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# ON *tgs*-CONVEX FUNCTION AND THEIR INEQUALITIES

**Mevlüt Tunç, Esra Göv* and Ümmügülsüm Şanal**

**Abstract.** In this paper, the authors define a new concept of the so-called *tgs*-convex function and establish some inequalities of the Hadamard type via ordinary and Riemann-Liouville integral.

**Keywords**: convexity; *tgs*-convexity; Hadamard's inequality; fractional integral.

## 1. Introduction

**Definition 1.1.** [18] A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on $I$ if the inequality

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v) \tag{1.1}$$

holds for all $u, v \in I$ and $t \in [0,1]$. We say that $f$ is concave if $(-f)$ is convex.

Geometrically, this means that if $P$, $Q$ and $R$ are three distinct points on the graph of $f$ with $Q$ between $P$ and $R$, then $Q$ is on or below the chord $PR$.

**Theorem 1.1.** ***The Hermite-Hadamard inequality:*** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $u, v \in I$ with $u < v$. The following double inequality:*

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v f(x)\,dx \leq \frac{f(u)+f(v)}{2} \tag{1.2}$$

*is known in the literature as Hadamard's inequality (or Hermite-Hadamard inequality) for convex functions. If $f$ is a positive concave function, then the inequality is reversed.*

The inequalities (1.1) and (1.2) which have numerous uses in a variety of settings, have become a significant groundwork in mathematical analysis and optimization. Many reports have provided new proofs, extensions, refinements, generalizations, numerous interpolations and applications, for example, in the theory of special means and information theory. For more details see [7]-[10], [14]-[20], [24]-[26].

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**Definition 1.2.** [25] Let $h : J \to R$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an $h$-convex function, or that $f$ belongs to the class $SX(h, I)$, if $f$ is non-negative and for all $x, y \in I$ ,$\alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha) y) \leq h(\alpha) f(x) + h(1-\alpha) f(y) \tag{1.3}$$

If inequality (1.3) is reversed, then $f$ is said to be $h$-concave, i.e. $f \in SV(h, I)$.

Throughout this paper we will use the following notations and conventions. Let $J = [0, \infty) \subset \mathbb{R} = (-\infty, +\infty)$, and $u, v \in J$ with $0 < u < v$ and $f' \in L[u, v]$ and

$$\begin{aligned} A(u,v) &= \frac{u+v}{2}, G(u,v) = \sqrt{uv}, K(u,v) = \frac{u^2+v^2}{2}, \\ L(u,v) &= \frac{v-u}{\ln v - \ln u}, u \neq v \end{aligned}$$

be the arithmetic mean, geometric mean, harmonic mean, logarithmic mean for $u, v > 0$ respectively.

Now we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper. For more result, one can see [11, 13, 21].

**Definition 1.3.** Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, x > a \tag{1.4}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-x)^{\alpha-1} f(t)\,dt, b > x \tag{1.5}$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$. Here is $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities see [1]-[6], [22]-[23].

In this paper we write a new definition concerning convexity and we establish new integral inequalities analogous to the well known Hermite-Hadamard's inequality. We also prove some fractional integral inequalities.

## 2. Main Results

We begin with the following new definition.

**Definition 2.1.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function. We say that $f : I \to \mathbb{R}$ is *tgs*−convex function on $I$ if the inequality

$$f(tu + (1-t)v) \leq t(1-t)[f(u) + f(v)] \tag{2.1}$$

holds for all $u, v \in I$ and $t \in (0,1)$. We say that $f$ is *tgs*-concave if $(-f)$ is *tgs*-convex.

We note that when $t = 0$ or $t = 1$, then according to the hypothesis the function equals to zero.

**Remark 2.1.** If we take $h(t) = t(1-t)$ in Definition 1.2, Definition 1.2 reduces to Definition 2.1. Namely, Definition of *tgs*-convex function may be regarded as a special case of *h*-convex function (see [25]).

