FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 4 (2023), 671–681
https://doi.org/10.22190/FUMI211022043Y
Original Scientific Paper

ON WIJSMAN DEFERRED STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF SETS

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Abstract. In this article, we introduce the concepts of Wijsman deferred statistical convergence and Wijsman strong deferred Cesaro summability for double sequences of sets. Additionally, some properties and based results have been established under a few restrictions.
Keywords: statistical convergence, Cesaro summability, double sequences of sets.

1. Introduction

The idea of statistical convergence was given by Zygmund [34] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [31] and Fast [14] and later reintroduced by Schoenberg [30] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Altın et al. [3], Bhardwaj et al. ([5],[6],[7]), Cakalli [8], Caserta et al. [9], Connor [10], Dagadur and Sezgek [11],

Received October 22, 2021, accepted: September 03, 2023
Communicated by Aleksandar Nastić
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2010 *Mathematics Subject Classification.* Primary 40A05, 40C05; Secondary 46A45

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Nuray et al. ([23],[24],[25]), Et et al. ([12],[13],[16],[17],[28]), Fridy [15], Işık and Akbaş ([4],[18],[19]), Küçükaslan and Yılmaztürk [20], Mursaleen et al. ([21, 22]), Salat [27], Savas [29] and many others.

Agnew [1] introduced the concept of deferred Cesaro mean of real (or complex) valued sequences $x=(x_k)$ defined by

$$(D_p^q(x))_n = \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} x_k, n = 1, 2, 3, ...$$

where $p=(p_n)$ and $q=(q_n)$ are two sequences of non-negative integers satisfying

$$p_n < q_n \text{ } and \text{ } \lim_{n\to\infty} q_n = \infty.$$

A sequence $x=(x_k)$ is said to be deferred statistically convergent to L provided that

$$\lim_{n\to\infty} \frac{|\{p_n < k \leq q_n : |x_k - L| \geq \varepsilon\}|}{q_n - p_n} = 0$$

for each $\varepsilon > 0$ and it is written by $S_p^q - \lim x_k = L$ [20].

Let (X,ρ) be a metric space. The distance $d(x,A)$ from a point x to a non-empty subset A of (X,ρ) is defined to be

$$d(x,A) = \inf_{y\in A} \rho(x,y)$$

If $\sup_k d(x,A_k) < \infty$ (for each $x \in X$), then we say that the sequence $\{A_k\}$ is bounded.

A set of sequence $\{A_k\}$ is said to be Wijsman statistical convergent to A provided that

$$d(x,A) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| = 0$$

if for each $\varepsilon > 0$ and for each $x \in X$. It is written by $st - \lim_W A_k = A$.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [26]. A double sequence $x=(x_{kj})_{k,j\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k,j > N_\varepsilon$. In this case we write $P - \lim_{k,j\to\infty} x_{kj} = L$ or $\lim_{k,j\to\infty} x_{kj} = L$.

A double sequence $x=(x_{kj})$ of real numbers is called to be bounded if there exists a positive real number M such that $|x_{kj}| < M$, for all $k,j \in \mathbb{N}$. In other words $\|x\|_\infty = \sup_{k,j} |x_{kj}| < \infty$.

A double sequence $x=(x_{kj})$ is said to be statistically convergent to L provided that

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{(k,j) : k \leq n, j \leq m : |x_{kj} - L| \geq \varepsilon\}| = 0,$$

Many worthwhile developments of double sequences in summability methods can be found in ([2],[11],[21],[22],[24],[25],[29],[32],[33]).

2. Main Results

In this section, Wijsman deferred statistical convergence and Wijsman strongly deferred Cesàro convergence of double sequences of sets will be defined and the relationship between them will be scrutinized.

Throughout this paper, we will suppose $p = (p_n)$, $q = (q_n), r = (r_m)$ and $t = (t_m)$ are sequences of non-negative integers satisfying the following condition:

$$p_n < q_n,\ t_m < r_m \text{ and} \lim_{n\to\infty} q_n = \infty,\ \lim_{m\to\infty} r_m = \infty \tag{2.1}$$

and

$$\psi_n = q_n - p_n,\ \omega_m = r_m - t_m, D = (p, q, ; r, t) \tag{2.2}$$

Definition 2.1. [11] Let (p_n), $(q_n), (r_m)$ and (t_m) be sequences of non-negative integers satisfying the conditions (2.1) and (2.2). The deferred double natural density of any subset S of $\mathbb{N} \times \mathbb{N}$ is denoted by $\delta_D(S)$ and defined as

$$\delta_D(S) = \lim_{m,n\to\infty} \frac{|\overline{S}_{nm}|}{\psi_n \omega_m},$$

provided the limit exists, where $\overline{S}_{nm} = \{(k, j) \in S : p_n < k \le q_n \text{ and } t_m < j \le r_m\}$.

