is closed under the statistical uo-convergence;*

(v) Suppose $0 \leq x_n \leq y_n$, *and so, we have* $x \leq y$.

Proof. The first three properties are straightforward. For (iv), take a non-negative and statistical *uo*-convergent sequence $x_n \xrightarrow{\text{st-uo}} x$ in E. Then it follows from Corollary 3.1 that $x_n = x_n^+ \xrightarrow{\text{st-uo}} x^+$. Moreover, by applying (iii), we obtain $x = x^+$. So, we get the desired, $x \in E_+$, result.

(v) Assume $0 \leq x_n \leq y_n$. Then, by applying Corollary 3.1, we have $x_n = |x_n| \xrightarrow{\text{st-uo}} |x|$. Thus, we can see that $x = |x| \geq 0$ due to the uniqueness of statistical *uo*-limit. As a result, we obtain $y \geq x$ because of $0 \leq y_n - x_n \xrightarrow{\text{st-uo}} y - x$. □

Be reminded that a positive vector $e > 0$ in a vector lattice is called *weak order unit* whenever, for each positive element x, we have $x \wedge ne \uparrow x$.

Theorem 3.3. *Let* e *be a weak order unit and* (x_n) *be a sequence in a* σ*-order complete vector lattice* E. *Then* $x_n \xrightarrow{\text{st-uo}} 0$ *if and only if* $|x_n| \wedge e$ *statistical order converges to zero.*

Proof. Suppose $x_n \xrightarrow{\text{st-uo}} 0$. Then it is not hard to see $|x_n| \wedge e \xrightarrow{\text{st-o}} 0$. For the converse, assume that $|x_n| \wedge e$ is statistical order convergent to 0 in E. So, following from Definition 2.1(b), there exists a sequence $q_n \downarrow^{st} 0$ and a subset $\delta(J) = 1$ such that $|x_{j_n}| \wedge e \leqslant q_{j_n}$ for all $j_n \in J$. Take a fixed $u \in E_+$. Thus, the inequality

$$\begin{aligned} |x_{j_n}| \wedge u &\leq |x_{j_n}| \wedge (u - u \wedge ne) + |x_{j_n}| \wedge (u \wedge ne) \\ &\leq (u - u \wedge ne) + n(|x_{j_n}| \wedge e), \end{aligned}$$

holds for any $j \in J$ and $n \in \mathbb{N}$. Since E is σ-order complete, we have

$$\limsup_j |x_{j_n}| \wedge u \leq (u - u \wedge ne) + n \limsup_j (|x_{j_n}| \wedge e)$$

for every $n \in \mathbb{N}$. On the other hand, there is another subset K of $\mathbb{N}$ such that $\delta(K) = 1$ and $(q_{k_n})_k \downarrow 0$ because of $q_n \downarrow^{\text{st}} 0$. Next, choose $M = J \cap K$. Then $\delta(M) = 1$ and $|x_{n_m}| \wedge e \leqslant q_{m_n}$ for all $m \in \mathbb{N}$, and so, we obtain $\limsup\limits_m (|x_{m_n}| \wedge e) = 0$. Also, we see

$$\limsup_m |x_{m_n}| \wedge u \leq (u - u \wedge ne)$$

holds for all $n \in \mathbb{N}$. Take $p_n := (u - u \wedge ne)$. Then one can see that $(p_n) \downarrow 0$, and so, $p_n \downarrow^{\text{st}} 0$ since e is the weak order unit. Therefore, we obtain the desired, $x_n \xrightarrow{\text{st-uo}} 0$, result because u is arbitrary. □

Recall that a subset A of a vector lattice E is called *solid* if, for each $x \in A$ and $y \in E$, $|y| \leq |x|$ implies $y \in A$. A solid vector subspace of a vector lattice is referred to as an *ideal*. An order closed ideal is called a *band* (cf. [1]).

Remark 3.1. Let A be an ideal in a vector lattice E and (a_n) be a sequence in A. One can observe that if $a_n \xrightarrow{o} 0$ in A then $a_n \xrightarrow{o} 0$ in E. Hence, it clear that $a_n \downarrow^{\text{st}} 0$ in A implies $a_n \downarrow^{\text{st}} 0$ in E. For the converse, if $a_n \xrightarrow{o} 0$ in E and order bounded then $a_n \xrightarrow{o} 0$ in A, and so, $a_n \downarrow^{\text{st}} 0$ in E implies $a_n \downarrow^{\text{st}} 0$ in A for order bounded sequences.