**Theorem 2.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a tgs-convex function and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{6}. \tag{2.2}$$

*Proof.* Since $f$ is *tgs*-convex on $[a,b]$, then we have

$$f(ta + (1-t)b) \leq t(1-t)[f(a) + f(b)]. \tag{2.3}$$

Integrating both sides of (2.3) over $[0,1]$, we get

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,dx \\
= &\int_0^1 f(ta + (1-t)b)\,dt \leq \int_0^1 t(1-t)[f(a)+f(b)]\,dt \\
= &\frac{f(a)+f(b)}{6}.
\end{aligned}$$

Using (2.1) and substituting $x = ta + (1-t)b$, $y = (1-t)a + tb$, we observe that

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{x+y}{2}\right) \\
&= f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \\
&\leq \frac{1}{4}f(ta+(1-t)b) + \frac{1}{4}f((1-t)a+tb).
\end{aligned}$$

Integrating both sides over $[0,1]$, it is easy to observe that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\int_a^b f(x)\,dx.$$

The proof is completed. □

**Theorem 2.2.** *Let $f$ and $g$ be real valued, nonnegative and tgs-convex functions on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \le \frac{1}{30}\left[M(a,b)+N(a,b)\right] \tag{2.4}$$

*where $M(a,b) = f(a)\,g(a) + f(b)\,g(b)$, $N(a,b) = f(a)\,g(b) + f(b)\,g(a)$.*

*Proof.* Since $f$ and $g$ are $tgs$-convex on $[a,b]$, then we have

$$f(ta+(1-t)b) \le t(1-t)\left[f(a)+f(b)\right] \tag{2.5}$$
$$g(ta+(1-t)b) \le t(1-t)\left[g(a)+g(b)\right]. \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$f(ta+(1-t)b)\,g(ta+(1-t)b) \le t^2(1-t)^2\left[f(a)+f(b)\right]\left[g(a)+g(b)\right]. \tag{2.7}$$

Integrating both sides of (2.7) over $[0,1]$, we have

$$\begin{aligned}\int_0^1 f(ta+(1-t)b)\,g(ta+(1-t)b)\,dt &\le \int_0^1 t^2(1-t)^2\left[f(a)+f(b)\right]\left[g(a)+g(b)\right]dt\\ &= \frac{1}{30}\left[f(a)+f(b)\right]\left[g(a)+g(b)\right]\\ &= \frac{1}{30}\left[M(a,b)+N(a,b)\right].\end{aligned}$$

By substituting $x = ta+(1-t)b$, we get the desired inequality. □

**Theorem 2.3.** *Let $f$ and $g$ be real valued, nonnegative and tgs-convex functions on $[a,b]$. Then*

$$8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{30}\left[M(a,b)+N(a,b)\right] \tag{2.8}$$

*where $M(a,b)$ and $N(a,b)$ are as defined in Theorem 2.2.*

*Proof.* Let

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)\\ g\left(\frac{a+b}{2}\right) &= g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right).\end{aligned}$$

Since $f$ and $g$ are $tgs$-convex on $[a,b]$, then we have

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\ \le\ &\frac{1}{4}\left[f(ta+(1-t)b)+f((1-t)a+tb)\right]\frac{1}{4}\left[g(ta+(1-t)b)+g((1-t)a+tb)\right]\\ =\ &\frac{1}{16}\{f(ta+(1-t)b)\,g(ta+(1-t)b)+f((1-t)a+tb)\,g((1-t)a+tb)\\ &+f((1-t)a+tb)\,g(ta+(1-t)b)+f(ta+(1-t)b)\,g((1-t)a+tb)\}.\end{aligned}$$

Integrating both sides of the above inequality over $[0,1]$ and using (2.1) we have

