FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 30, No 5 (2015), 693–705

RADIUS CONSTANTS FOR A CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH A MULTIPLIER LINEAR OPERATOR

Poonam Sharma and Ankita *

Abstract. The purpose of this paper is to find radius constants for a Janowski type class $H^m_{k,\mu}(\lambda, A, B)$ involving a multiplier linear operator for functions f satisfying certain conditions on its coefficients. The sharpness of the results are verified. Some consequent results are also mentioned.

Keywords: Univalent functions, subclasses of univalent functions, multiplier operator, subordination, coefficient inequality, radius constant.

1. Introduction

Let $\mathcal{A}$ denotes a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. A subclass of univalent functions $f \in \mathcal{A}$ is denoted by $\mathcal{S}$. Bieberbach conjectured that a function $f \in \mathcal{S}$ of the form (1.1) satisfies the coefficient condition: $|a_n| \le n$ $(n \ge 2)$ which was proved by de Branges [4]. But it was observed that this coefficient condition is not sufficient for the functions f to be in the class $\mathcal{S}$. For example, functions

$$f_1(z) = z + 2z^2, \; f_2(z) = 2z - \frac{z}{(1-z)^2}$$

satisfy coefficient condition $|a_n| \le n$ but their derivatives vanish inside $\mathbb{U}$, hence, the functions f_1 and f_2 are not in the class $\mathcal{S}$. Thus, we needed to find the least upper bound $r(f)$ of $r \in (0, 1)$ such that $f \in \mathcal{A}$ satisfying the condition $|a_n| \le n$ be univalent in $\mathbb{U}_r = \{z : |z| < r\}$ and is called the radius of univalence or the radius constant for $f \in \mathcal{S}$ or $\mathcal{S}-$ radius. Gavrilov [10] showed that radius of univalence

Received March 24, 2015.; Accepted April 08, 2015.
2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C80
*The author was supported by SRF of CSIR, New Delhi India

for functions $f \in \mathcal{A}$ of the form (1.1) satisfying $|a_n| \le n$, is the real root $r_0 = 0.164$ (approx.) of the equation $2(1-r)^3 - (1+r) = 0$ and the result is sharp for the function f_2. Gavrilov also obtained the radius of univalence of functions $f \in \mathcal{A}$ satisfying another inequality $|a_n| \le M$ $(M > 0, n \ge 2)$. Landau [14] obtained the radius of univalence for functions $f \in \mathcal{A}$ satisfying $\left|f(z)\right| \le M$. Various subclasses of $\mathcal{S}$ have been defined and studied so far, well known out of which are the classes of starlike and convex functions, denoted, respectively, by $\mathcal{ST}$ and $\mathcal{CV}$ (see Duren [7]). Yamashita [28] showed that the radius of univalence obtained by Gavrilov is same as the radius of starlikeness for functions $f \in \mathcal{A}$ satisfying $|a_n| \le n$ or $|a_n| \le M$. Yamashita [28] also determined the radius of convexity, for functions $f \in \mathcal{A}$ satisfying $|a_n| \le n$, which is the real root $r_0 = 0.090$ of the equation $2(1-r)^4 - (1+4r+r^2) = 0$, while the radius of convexity for functions $f \in \mathcal{A}$ satisfying $|a_n| \le M$ is the real root of $(M+1)(1-r)^3 - M(1+r) = 0$.

The second coefficient a_2 of $f \in \mathcal{A}$ given by (1.1), determines some important properties such as growth and distortion estimates of the function f. By fixing the second coefficient, let $\mathcal{A}_b$ denotes a subclass of the class $\mathcal{A}$ whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (|a_2| = 2b, 0 \le b \le 1)\,.$$

Several authors have investigated various properties of univalent functions and its subclasses by fixing the second coefficient; for detail see [1, 2, 11, 15, 16, 23, 26]. In [23], Ravichandran obtained the sharp radii of starlikeness and convexity of order α $(0 \le \alpha < 1)$ for functions $f \in \mathcal{A}_b$ satisfying the condition $|a_n| \le n$ or $|a_n| \le M$ or $|a_n| \le M/n$ for $n \ge 3$. Further, in [16], radius constants are obtained for functions $f \in \mathcal{A}_b$ satisfying the condition $|a_n| \le cn + d$ $(c, d \ge 0)$ or $|a_n| \le c/n$ $(c > 0)$ for $n \ge 3$.

Let f and g be analytic in $\mathbb{U}$. Then we say f is subordinate to g, written $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there is an analytic function w with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{U}$, then f is subordinate to g provided $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$. The concept of subordination can be found in [17]. Involving subordination, a brief history for various subclasses of $\mathcal{S}$ may be found in [1].