It is obvious that the deferred double natural density of any finte subset of $\mathbb{N}\times\mathbb{N}$ is zero and $\delta_D(S) + \delta_D(\mathbb{N} \times \mathbb{N} - S) = 1$ for any set $S \subset \mathbb{N} \times \mathbb{N}$.

Before proceeding further, we recall a double sequence (A_{kj}) is Wijsman convergent to A if for each $x \in X$, $P-\lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$ or $\lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$, where the convergence is in Pringsheim's sense.

Definition 2.2. Let (p_n), $(q_n), (r_m)$ and (t_m) be sequences of non-negative integers satisfying the conditions (2.1) and (2.2). A double sequence $\{A_{kj}\}$ is said to be Wijsman deferred statistically convergent to A provided that

$$\lim_{m,n\to\infty} \frac{1}{\psi_n \omega_m} \left|\left\{(k, j) \in \overline{S}_{nm} : |d(x, A_{kj}) - d(x, A)| \ge \varepsilon\right\}\right| = 0$$

for each $\varepsilon > 0$ and for each $x \in X$ and it is written by $A_{kj} \to A$ (WS_d^2) or $WS_d^2 - \lim A_{kj} = A$. The set of all Wijsman deferred statistically convergent sequences will be denoted by WS_d^2. If $q_n = n, p_n = 0, r_m = m$ and $t_m = 0$, then we write WS^2 instead of WS_d^2.

Definition 2.3. Let (p_n), $(q_n), (r_m)$ and (t_m) be sequences of non-negative integers satisfying the conditions (2.1) and (2.2). A double sequence (A_{kj}) is said to be Wijsman strongly deferred convergent to A provided that

$$\lim_{m,n\to\infty} \frac{1}{\psi_n \omega_m} \sum_{k=p_n+1}^{q_n} \sum_{j=t_m+1}^{r_m} |d\,(x, A_{kj} - d\,(x, A))\,| = 0$$

for each $\varepsilon > 0$ and for each $x \in X$, and it is written by $A_{kj} \to A \left(WN_d^2\right)$ or $WN_d^2 - \lim A_{kj} = A$. The set of all Wijsman strongly deferred convergent sequences will be denoted by WN_d^2. If $q_n = n$, $p_n = 0$, $r_m = m$, $t_m = 0$, then we write WN^2 instead of WN_d^2.

If we take $q_n = k_n$, $p_n = k_{n-1}$, $r_m = j_m$, $t_m = j_{m-1}$, where $\theta = \{(k_n, j_m)\}$ are double lacunary sequences, then WS_d^2-convergence is the same as Wijsman lacunary statistical convergence of double sequences of sets and WN_d^2-convergence coincides with Wijsman lacunary strongly convergent of double sequences of sets [25].

We first show that a double sequence which is Wijsman strongly deferred Cesaro summable is Wijsman deferred statistically convergent. However, the converse is not true, in general.

Theorem 2.1. *Let (p_n), (q_n), (r_m) and (t_m) be sequences of non-negative integers satisfying the conditions (2.1) and (2.2). If $WN_d^2 - \lim A_{kj} = A$, then $WS_d^2 - \lim A_{kj} = A$.*

Proof. We assume that $WN_d^2 - \lim A_{kj} = A$. Then for an arbitrary $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{\psi_n \omega_m} \sum_{\substack{k=p_n+1 \\ j=t_m+1}}^{q_n, r_m} |d(x, A_{kj}) - d(x, A)| \\
&= \frac{1}{\psi_n \omega_m} \left(\sum_{\substack{k=p_n+1 \\ j=t_m+1 \\ |d(x,A_{kj})-d(x,A)| \geq \varepsilon}} + \sum_{\substack{k=p_n+1 \\ j=t_m+1 \\ |d(x,A_{kj})-d(x,A)| < \varepsilon}} \right) |d(x, A_{kj}) - d(x, A)| \\
&\geq \frac{1}{\psi_n \omega_m} \sum_{\substack{k=p_n+1 \\ j=t_m+1 \\ |d(x,A_{kj})-d(x,A)| \geq \varepsilon}} |d(x, A_{kj}) - d(x, A)| \\
&\geq \varepsilon \frac{\left|\left\{(k, j) \in \overline{S}_{nm} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right|}{\psi_n \omega_m}.
\end{aligned}$$