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
\leq\ & \frac{1}{16}\left\{\int_0^1 f(ta+(1-t)b)\, g(ta+(1-t)b)\, dt+\int_0^1 f((1-t)a+tb)\, g((1-t)a+tb)\, dt\right. \\
& \left.+\int_0^1 f((1-t)a+tb)\, g(ta+(1-t)b)\, dt+\int_0^1 f(ta+(1-t)b)\, g((1-t)a+tb)\, dt\right\} \\
\leq\ & \frac{1}{16}\left\{\frac{2}{(b-a)}\int_a^b f(x)\, g(x)\, dx+2\int_0^1 t^2(1-t)^2\,[f(a)+f(b)]\,[g(a)+g(b)]\, dt\right\} \\
=\ & \frac{1}{16}\left\{\frac{2}{(b-a)}\int_a^b f(x)\, g(x)\, dx+2\int_0^1 t^2(1-t)^2\,[f(a)+f(b)]\,[g(a)+g(b)]\, dt\right\} \\
=\ & \frac{1}{16}\left\{\frac{2}{(b-a)}\int_a^b f(x)\, g(x)\, dx+\frac{2}{30}\,[M(a,b)+N(a,b)]\right\} \\
=\ & \frac{1}{8}\left\{\frac{1}{(b-a)}\int_a^b f(x)\, g(x)\, dx+\frac{1}{30}\,[M(a,b)+N(a,b)]\right\}.
\end{aligned}$$

The proof is completed. □

**Theorem 2.4.** *Let $f,g: I\subseteq\mathbb{R}\to\mathbb{R}$ be real valued, nonnegative and tgs-convex function and $f,g\in L_1[a,b]$, then*

$$\begin{aligned}
& \frac{15}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\, g(tx+(1-t)y)\, dtdydx \\
\leq\ & \frac{1}{b-a}\int_a^b f(x)\, g(x)\, dx+\frac{1}{36}\,[M(a,b)+N(a,b)]
\end{aligned}\tag{2.9}$$

*Proof.* Since $f$ and $g$ are *tgs*-convex on $[a,b]$, then we have

$$f(tx+(1-t)y) \leq t(1-t)\,[f(x)+f(y)]\tag{2.10}$$
$$g(ta+(1-t)b) \leq t(1-t)\,[g(x)+g(y)].\tag{2.11}$$

Multiplying both sides of (2.10) and (2.11) and integrating over $[0,1]$, we have

$$\begin{aligned}
& \int_0^1 f(tx+(1-t)y)\, g(tx+(1-t)y)\, dt \\
\leq\ & \int_0^1 t^2(1-t)^2\,[f(x)+f(y)]\,[g(x)+g(y)]\, dt \\
=\ & \frac{1}{30}\,[f(x)+f(y)]\,[g(x)+g(y)].
\end{aligned}\tag{2.12}$$

Integrating both sides of (2.12) on $[a,b]\times[a,b]$, we obtain

$$\begin{aligned}&\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dtdydx\\ \leq\ &\frac{1}{30}\int_a^b\int_a^b[f(x)+f(y)][g(x)+g(y)]\,dydx\\ =\ &\frac{1}{30}\int_a^b\int_a^b[f(x)g(x)+f(y)g(y)+f(x)g(y)+f(y)g(x)]\,dydx.\end{aligned}$$

From (2.2), we have

$$\begin{aligned}&\frac{1}{30}\left\{\int_a^b\int_a^b[f(x)g(x)+f(y)g(y)]\,dydx\right.\\ &\left.+\int_a^b f(x)\,dx\int_a^b g(y)\,dy+\int_a^b f(y)\,dy\int_a^b g(x)\,dx\right\}\\ \leq\ &\frac{1}{30}\left[2(b-a)\int_a^b f(x)g(x)\,dx+2(b-a)^2\frac{f(a)+f(b)}{6}\frac{g(a)+g(b)}{6}\right]\\ =\ &\frac{b-a}{15}\int_a^b f(x)g(x)\,dx+\frac{(b-a)^2}{540}[M(a,b)+N(a,b)].\end{aligned}$$

Now multiplying both sides by $\frac{15}{(b-a)^2}$ we get the desired inequality. □