In geometric function theory, various linear operators, associated with some geometric properties of the image domain are studied. For the purpose of this paper, we consider a multiplier linear operator $\mathcal{J}^m_{k,\mu} : \mathcal{A} \to \mathcal{A}$, defined recently in [21] (see also [22], [25]), for $m \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and for $\mu > -1,\ k > 0$, by

$$(1.2) \qquad \begin{cases} \mathcal{J}^m_{k,\mu} f(z) &= f(z), \qquad\qquad m = 0, \\ \mathcal{J}^m_{k,\mu} f(z) &= \frac{\mu+1}{k} z^{1-\frac{\mu+1}{k}} \int\limits_0^z t^{\frac{\mu+1}{k}-2} \mathcal{J}^{m+1}_{k,\mu} f(t) \mathrm{d}t, \qquad m \in \mathbb{Z}^- = \{-1, -2, \ldots\}\,, \\ \mathcal{J}^m_{k,\mu} f(z) &= \frac{k}{\mu+1} z^{2-\frac{\mu+1}{k}} \frac{\mathrm{d}}{\mathrm{d}t} \left(z^{\frac{\mu+1}{k}-1} \mathcal{J}^{m-1}_{k,\mu}\ f(z) \right), \qquad m \in \mathbb{Z}^+ = \{1, 2, \ldots\} \end{cases}$$

The series representation of $\mathcal{J}_{k,\mu}^{m} f(z)$ for $f(z)$ of the form (1.1) is given by

$$\mathcal{J}_{k,\mu}^{m} f(z) = z + \sum_{n=2}^{\infty} \left(1 + \frac{k(n-1)}{\mu+1}\right)^{m} a_n z^n. \tag{1.3}$$

The multiplier operator $\mathcal{J}_{k,\mu}^{m}$ generalizes several previously studied operators in various papers some of which are as follows:

(i) $\mathcal{J}_{k,0}^{m} = D_k^m$ $(m \in \mathbb{N}_0 = \{0, 1, 2, ...\})$ [18]

(ii) $\mathcal{J}_{1,0}^{m} = D^m$ $(m \in \mathbb{N}_0)$ [24]

(iii) $\mathcal{J}_{1,1}^{m} = \mathcal{D}^m$ [27]

(iv) $\mathcal{J}_{1,\mu}^{m} = I_\mu^m$ $(m \in \mathbb{N}_0,\ \mu \geq 0)$ [5, 6]

(v) $\mathcal{J}_{k,0}^{-n} = \mathcal{I}_k^{-n}$ $(n \in \mathbb{N}_0,\ k > 0)$ [3, 20]

(vi) $\mathcal{J}_{1,a}^{-n} = L_{a+1}^{n}$ $(n \in \mathbb{N}_0,\ a \geq 0)$ [13]

(vii) $\mathcal{J}_{1,1}^{-n} = I^{-n}$ $(n \in \mathbb{N}_0)$ [8]

(viii) $\mathcal{J}_{1,0}^{-n} f(z) = \mathcal{I}^{-n}$ $(n \in \mathbb{N}_0, \lambda > 0)$ [24]

Involving the operator $\mathcal{J}_{k,\mu}^{m}$, we define a Janowski type class $H_{k,\mu}^{m}(\lambda, A, B)$ as follows:

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in class $H_{k,\mu}^{m}(\lambda, A, B)$, if it satisfies for $\lambda \geq 0, -1 \leq B < A \leq 1$, a subordination:

$$\frac{(1-\lambda)\,\mathcal{J}_{k,\mu}^{m+1} f(z) + \lambda z \left(\mathcal{J}_{k,\mu}^{m+1} f(z)\right)'}{\mathcal{J}_{k,\mu}^{m} f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U})\,. \tag{1.4}$$

Note that on giving appropriate values to the parameters involved in the aforementioned class $H_{k,\mu}^{m}(\lambda, A, B)$, we find several previously defined classes. Some of these are as follows:

(i) $H_{1,0}^{0}(0, A, B) = \mathcal{ST}[A, B]\,, H_{1,0}^{1}(0, A, B) = \mathcal{CV}[A, B]$ studied by Janowski [12].

(ii) $H_{1,0}^{0}(\alpha, 1-2\beta, -1) = \mathcal{L}(\alpha, \beta)$ $(\alpha \geq 0, \beta \in \mathbb{R}\backslash\{1\})$ studied by Nargesi *et al.* [16] ([19]).