Thanks to Remark 3.1, we give the following two results.

Theorem 3.4. *Let A be an ideal in an σ-order complete vector lattice and (x_n) be a sequence in A. Then $x_n \xrightarrow{\text{st-uo}} 0$ in A if and only if $x_n \xrightarrow{\text{st-uo}} 0$ in E.*

Proof. Suppose that $x_n \xrightarrow{\text{st-uo}} 0$ hold in A. Then, for arbitrary $u \in A_+$, there exists a sequence $q_n \downarrow^{st} 0$ in A and a subset J of the natural numbers with $\delta(J) = 1$ such that $|x_{j_n}| \wedge u \leq q_{j_n}$ for all $j \in \mathbb{N}$. Also, since $q_n \downarrow^{st} 0$ holds, there is a subset K in $\mathbb{N}$ such that $\delta(K) = 1$ and $(q_{k_n})_k \downarrow 0$ in A. Then, by previous remark, $(q_{k_n})_k \downarrow 0$ in E. Now, let's take $0 \leq w \in A^d$ and $H = J \cap K$. Then we have

$$|x_{n_h}| \wedge (u + w) = |x_{n_h}| \wedge u \leq q_{n_h}$$

for each $h \in \mathbb{N}$. Now, choose arbitrary $z \in E_+$ and $a \in (A \oplus A^d)_+$. Then we have $z \wedge a \in (A \oplus A^d)_+$. So, there exists a sequence $t_n \downarrow^{st} 0$ in E and a subset $\delta(M) = 1$ such that

$$|x_{n_m}| \wedge (z \wedge a) \leq t_{n_m}$$

for all $n_m \in M$. It means that

$$\left(\inf_m \sup_{i \geq m} |x_{n_i}| \wedge z\right) \wedge a = \inf_m \sup_{i \geq m} (|x_{n_i}| \wedge (z \wedge a)) \leq t_{n_m}$$

holds for every $n_m \in M$ in E. Since $(A \oplus A^d)$ is order dense in E for any ideal A in E, it follows from [2, Thm.1.36.] that $(A \oplus A^d)^d = \{0\}$. So, we obtain

$$\inf_m \sup_{i \geq m} (|x_{n_i}| \wedge z) \leq t_{n_m}$$

for each $m \in \mathbb{N}$ in E because a is arbitrary. As a result, we obtain the desired, $x_n \xrightarrow{\text{st-uo}} 0$ in E, result.

For the converse implication, assume (x_n) is statistical *uo*-converges to 0 in E. Fix any $u \in A_+$. Then there exists a sequence $r_n \downarrow^{st} 0$ in E and a subset J of the natural numbers with $\delta(J) = 1$ such that $|x_{j_n}| \wedge u \leq r_{j_n}$ for all $j \in \mathbb{N}$. Thus, the previous remark implies $r_n \downarrow^{st} 0$ in A, and so, we get $x_n \xrightarrow{\text{st-uo}} 0$ in A whenever we take, in a special case $u \in A$. □

Proposition 3.1. *Let E be a vector lattice. If $x_n \xrightarrow{\text{st-uo}} x$ in E then $P_B(x_n) \xrightarrow{\text{st-uo}} P_B(x)$ for every the corresponding band projection of a projection band B in E.*

Proof. It is well known that $0 \leq P_B \leq I$ holds, and P_B is a lattice homomorphism (cf. [2, Thm.1.144]). Following from the inequality

$$|P_B(x_\alpha) - P_B(x)| = P_B|x_\alpha - x| \leq |x_\alpha - x|,$$

it is clear that $P_B(x_\alpha) \xrightarrow{\text{st-uo}} P_B(x)$. □

Theorem 3.5. *Let E be a vector lattice, F be a sublattice of E and (f_n) be a sequence in F. If $f_n \xrightarrow{\text{st-uo}} 0$ in F then $f_n \xrightarrow{\text{st-uo}} 0$ in E in each of cases; F is majorizing in E, or F is a projection band in E.*

Proof. Fix an element $u \in E_+$. Assume F is majorizing in E. Then there exists $w \in F$ with $u \leq w$. On the other hand, there exists a sequence $q_n \downarrow^{\text{st}} 0$ and a subset $\delta(J) = 1$ such that $|f_{j_n}| \wedge w \leqslant q_{j_n}$ for all $j_n \in J$ in consequence of $f_n \xrightarrow{\text{st-uo}} 0$. Hence, it follows from the inequality

$$|f_{j_n}| \wedge u \leq |f_{j_n}| \wedge w$$

that $f_n \xrightarrow{\text{st-uo}} 0$ in E.