**Theorem 2.5.** *Let $f,g:I\subseteq\mathbb{R}\to\mathbb{R}$ be real valued, nonnegative and tgs-convex function and $f,g\in L_1[a,b]$, then*

$$\begin{aligned}&\frac{30}{b-a}\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dtdx\\ \leq\ &\frac{1}{b-a}\int_a^b f(x)g(x)\,dx+\frac{35}{8}[M(a,b)+N(a,b)].\end{aligned}$$

*Proof.* Since $f$ and $g$ are *tgs*-convex on $[a,b]$, then we have

$$\begin{aligned}f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)&\leq t(1-t)\left[f(x)+f\left(\frac{a+b}{2}\right)\right]\\ g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)&\leq t(1-t)\left[g(x)+g\left(\frac{a+b}{2}\right)\right].\end{aligned}$$

Multiplying both sides of the above inequalities and integrating over $[0,1]$, we get

$$\begin{aligned}&\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dt\\ \leq\ &\int_0^1 t^2(1-t)^2\left[f(x)+f\left(\frac{a+b}{2}\right)\right]\left[g(x)+g\left(\frac{a+b}{2}\right)\right]dt\\ =\ &\frac{1}{30}\left[f(x)+f\left(\frac{a+b}{2}\right)\right]\left[g(x)+g\left(\frac{a+b}{2}\right)\right].\end{aligned}\tag{2.13}$$

Integrating both sides of (2.13) on $[a,b]$, we obtain

$$\begin{aligned}
&\int_a^b \int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) dtdx \\
\leq\ & \frac{1}{30}\int_a^b \left[f(x)+f\left(\frac{a+b}{2}\right)\right]\left[g(x)+g\left(\frac{a+b}{2}\right)\right] dx \\
=\ & \frac{1}{30}\int_a^b \left[f(x)g(x)+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)+f(x)g\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)g(x)\right] dx.
\end{aligned}$$

From (2.2) and Theorem 2.2 we see that

$$\begin{aligned}
&\frac{1}{30}\left\{\int_a^b f(x)g(x)\,dx+(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\right. \\
&\left. +g\left(\frac{a+b}{2}\right)(b-a)\frac{f(a)+f(b)}{6}+f\left(\frac{a+b}{2}\right)(b-a)\frac{g(a)+g(b)}{6}\right\} \\
\leq\ & \frac{1}{30}\left\{\int_a^b f(x)g(x)\,dx+\frac{g(a)+g(b)}{4}(b-a)\frac{f(a)+f(b)}{6}\right. \\
&\left. +\frac{f(a)+f(b)}{4}(b-a)\frac{g(a)+g(b)}{6}+(b-a)\frac{f(a)+f(b)}{4}\frac{g(a)+g(b)}{4}\right\} \\
=\ & \frac{1}{30}\left\{\int_a^b f(x)g(x)\,dx+\frac{7(b-a)}{48}\left[M(a,b)+N(a,b)\right]\right\}.
\end{aligned}$$

Now multiplying both sides by $\frac{30}{b-a}$, we get the desired inequality. □

**Theorem 2.6.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a tgs-convex function on $[a,b]$ with $a < b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)\,dx \leq \frac{1}{30}\left[f(a)+f(b)\right]^2. \tag{2.14}$$

*Proof.* Since $f$ is *tgs*-convex function on $I$, then we have

$$f(ta+(1-t)b) \leq t(1-t)\left[f(a)+f(b)\right] \tag{2.15}$$

and

$$f((1-t)a+tb) \leq t(1-t)\left[f(a)+f(b)\right] \tag{2.16}$$

for all $a, b \in I$. It is easy to observe that

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)\,dx = \int_0^1 f(ta+(1-t)b)\,f((1-t)a+tb)\,dt. \tag{2.17}$$

Using the elementary inequality $G(p,q) \leq K(p,q)\ (p,q \geq 0)$, making the change of variable and using (2.1), we get