(iii) $H_{1,0}^{0}(0, 1-\alpha, 0), H_{1,0}^{1}(0, 1-\alpha, 0)$ $(0 \leq \alpha < 1)$ studied by Ravichandran [23].

Denote $H^m_{k,\mu}(\lambda, 1-2\beta, -1) = H^m_{k,\mu}(\lambda, \beta)$ $(0 \leq \beta < 1)$ and $H^m_{k,\mu}(\lambda, 0) = H^m_{k,\mu}(\lambda)$. Functions in the class $H^m_{k,\mu}(\lambda, \beta)$ satisfy

$$\text{(1.5)} \qquad \operatorname{Re}\left\{ \frac{(1-\lambda)\,\mathcal{J}^{m+1}_{k,\mu} f(z) + \lambda z \left(\mathcal{J}^{m+1}_{k,\mu} f(z)\right)'}{\mathcal{J}^{m}_{k,\mu} f(z)} \right\} > \beta \ (z \in \mathbb{U}).$$

Since, for $-1 \leq D \leq B < A \leq C \leq (1-2\beta) \leq 1$,

$$\frac{1+Az}{1+Bz} \prec \frac{1+Cz}{1+Dz} \prec \frac{1+(1-2\beta)\,z}{1-z} \prec \frac{1+z}{1-z} \ (0 \leq \beta < 1; z \in \mathbb{U}),$$

we observe that

$$H^m_{k,\mu}(\lambda, A, B) \subset H^m_{k,\mu}(\lambda, C, D),$$

and

$$H^m_{k,\mu}(\lambda, A, B) \subset H^m_{k,\mu}(\lambda, \beta) \subset H^m_{k,\mu}(\lambda).$$

But the reverse inclusion is true in some disk $\mathbb{U}_r$. According to [9], we have following inclusions:

(i) $H^m_{k,\mu}(\lambda, C, D) \subset H^m_{k,\mu}(\lambda, A, B)$ in $\mathbb{U}_{r_1}$, where $r_1 = \min\left(\frac{A-B}{C-D-|AD-BC|}, 1\right)$.

(ii) $H^m_{k,\mu}(\lambda, \beta) \subset H^m_{k,\mu}(\lambda, A, B)$ in $\mathbb{U}_{r_2}$, where $r_2 = \min\left(\frac{A-B}{2(1-\beta)-|A+B(1-2\beta)|}, 1\right)$.

(iii) $H^m_{k,\mu}(\lambda) \subset H^m_{k,\mu}(\lambda, A, B)$ in $\mathbb{U}_{r_3}$, where $r_3 = \min\left(\frac{A-B}{2-|A+B|}, 1\right)$.

We note that the functions belonging to a class, satisfy certain coefficient condition, for example, if $f \in \mathcal{A}$ of the form (1.1) is convex (univalent) in $\mathbb{U}$, then $|a_n| \leq n$ $(n \geq 2)$ and if it is starlike in $\mathbb{U}$, then $|a_n| \leq 1$ $(n \geq 2)$. Also, if f satisfies $\left|f(z)\right| \leq M$ $(M > 0; z \in \mathbb{U})$, then $|a_n| \leq M$ $(n \geq 2)$, and if $\operatorname{Re}(f'(z)) > 0$ in $\mathbb{U}$, then $|a_n| \leq 2/n$ $(n \geq 2)$.

The purpose of this paper is to find results on $H^m_{k,\mu}(\lambda, A, B)-$ radius for the functions satisfying certain conditions on the coefficients a_n $(n \geq 2)$, which presumingly arise for the functions belonging to various classes. Motivated with the work [16] and [23], for $f \in \mathcal{A}$ of the form (1.1), satisfying certain conditions on the coefficients a_n $(n \geq 2)$, $H^m_{k,\mu}(\lambda, A, B)-$ radius is obtained by using the sufficient coefficient condition for the class $H^m_{k,\mu}(\lambda, A, B)$ which is also obtained in this paper. The sharpness of the radii results are verified. Some consequent results are also mentioned.

2. Coefficient Inequality

Theorem 2.1. *Let $\mu > -1, k > 0, \lambda \geq 0$ and let $-1 \leq B < 0, B < A \leq 1$. If $f \in \mathcal{A}$ of the form (1.1) satisfies the inequality*

$$\sum_{n=2}^{\infty}\left[A-1+(1-B)(1-\lambda+\lambda n)\left(1+\frac{k(n-1)}{\mu+1}\right)\right]\theta_{k,\mu}^{m}(n)\,|a_n| \leq A-B, \tag{2.1}$$

where

$$\theta_{k,\mu}^{m}(n)=\left(1+\frac{k(n-1)}{\mu+1}\right)^{m}, \tag{2.2}$$

then $f \in H_{k,\mu}^{m}(\lambda, A, B)$.