By taking limit as $n, m \to \infty$, we obtain

$$\lim_{m,n\to\infty} \frac{\left|\left\{(k, j) \in \overline{S}_{nm} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right|}{\psi_n \omega_m} = 0.$$

The converse of Theorem 2.1. is not true, in general. For this; $q_n = k_n$, $p_n = k_{n-1}$, $r_m = j_m$, $t_m = j_{m-1}$, where $\theta = \{(k_n, j_m)\}$ are double lacunary sequences and define a sequence $\{A_{kj}\}$ as follows:

$$A_{kj} = \begin{cases} \{(k, j)\}, & \text{if } k_{n-1} < k \leq k_{n-1} + \left[\sqrt{h_n}\right], j_{m-1} < j \leq j_{m-1} + \left[\sqrt{h_m}\right] \\ & (n, m = 1, 2, ...) \\ \{(0, 0)\}, & \text{otherwise.} \end{cases}.$$

Note that $\{A_{kj}\}$ is not bounded. For every $\varepsilon > 0$ and for each $x \in X$, we get

$$\begin{aligned} & \frac{1}{h_n h_m} \left|\{(k,j) \in I_{nm} : |d(x, A_{kj}) - d(x, \{(0,0)\})| \geq \varepsilon\}\right| \\ = & \frac{\left[\sqrt{h_n}\right]\left[\sqrt{h_m}\right]}{h_n h_m} \to 0 \text{ as } n, m \to \infty, \end{aligned}$$

that is, $A_{kj} \to \{(0,0)\}\,(WS_d^2)$. But

$$\begin{aligned} & \frac{1}{h_n h_m} \sum_{(k,j) \in I_{nm}} |d(x, A_{kj}) - d(x, \{(0,0)\})| \\ = & \frac{1}{h_n h_m} \left[\frac{\left(\left[\sqrt{h_n}\right]\left(\left[\sqrt{h_n}\right]+1\right)\right)\left(\left[\sqrt{h_m}\right]\left(\left[\sqrt{h_m}\right]+1\right)\right)}{4} \right] \\ \to & \frac{1}{4} \end{aligned}$$

Therefore, $A_{kj} \nrightarrow \{(0,0)\}\,(WN_d^2)$. $\square$

The following theorem establishes that for bounded double sequences $\{A_{kj}\}$, the converse of Theorem 2.1. is also true.

Theorem 2.2. *Let* (p_n), (q_n), (r_m) *and* (t_m) *be sequences of non-negative integers satisfying the conditions* (2.1), (2.2) *and let* $\{A_{kj}\}$ *be a bounded double sequence. If* $WS_d^2 - \lim A_{kj} = A$, *then* $WN_d^2 - \lim A_{kj} = A$.

Proof. Suppose that $\{A_{kj}\}$ is bounded and $WS_d^2 - \lim A_{kj} = A$. In this case, there exists a real number $M > 0$ such that

$$|d(x, A_{kj}) - d(x, A)| \leq M$$

for all $k, j \in \mathbb{N}$. For an arbitrary $\varepsilon > 0$, we have

$$\begin{aligned} & \frac{1}{\psi_n \omega_m} \sum_{\substack{k=p_n+1 \\ j=t_m+1}}^{q_n, r_m} |d(x, A_{kj}) - d(x, A)| \\ = & \frac{1}{\psi_n \omega_m} \left(\sum_{\substack{k=p_n+1 \\ j=t_m+1 \\ |d(x,A_{kj})-d(x,A)| \geq \varepsilon}} |d(x, A_{kj}) - d(x, A)| \right) \\ & + \frac{1}{\psi_n \omega_m} \left(\sum_{\substack{k=p_n+1 \\ j=t_m+1 \\ |d(x,A_{kj})-d(x,A)| < \varepsilon}} |d(x, A_{kj}) - d(x, A)| \right) \\ \leq & M \frac{\left|\left\{(k,j) \in \overline{S}_{nm} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right|}{\psi_n \omega_m} + \varepsilon \end{aligned}$$