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)\,dx$$

$$\leq \frac{1}{2}\int_0^1 \left\{\left[f\left(ta+(1-t)b\right)\right]^2+\left[f\left((1-t)a+tb\right)\right]^2\right\}dt$$

$$\leq \frac{1}{2}\int_0^1 \left\{t^2(1-t)^2\left[f(a)+f(b)\right]^2+(1-t)^2t^2\left[f(a)+f(b)\right]^2\right\}dt$$

$$= \int_0^1 t^2(1-t)^2\left[f(a)+f(b)\right]^2dt$$

$$= \frac{1}{30}\left[f(a)+f(b)\right]^2.$$

The proof is completed. □

## 3. Fractional Integral Inequalities of $tgs$-convex Functions

In this section, we prove some Hermite-Hadamard type fractional integral inequalities for $tgs$-convex functions.

**Theorem 3.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function with $a < b$ and $L_1[a,b]$. If $f$ is a $tgs$-convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$2f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \leq \frac{\alpha\left[f(a)+f(b)\right]}{(\alpha+1)(\alpha+2)} \tag{3.1}$$

with $\alpha > 0$.

*Proof.* Since $f$ is $tgs$-convex on $[a,b]$, then we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{4}, \tag{3.2}$$

for $x, y \in [a,b]$. Now, let $x = ta+(1-t)b$ and $y = (1-t)a+tb$ with $t \in (0,1)$. Then we obtain by (3.2) that;

$$4f\left(\frac{a+b}{2}\right) \leq f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right) \tag{3.3}$$

for all $t \in (0,1)$. Multiplying both side of (3.3) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to $t$ over $[0,1]$, we get

$$\frac{4}{\alpha}f\left(\frac{a+b}{2}\right) \tag{3.4}$$

$$\leq \int_0^1 t^{\alpha-1}f\left(ta+(1-t)b\right)dt+\int_0^1 t^{\alpha-1}f\left((1-t)a+tb\right)dt$$

$$= \frac{1}{(b-a)^{\alpha}}\int_a^b (b-u)^{\alpha-1}f(u)\,du+\frac{1}{(b-a)^{\alpha}}\int_a^b (a-v)^{\alpha-1}f(v)\,dv$$

$$= \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right].$$

Multiplying both side of (3.4) by $\frac{\alpha}{2}$, then

$$2f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]. \tag{3.5}$$

For the proof of the second inequality in (3.1) we first note that if $f$ is a *tgs*-convex function, then, it yields

$$f(ta+(1-t)b) \leq t(1-t)[f(a)+f(b)] \tag{3.6}$$

and

$$f((1-t)a+tb) \leq t(1-t)[f(a)+f(b)]. \tag{3.7}$$

By adding these inequalities we have

$$f(ta+(1-t)b) + f((1-t)a+tb) \leq 2t(1-t)[f(a)+f(b)]. \tag{3.8}$$

Then multiplying both sides of (3.8) by $t^{\alpha-1}$, and integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\,dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)\,dt \leq 2\int_0^1 t^{\alpha}(1-t)[f(a)+f(b)]\,dt \tag{3.9}$$

i.e.

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{2[f(a)+f(b)]}{(\alpha+1)(\alpha+2)} \tag{3.10}$$

From (3.5) and (3.10), we complete the proof. □

**Remark 3.1.** If in Theorem 3.1, we let $\alpha = 1$, then (3.1) reduces to (2.2).

**Theorem 3.2.** *Let $f$ and $g$ be real-valued, symmetric about $\frac{a+b}{2}$, nonnegative and tgs-convex functions on $[a,b]$. Then for all $a,b>0$, $\alpha>0$, we have*

$$\frac{J_{a+}^{\alpha}[f(b)g(b)]}{(b-a)^{\alpha}} \leq \frac{2\alpha(\alpha+1)[M(a,b)+N(a,b)]}{\Gamma(\alpha+5)} \tag{3.11}$$

*and*

$$8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^{\alpha}} J_{a+}^{\alpha}[f(b)g(b)] + \frac{2\alpha(\alpha+1)[M(a,b)+N(a,b)]}{\Gamma(\alpha+5)}. \tag{3.12}$$