Proof. To prove $f \in H_{k,\mu}^{m}(\lambda, A, B)$, from the class condition (1.4), we need to show

$$S_1 := \left|\frac{1-P(z)}{BP(z)-A}\right| < 1, \tag{2.3}$$

where

$$P(z)=\frac{(1-\lambda)\mathcal{J}_{k,\mu}^{m+1}f(z)+\lambda z\left(\mathcal{J}_{k,\mu}^{m+1}f(z)\right)'}{\mathcal{J}_{k,\mu}^{m}f(z)}. \tag{2.4}$$

Observe from (1.1) that if $a_n = 0\ (n \geq 2)$, then $P(z) = 1\ (z \in \mathbb{U})$ which verifies (2.3), and if there is some $a_n \neq 0\ (n \geq 2)$, then from (2.1) it follows that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left\{A-B(1-\lambda+\lambda n)\left(1+\frac{k(n-1)}{\mu+1}\right)\right\}\theta_{k,\mu}^{m}(n)\,|a_n| \\
<\ &\sum_{n=2}^{\infty}\left[A-1+(1-B)(1-\lambda+\lambda n)\left(1+\frac{k(n-1)}{\mu+1}\right)\right]\theta_{k,\mu}^{m}(n)\,|a_n| \\
\leq\ & A-B.
\end{aligned} \tag{2.5}$$

Now, on writing the series expressions from (1.3) in (2.4), we get

$$S_1=\left|\frac{\sum_{n=2}^{\infty}\left\{(1-\lambda+\lambda n)\left(1+\frac{k(n-1)}{\mu+1}\right)-1\right\}\theta_{k,\mu}^{m}(n)\,a_n z^{n-1}}{A-B+\sum_{n=2}^{\infty}\left\{A-B(1-\lambda+\lambda n)\left(1+\frac{k(n-1)}{\mu+1}\right)\right\}\theta_{k,\mu}^{m}(n)\,a_n z^{n-1}}\right|$$

which in view of (2.5), proves

$$S_1<\frac{\sum_{n=2}^{\infty}\left\{(1-\lambda+\lambda n)\left(1+\frac{k(n-1)}{\mu+1}\right)-1\right\}\theta_{k,\mu}^{m}(n)\,|a_n|}{A-B-\sum_{n=2}^{\infty}\left\{A-B(1-\lambda+\lambda n)\left(1+\frac{k(n-1)}{\mu+1}\right)\right\}\theta_{k,\mu}^{m}(n)\,|a_n|}\leq 1$$

if (2.1) holds. This completes the proof of Theorem 2.1. □

3. Radius Constant

Theorem 3.1. *Let $f \in \mathcal{A}$ be of the form (1.1) and let for some $m \in \mathbb{Z}$, $\mu > -1$, $k > 0$, $\theta^m_{k,\mu}(n)$ $(n \geq 2)$ be given by (2.2). If $|a_2| = \frac{2b}{\theta^m_{k,\mu}(2)}$ $(0 \leq b \leq 1)$ and $|a_n| \leq \frac{cn+d}{\theta^m_{k,\mu}(n)}$ $(n \geq 3, c \geq 0, d \geq 0)$, then $H^m_{k,\mu}(\lambda, A, B)-$ radius is the real root in $(0,1)$, given by the equation*

$$
\begin{aligned}
&[(c+d+1)(A-B)+(2c-2b+d)\{(1-B)(1+\lambda)(1+K)+A-1\}r](1-r)^4 \\
&\quad = (1-B)\lambda cK\left(1+4r+r^2\right)+(1-B)\{c(\lambda+K-2\lambda K)+\lambda dK\}(1-r^2) \\
&\quad\quad +[\{c(1-\lambda)(1-K)+d(\lambda+K-2\lambda K)\}(1-B)+c(A-1)](1-r)^2 \\
&\quad\quad +d\{(1-\lambda)(1-K)(1-B)+A-1\}(1-r)^3, \qquad (3.1)
\end{aligned}
$$

where $K = \frac{k}{\mu+1}$. The result is sharp.