Since

$$\lim_{m,n\to\infty} \frac{\left|\left\{(k,j)\in \overline{S}_{nm} : |d(x,A_{kj})-d(x,A)|\geq \varepsilon\right\}\right|}{\psi_n\omega_m} = 0 \; as \; n,m\to\infty$$

we get

$$\lim_{m,n\to\infty} \frac{1}{\psi_n\omega_m} \sum_{\substack{k=p_n+1\\ j=t_m+1}}^{q_n,r_m} |d(x,A_{kj})-d(x,A)| = 0 \text{ as } n,m\to\infty.$$

□

Theorem 2.3. *Let (p_n), (q_n), (r_m) and (t_m) be sequences of non-negative integers satisfying the conditions* (2.1) *and* (2.2). *A Wijsman convergent double sequence (A_{kj}) is Wijsman deferred statistically convergent, but converse need not be true.*

Proof. The proof follows in view of the fact that the deferred double natural density of any finite set is zero. However, the converse is not true, in general. Example in Theorem 2.1. provides a double sequence (A_{kj}) of sets which is Wijsman deferred statistically convergent but fails to be Wijsman convergent. □

We next show that the under certain condition Wijsman statistically convergent double sequences are Wijsman deferred statistically convergent.

Theorem 2.4. *Let (p_n), (q_n), (r_m) and (t_m) be sequences of non-negative integers satisfying the conditions* (2.1), (2.2) *and* $\lim_{n,m\to\infty}\frac{nm}{\psi_n\omega_m} = a > 0$. *If $WS^2 - \lim_W A_{kj} = A$, then $WS_d^2 - \lim A_{kj} = A$.*

Proof. If $WS^2 - \lim_W A_{kj} = A$, then we have

$$\lim_{n,m\to\infty} \frac{1}{nm} \left|\{(k,j) : k\leq n, j\leq m : |d(x,A_{kj})-d(x,A)|\geq\varepsilon\}\right| = 0.$$

Since

$$\begin{aligned} & \left\{(k,j)\in\overline{S}_{nm}\colon\; |d(x,A_{kj})-d(x,A)|\geq\varepsilon\right\} \\ \subset \; & \{(k,j) : k\leq n,\; j\leq m,\; |d(x,A_{kj})-d(x,A)|\geq\varepsilon\} \end{aligned}$$

we have

$$\begin{aligned} & \left|\left\{(k,j)\in\overline{S}_{nm}\colon\; |d(x,A_{kj})-d(x,A)|\geq\varepsilon\right\}\right| \\ \leq \; & \left|\{(k,j) : k\leq n,\; j\leq m,\; |d(x,A_{kj})-d(x,A)|\geq\varepsilon\}\right| \end{aligned}$$

and so

$$\begin{aligned} & \frac{\left|\left\{(k,j)\in\overline{S}_{nm}\colon\; |d(x,A_{kj})-d(x,A)|\geq\varepsilon\right\}\right|}{\psi_n\omega_m} \\ \leq \; & \frac{nm}{\psi_n\omega_m}\frac{\left|\{(k,j) : k\leq n,\; j\leq m : |d(x,A_{kj})-d(x,A)|\geq\varepsilon\}\right|}{nm} \end{aligned}$$

Hence

$$\lim_{n,m\to\infty} \frac{\left|\left\{(k,j)\in \overline{S}_{nm}\colon\ |d(x,A_{kj})-d(x,A)|\geq \varepsilon\right\}\right|}{\psi_n\omega_m}=0$$

that is $WS_d^2 - \lim A_{kj} = A$. $\square$

In the next theorem, we arrive at the same result as established in Theorem 4, but under a different condition.