*where $M(a,b)$ and $N(a,b)$ are as defined in Theorem 2.2.*

*Proof.* Since $f$ and $g$ are $tgs$-convex functions on $[a,b]$, then we have

$$f(ta+(1-t)b) \le t(1-t)[f(a)+f(b)] \tag{3.13}$$

and

$$g(ta+(1-t)b) \le t(1-t)[g(a)+g(b)]. \tag{3.14}$$

From (3.13)-(3.14), we obtain

$$\begin{aligned} & f(ta+(1-t)b)\, g(ta+(1-t)b) \\ \le\ & t^2(1-t)^2[f(a)+f(b)][g(a)+g(b)]. \end{aligned} \tag{3.15}$$

Now multiplying both sides of (3.15) by $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, then integrating the resulting inequality with respect to $t$ over $[0,1]$, we get

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\, g(ta+(1-t)b)\, dt \\ \le\ & \frac{[f(a)+f(b)][g(a)+g(b)]}{\Gamma(\alpha)}\int_0^1 t^{\alpha+1}(1-t)^2\, dt \\ =\ & \frac{2\alpha(\alpha+1)[M(a,b)+N(a,b)]}{\Gamma(\alpha+5)}. \end{aligned} \tag{3.16}$$

Let $x = ta+(1-t)b$. Then we have

$$\frac{1}{(b-a)^{\alpha}} J_{a+}^{\alpha}[f(b)\, g(b)] \le \frac{2\alpha(\alpha+1)[M(a,b)+N(a,b)]}{\Gamma(\alpha+5)}. \tag{3.17}$$

Since $f$ and $g$ are $tgs$-convex functions on $[a,b]$, then we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ =\ & f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\ \le\ & \frac{1}{4}[f(ta+(1-t)b)+f((1-t)a+tb)]\,\frac{1}{4}[g(ta+(1-t)b)+g((1-t)a+tb)] \\ =\ & \frac{1}{16}\{f(ta+(1-t)b)\, g(ta+(1-t)b)+f((1-t)a+tb)\, g((1-t)a+tb) \\ & +f((1-t)a+tb)\, g(ta+(1-t)b)+f(ta+(1-t)b)\, g((1-t)a+tb)\}. \end{aligned} \tag{3.18}$$

Now multiplying both sides of (3.18) by $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, then integrating the resulting inequality with respect to $t$ over $[0,1]$, we get

$$\frac{1}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)$$

$$
\begin{aligned}
\leq\ & \frac{1}{16}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\, g(ta+(1-t)b)\, dt \\
& +\frac{1}{16}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1} f((1-t)a+tb)\, g((1-t)a+tb)\, dt \\
& +\frac{1}{16}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1} f((1-t)a+tb)\, g(ta+(1-t)b)\, dt \\
& +\frac{1}{16}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\, g((1-t)a+tb)\, dt.
\end{aligned}
$$

i.e.

$$
\begin{aligned}
& 8 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
\leq\ & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} J_{a+}^{\alpha}\left[f(b)\,g(b) + f(b)\,g(a)\right] + J_{b-}^{\alpha}\left[f(a)\,g(a) + f(a)\,g(b)\right].
\end{aligned}
$$

Consequently, we complete the proof. □

**Remark 3.2.** If we take $\alpha = 1$ in (3.11), then it reduces to (2.4).

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Mevlüt Tunç
Mustafa Kemal University
Faculty of Science and Arts
Department of Mathematics
31000, Hatay, Turkey
mevluttttunc@gmail.com

Esra Göv
Mustafa Kemal University
Faculty of Science and Arts
Department of Mathematics
31000, Hatay, Turkey
esordulu@gmail.com
*Corresponding Author

Ümmügülsüm Şanal
Mustafa Kemal University
Faculty of Science and Arts
Department of Mathematics
31000, Hatay, Turkey
gsanal020@gmail.com