Proof. Let $r_0 \in (0,1)$ be the $H^m_{k,\mu}(\lambda, A, B)-$ radius. Then, we show that $\frac{f(r_0 z)}{r_0} \in H^m_{k,\mu}(\lambda, A, B)$. Hence, from the coefficient inequality (2.1), we show

$$
S_2 := \sum_{n=2}^{\infty}\left[A-1+(1-B)(1-\lambda+\lambda n)\left(1+\frac{k(n-1)}{\mu+1}\right)\right]\theta^m_{k,\mu}(n)\,|a_n|\,r_0^{n-1} \leq A-B.
$$

Applying conditions $|a_2| = \frac{2b}{\theta^m_{k,\mu}(2)}$ $(0 \leq b \leq 1)$ and $|a_n| \leq \frac{cn+d}{\theta^m_{k,\mu}(n)}$ $(n \geq 3, c \geq 0, d \geq 0)$, on putting $\frac{k}{\mu+1} = K$, we obtain

$$
\begin{aligned}
S_2 \leq\ & \{A-1+(1-B)(1+\lambda)(1+K)\}2br_0+\lambda cK(1-B)\sum_{n=3}^{\infty}n^3 r_0^{n-1} \\
&+(1-B)[c\{\lambda(1-2K)+K\}+d\lambda K]\sum_{n=3}^{\infty}n^2 r_0^{n-1} \\
&+[\{c(1-\lambda)(1-K)+d(\lambda+K-2\lambda K)\}(1-B)+c(A-1)]\sum_{n=3}^{\infty}n r_0^{n-1} \\
&+d\{A-1+(1-\lambda)(1-K)(1-B)\}\sum_{n=3}^{\infty}r_0^{n-1}
\end{aligned}
$$

and on using the expansions

$$
\frac{1}{1-r_0} = \sum_{n=1}^{\infty} r_0^{n-1}, \qquad (3.2)
$$

$$\frac{1}{(1-r_0)^2} = \sum_{n=1}^{\infty} n r_0^{n-1}, \tag{3.3}$$

$$\frac{1+r_0}{(1-r_0)^3} = \sum_{n=1}^{\infty} n^2 r_0^{n-1}, \tag{3.4}$$

$$\frac{1+4r_0+r_0^2}{(1-r_0)^4} = \sum_{n=1}^{\infty} n^3 r_0^{n-1}, \tag{3.5}$$

we get

$$\begin{aligned} S_2 \leq\ & \{A-1+(1-B)(1+\lambda)(1+K)\}\, 2br_0 \\ & +\lambda cK(1-B)\left\{\frac{1+4r_0+r_0^2}{(1-r_0)^4}-1-8r_0\right\} \\ & +(1-B)\left[c\{\lambda(1-2K)+K\}+d\lambda K\right]\left\{\frac{1+r_0}{(1-r_0)^3}-1-4r_0\right\} \\ & +\left[\{c(1-\lambda)(1-K)+d(\lambda+K-2\lambda K)\}(1-B)+c(A-1)\right] \\ & \left\{\frac{1}{(1-r_0)^2}-1-2r_0\right\} \\ & +d\{A-1+(1-\lambda)(1-K)(1-B)\}\left\{\frac{1}{(1-r_0)}-1-r_0\right\} \end{aligned}$$

$$\begin{aligned} =\ & (c+d)(B-A)+(2b-2c-d)\{(1-B)(1+\lambda)(1+K)+A-1\}\, r_0 \\ & +\Big[\lambda cK(1-B)\left(1+4r_0+r_0^2\right)+(1-B)\{c\{\lambda(1-2K)+K\}+d\lambda K\}(1-r_0^2) \\ & +\left[\{c(1-\lambda)(1-K)+d(\lambda+K-2\lambda K)\}(1-B)+c(A-1)\right](1-r_0)^2 \\ & +d\{(1-\lambda)(1-K)(1-B)+A-1\}(1-r_0)^3\Big]\frac{1}{(1-r_0)^4} \\ =\ & A-B \end{aligned}$$

if r_0 satisfy (3.1). Sharpness can be verified for the function $f_0(z)$ such that

$$\mathcal{J}_{k,\mu}^{m}(f_0(z)) = z - 2bz^2 - \sum_{n=3}^{\infty}(cn+d)z^n.$$

Since, for this function

$$\mathcal{J}_{k,\mu}^{m+1}(f_0(z)) = z - 2b(1+K)z^2 - \sum_{n=3}^{\infty}\{1+K(n-1)\}(cn+d)z^n,$$

where $K = \frac{k}{\mu+1}$ and at $z = r_0 \in (0,1)$, satisfying (3.1), we get

$$1 - \frac{(1-\lambda)\mathcal{J}_{k,\mu}^{m+1} f_0(z) + \lambda z \left(\mathcal{J}_{k,\mu}^{m+1} f_0(z)\right)'}{\mathcal{J}_{k,\mu}^{m} f_0(z)} = \frac{N_{r_0}}{D_{r_0}} = \frac{A-B}{1-B} > 0, \tag{3.6}$$