Now, assume that F is a projection band in E. Chose $F = F^{dd}$. Then we have $E = F \oplus F^d$. Hence, we can write $u = u_1 + u_2$ with $u_1 \in F$ and $u_2 \in F^d$. Hence, we obtain

$$|f_n| \wedge u = |f_n| \wedge (u_1 + u_2) = |f_n| \wedge u_1$$

because of $f_n \wedge u_2 = 0$. Hence $f_n \xrightarrow{\text{st-uo}} 0$ in E. □

Proposition 3.2. *Let (x_n) be a disjoint sequence in a vector lattice E. Then (x_n) statistical uo-converges to zero.*

Proof. Suppose (x_n) is a disjoint sequence. Then it follows from [10, Cor.3.6.] that (x_n) is *uo*-convergent to 0 in E. Now, by using Proposition 2.1, we can get $x_n \xrightarrow{\text{st-uo}} 0$. □

REFERENCES

1. C. D. Aliprantis and O. Burkinshaw: *Locally Solid Riesz Spaces with Applications to Economics*. American Mathematical Society, 2003.
2. C. D. Aliprantis and O. Burkinshaw: *Positive Operators*. Springer, Dordrecht, 2006.
3. A. Aydin: *The statistically unbounded τ-convergence on locally solid Riesz spaces.* Turkish J. Math. **44** (2020), 949–956.
4. A. Aydin: *Multiplicative order convergence in f-algebras.* Hacet. J. Math. Stat. **49** (2020), 998–1005.
5. A. Aydin: *The statistical multiplicative order convergence in Riesz spaces.* Facta Uni. Series: Math. Info. **36** (2021), 409–417.
6. A. Aydin and M. Et: *Statistically multiplicative convergence on locally solid Riesz algebras.* Turkish J. Math. **45** (2021), 1506–1516.
7. Z. Ercan: *A characterization of u-uniformly completeness of Riesz spaces in terms of statistical u-uniformly pre-completeness.* Demon. Math. **42** (2009), 383–387.
8. H. Fast: *Sur la convergence statistique.* Colloq. Math. **2** (1951), 3–4.
9. J. Fridy: *On statistical convergence.* Analysis, **5** (1985), 301–13.
10. N. Gao, V. G. Troitisky and F. Xanthos: *Uo-convergence and its applications to Cesáro means in Banach lattices.* Israel J. of Math. **220** (2017), 649–689.
11. W. A. J. Luxemburg and A. C. Zaanen: *Riesz spaces I.* North-Holland Pub. Co., Amsterdam, 1971.
12. I. J. Maddox: *Statistical convergence in a locally convex space.* Math. Proc. Cambridge Phil. Soc. **104** (1967), 141–145.
13. H. Nakano: *Ergodic theorems in semi-ordered linear spaces.* Ann. Math. **49** (1948), 538–556.
14. I. J. Maddox: *Statistical convergence in a locally convex space.* Math. Proc. Cambr. Phil. Soc. **104** (1988), 141–145.
15. F. Riesz: *Sur la décomposition des opérations fonctionelles linéaires.* Atti D. Congr. Inter. D. Math., Bologna, 1928.
16. T. Salat: *On statistically convergent sequences of real numbers.* Math. Slov. **30** (1980), 139–150.
17. H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique.* Colloq. Math. **2** (1951), 73–74.
18. C. Şencimen and S. Pehlivan: *Statistical order convergence in Riesz spaces.* Math. Slov. **62** (2012), 557–570.

19. Z. Wang and Z. Chen: *Applications for unbounded convergences in Banach lattices.* Frac. Fractional, **6**(4) (2022) 199.

20. Z. Wang, Z. Chen and J. Chen: *Statistical unbounded convergence in Banach lattices.* arXiv:1903.10734v8

21. A. C. Zaanen: *Riesz Spaces II.* North-Holland Publishing Co., Amsterdam, 1983.