where N_{r_0} and D_{r_0} are given by

$$\begin{aligned} N_{r_0} \quad = \quad & (2b-2c-d)\{(1+\lambda)K+\lambda\} r_0 + \lambda c K \frac{1+4r_0+r_0^2}{(1-r_0)^4} \\ & + \{c(\lambda+K-2\lambda K) + d\lambda K\} \frac{1+r_0}{(1-r_0)^3} \\ & - \{c(\lambda+K-\lambda K) - d(\lambda+K-2\lambda K)\} \frac{1}{(1-r_0)^2} \\ & - d(\lambda+K-\lambda K) \frac{1}{1-r_0} \end{aligned}$$

and

$$D_{r_0} = \mathcal{J}_{k,\mu}^{m}(f_0(z)) = (1+c+d) - (2b-2c-d) r_0 - \frac{c}{(1-r_0)^2} - \frac{d}{1-r_0}.$$

Thus, for the function $f_0(z)$ at $z = r_0$, satisfying (3.1),

$$P_1(z) := \frac{(1-\lambda)\mathcal{J}_{k,\mu}^{m+1} f_0(z) + \lambda z \left(\mathcal{J}_{k,\mu}^{m+1} f_0(z)\right)'}{\mathcal{J}_{k,\mu}^{m} f_0(z)} = \frac{1+Aw(z)}{1+Bw(z)}$$

where

$$w(z) = \frac{1-P_1(z)}{BP_1(z)-A} = -1.$$

This completes the proof of Theorem 3.1. □

Remark 3.1.

(i) Taking $m = 0, k = 1, \mu = 0, \lambda = 0$ in Theorem 3.1, we get the radius result obtained by Nargesi *et al.* [16, Theorem 6, p. 4].

(ii) Taking $m = 0, k = 1, \mu = 0, A = 1 - 2\beta$ $(0 \le \beta < 1)$, $B = -1$ in Theorem 3.1, we get the radius result obtained by Nargesi *et al.* [16, Theorem 2, p. 2].

On giving special values: $c = 1, d = 0$ in Theorem 3.1, we get following result.

Corollary 3.1. *Let $f \in \mathcal{A}$ be of the form (1.1) and let for some $m \in \mathbb{Z}$, $\mu > -1$, $k > 0$, $\Theta_{k,\mu}^{m}(n)$ $(n \ge 2)$ be given by (2.2). If $|a_2| = \frac{2b}{\Theta_{k,\mu}^{m}(2)}(0 \le b \le 1)$, $|a_n| \le \frac{n}{\Theta_{k,\mu}^{m}(n)}$ $(n \ge 3)$, then $H_{k,\mu}^{m}(\lambda, A, B)-$ radius is the real root in $(0,1)$, given by the equation*

$$\begin{aligned} & [2(A-B) + 2(1-b)\{(1-B)(1+\lambda)(1+K) + A - 1\} r](1-r)^4 \\ = \quad & \lambda K(1-B)\left(1+4r+r^2\right) + (1-B)(\lambda+K-2\lambda K)(1-r^2) \\ & + \{(1-\lambda)(1-K)(1-B) + A - 1\}(1-r)^2, \end{aligned}$$

where $K = \frac{k}{\mu+1}$. *The result is sharp.*

Further, giving special values: $c = 0, d = M$ in Theorem 3.1, we get the following result.

Corollary 3.2. *Let* $f \in \mathcal{A}$ *be of the form (1.1) and let for some* $m \in \mathbb{Z}$, $\mu > -1$, $k > 0, \Theta^m_{k,\mu}(n)$ $(n \geq 2)$ *be given by (2.2). If* $|a_2| = \frac{2b}{\Theta^m_{k,\mu}(2)}(0 \leq b \leq 1)$, $|a_n| \leq \frac{M}{\Theta^m_{k,\mu}(n)}$ $(n \geq 3, M \geq 0)$, *then* $H^m_{k,\mu}(\lambda, A, B)-$ *radius is the real root in* $(0, 1)$, *given by the equation*

$$\begin{aligned} & [(M+1)(A-B) + (M-2b)\{(1-B)(1+\lambda)(1+K) + A - 1\} r](1-r)^4 \\ = \; & \lambda MK(1-B)(1-r^2) + M(\lambda + K - 2\lambda K)(1-B)(1-r)^2 + \\ & M\{(1-\lambda)(1-K)(1-B) + A - 1\}(1-r)^3, \end{aligned}$$

where $K = \frac{k}{\mu+1}$. *The result is sharp.*

Remark 3.2.

(i) For $b = 1$ Corollary 3.1 provides the $H^m_{k,\mu}(\lambda, A, B)-$ radius if the function $\mathcal{J}^m_{k,\mu} f(z)$ is univalent (convex) in $\mathbb{U}$.

(ii) Taking $m = 0, k = 1, \mu = 0, \lambda = 0, A = 1 - \alpha$ $(0 \leq \alpha < 1)$, $B = 0$ in Corollary 3.1, we get the radius result obtained by Ravichandran [23, Theorem 2.1, p. 29] for starlikeness of order α and for parabolic-starlikeness, which also includes the cases when $b = 0$ and 1, respectively, [23, Corollaries 2.1.1 and 2.1.2, p. 31, 32], and when $b = 1, \alpha = 0$ [28, Theorem 2, p. 454].

(iii) Taking $m = 0, k = 1, \mu = 0, \lambda = 0, A = 1 - \alpha$ $(0 \leq \alpha < 1)$, $B = 0$ in Corollary 3.2, we get the radius result obtained by Ravichandran *[23, Theorem 2.2, p. 32]* which also includes the case when $b = \frac{M}{2}$ *[23, Corollary 2.2.1, p. 33]* ([28, Theorem 2, p. 454] if $b = \frac{M}{2}, \alpha = 0$).

(iv) Taking $m = 1, k = 1, \mu = 0, \lambda = 0, A = 1 - \alpha$ $(0 \leq \alpha < 1)$, $B = 0$ in Corollary 3.1, we get result [23, Theorem 3.1, p. 34] for convexity of order α and for uniform convexity, which includes the cases when $b = 1$ and 0, respectively, *[23, Corollaries 3.1.1 and 3.1.2, p. 35, 36]*, and when $b = 1, \alpha = 0$ [28, Theorem 2, p. 454].

(v) On taking $m = 1, k = 1, \mu = 0, \lambda = 0, A = 1 - \alpha$ $(0 \leq \alpha < 1)$, $B = 0$ in Corollary 3.2, we get result *[23, Theorem 3.2, p. 36] which* includes the cases when $b = \frac{M}{2}$ and $\alpha = 0$, respectively, in *[23, Corollary 3.2.1, p. 37] and* [28, Theorem 2, p. 454].

Theorem 3.2. *Let* $f \in \mathcal{A}$ *be of the form (1.1) and let for some* $m \in \mathbb{Z}$, $\mu > -1$, $k > 0, \Theta^m_{k,\mu}(n)$ $(n \geq 2)$ *be given by (2.2). If* $|a_2| = \frac{2b}{\Theta^m_{k,\mu}(2)}(0 \leq b \leq 1)$,
$|a_n| \leq \frac{c}{n\,\Theta^m_{k,\mu}(n)}$ $(n \geq 3, c \geq 0)$, *then* $H^m_{k,\mu}(\lambda, A, B)-$ *radius is the real root in* $(0, 1)$, *given by the equation*

$$\begin{aligned} & \left[(c+1)(A-B) - \left(2b - \frac{c}{2}\right)\{(1+\lambda)(1+K)(1-B) + A - 1\} r\right](1-r)^2 \\ = \; & \lambda cK(1-B) + c(\lambda + K - 2\lambda K)(1-B)(1-r) \\ & -c\{(1-\lambda)(1-K)(1-B) + A - 1\}\frac{\log(1-r)}{r}(1-r)^2, \end{aligned} \tag{3.7}$$

where $K = \frac{k}{\mu+1}$. *The result is sharp.*

Proof. Let r_0 be $H^m_{k,\mu}(\lambda, A, B)-$ radius. Then, we show that $\frac{f(r_0 z)}{r_0} \in H^m_{k,\mu}(\lambda, A, B)$. From the coefficient inequality (2.1), we show that

$$S_3 := \sum_{n=2}^{\infty}\left[A - 1 + (1-B)(1-\lambda+\lambda n)\left(1 + \frac{k(n-1)}{\mu+1}\right)\right]\theta^m_{k,\mu}(n)\,|a_n|\,r_0^{n-1} \leq A - B.$$

Applying the conditions $|a_2| = \frac{2b}{\theta^m_{k,\mu}(2)}(n \geq 2, 0 \leq b \leq 1)$ and $|a_n| \leq \frac{c}{n\theta^m_{k,\mu}(n)}\,(n \geq 3, c \geq 0)$, a calculation shows on using the expansions (3.2), (3.3), (3.4) and (on integrating (3.2)):

$$-\frac{\log(1-r_0)}{r_0} = \sum_{n=1}^{\infty}\frac{r_0^{n-1}}{n},$$

on putting $\frac{k}{\mu+1} = K$, that

$$\begin{aligned}
S_3 &\leq \{(1+\lambda)(1+K)(1-B)+A-1\}\,2br_0 \\
&\quad + \sum_{n=3}^{\infty}\{(1-\lambda+\lambda n)(1-K+Kn)(1-B)+A-1\}\frac{c}{n}r_0^{n-1} \\
&= \{(1+\lambda)(1+K)(1-B)+A-1\}\,2br_0 \\
&\quad + \lambda cK(1-B)\left\{\frac{1}{(1-r_0)^2} - 1 - 2r_0\right\} \\
&\quad + c(\lambda+K-2\lambda K)(1-B)\left\{\frac{1}{(1-r_0)} - 1 - r_0\right\} \\
&\quad - c\{(1-\lambda)(1-K)(1-B)+A-1\}\left\{\frac{\log(1-r_0)}{r_0} + 1 + \frac{r_0}{2}\right\} \\
&= c(B-A) + \left(2b - \frac{c}{2}\right)\{(1+\lambda)(1+K)(1-B)+A-1\}\,r_0 \\
&\quad + \lambda cK(1-B)\frac{1}{(1-r_0)^2} + c(\lambda+K-2\lambda K)(1-B)\frac{1}{1-r_0} \\
&\quad - c\{(1-\lambda)(1-K)(1-B)+A-1\}\frac{\log(1-r_0)}{r_0} \\
&= A - B
\end{aligned}$$

if r_0 satisfy (3.1). Sharpness can be verified for the function $f_1(z)$ such that

$$\mathcal{J}^m_{k,\mu}(f_1(z)) = z - 2bz^2 - \sum_{n=3}^{\infty}\frac{c}{n}z^n.$$

Since, for the function $f_1(z)$,

$$\mathcal{J}_{k,\mu}^{m+1}(f_1(z)) = z - 2b\,(1+K)\,z^2 - \sum_{n=3}^{\infty}(1-K+Kn)\,\frac{c}{n}z^n,$$

where $K = \frac{k}{\mu+1}$, at $z = r_0 \in (0,1)$, satisfying (3.7), we get

$$1 - \frac{(1-\lambda)\,\mathcal{J}_{k,\mu}^{m+1} f_1(z) + \lambda z\left(\mathcal{J}_{k,\mu}^{m+1} f_1(z)\right)'}{\mathcal{J}_{k,\mu}^{m} f_1(z)} = \frac{\mathcal{N}_{r_0}}{\mathcal{D}_{r_0}} = \frac{A-B}{1-B} > 0,$$

$\mathcal{N}_{r_0}$ and $\mathcal{D}_{r_0}$ are given by

$$\begin{aligned}\mathcal{N}_{r_0} &= \left(2b - \frac{c}{2}\right)(\lambda + K + \lambda K)\,r_0 + \lambda c K\frac{1}{(1-r_0)^2}\\ &\quad + c\,(\lambda + K - 2\lambda K)\,\frac{1}{1-r_0} + c\,(\lambda + K - \lambda K)\,\frac{\log(1-r_0)}{r_0}\end{aligned}$$

$$\mathcal{D}_{r_0} = 1 + c - \left(2b - \frac{c}{2}\right)r_0 + c\frac{\log(1-r_0)}{r_0}.$$

This completes the proof of Theorem 3.2. □

Remark 3.3.

(i) Taking $m = 0$, $k = 1$, $\mu = 0$, $\lambda = \alpha$, $A = 1-2\beta$, $B = -1$ in Theorem 3.2, we get the result of Nargesi *et al.* [16, Theorem 3, p. 3] for the class $\mathcal{L}(\alpha,\beta)$

(ii) Taking $m = 0$, $k = 1$, $\mu = 0$, $\lambda = 0$ in Theorem 3.2, we get result [16, Theorem 7, p. 5] for the class $\mathcal{ST}\,[A,B]$.

(iii) Taking $m = 0$, $k = 1$, $\mu = 0$, $\lambda = 0$, $A = 1-\alpha$ $(0 \le \alpha < 1)$, $B = 0$, $c = M$ in Theorem 3.2, we get a result of Ravichandran *[23, Theorem 2.3, p. 34]*.

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Poonam Sharma
Department of Mathematics & Astronomy
University of Lucknow
Lucknow 226007 UP, India
sharma_poonam@lkouniv.ac.in

Ankita
Department of Mathematics & Astronomy
University of Lucknow
Lucknow 226007 UP, India
ankita.maths06@gmail.com