Theorem 2.5. *Let (p_n), (q_n), (r_m) and (t_m) be sequences of non-negative integers satisfying the conditions (2.1), (2.2) and $\{A_{kj}\}$ be double sequence of sets. If $\liminf_n \frac{q_n}{p_n} > 1$ and $\liminf_m \frac{r_m}{t_m} > 1$, then $WS^2 - \lim_W A_{kj} = A$ implies $WS_d^2 - \lim A_{kj} = A$.*

Proof. Assume that $\liminf_n \frac{q_n}{p_n} > 1$ and $\liminf_m \frac{r_m}{t_m} > 1$, then there exist $\alpha,\beta > 0$ such that $\frac{q_n}{p_n} \geq 1+\alpha$ and $\frac{r_m}{t_m} \geq 1+\beta$ for sufficiently large n, m which implies that

$$\begin{aligned}
\frac{q_n}{p_n} &\geq 1+\alpha \Rightarrow \frac{q_n-p_n}{q_n} \geq \frac{\alpha}{1+\alpha}\\
\frac{r_m}{t_m} &\geq 1+\beta \Rightarrow \frac{r_m-t_m}{r_m} \geq \frac{\beta}{1+\beta}\\
\frac{(q_n-p_n)}{q_n}\frac{(r_m-t_m)}{r_m} &\geq \frac{\alpha\beta}{(1+\alpha)(1+\beta)}\\
&\Rightarrow \frac{\psi_n}{q_n}\frac{\omega_m}{r_m} \geq \frac{\alpha\beta}{(1+\alpha)(1+\beta)}
\end{aligned}$$

If $WS^2 - \lim_W A_{kj} = A$, then for every $\varepsilon > 0$ and for sufficiently larger n, m we get

$$\begin{aligned}
&\frac{1}{q_n r_m}\left|\{(k,j): k\leq q_n,\ j\leq r_m : |d(x,A_{kj})-d(x,A)|\geq\varepsilon\}\right|\\
\geq\ &\frac{1}{q_n r_m}\left|\left\{(k,j)\in\overline{S}_{nm} : |d(x,A_{kj})-d(x,A)|\geq\varepsilon\right\}\right|\\
\geq\ &\frac{\alpha\beta}{(1+\alpha)(1+\beta)}\left(\frac{1}{\psi_n\omega_m}\left|\left\{(k,j)\in\overline{S}_{nm}\colon\ |d(x,A_{kj})-d(x,A)|\geq\varepsilon\right\}\right|\right)
\end{aligned}$$

for each $x \in X$. Hence, $WS_d^2 - \lim A_{kj} = A$. $\square$

Following the same technique , as that of Lemma 1,1 of Salat [27], we have

Theorem 2.6. *Let (p_n), (q_n), (r_m) and (t_m) be sequences of non-negative integers satisfying the conditions (2.1) and (2.2). A double sequence $\{A_{kj}\}$ of sets is Wijsman deferred statistically convergent to A if and only if there exists a set $K \subset \mathbb{N}\times\mathbb{N}$ such that $\delta_D(K) = 1$ and $\lim_{\substack{k,j\to\infty\\(k,j)\in K}} A_{kj} = A$.*

Proof. For $r \in \mathbb{N}$, let $K^r = \{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x, A_{kj}) - d(x, A)| < \frac{1}{r}\}$. As $K^r = \mathbb{N}\times\mathbb{N} - \{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \frac{1}{r}\}$ and $WS_d^2 - \lim A_{kj} = A$ so $\delta_D(K^r) = 1$. As

$$\begin{aligned} & \left\{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x, A_{kj}) - d(x, A)| < \frac{1}{r+1}\right\} \\ \subset \quad & \left\{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x, A_{kj}) - d(x, A)| < \frac{1}{r}\right\} \end{aligned}$$

so $K^1 \supset K^2 \supset K^3 \cdots \supset K^r \supset K^{(r+1)} \cdots$ and $\delta_D(K^r) = 1$. Let us choose $(n_1, m_1) \in K^1$. Then there exists $(n_2, m_2) > (n_1, m_1)$, $(n_2, m_2) \in K^2$ such that for all $(n, m) \geq (n_2, m_2)$ we have

$$\frac{1}{\psi_n \omega_m}\left|\left\{(k,j) \in \mathbb{N}\times\mathbb{N} : p_n < k \leq q_n \text{ and } t_m < j \leq r_m \ \ |d(x, A_{kj}) - d(x, A)| < \frac{1}{2}\right\}\right| > \frac{1}{2}.$$

Choose $(n_3, m_3) > (n_2, m_2)$,$(n_3, m_3) \in K^3$ such that for all $(n, m) \geq (n_3, m_3)$ we have

$$\frac{1}{\psi_n \omega_m}\left|\left\{(k,j) \in \mathbb{N}\times\mathbb{N} : p_n < k \leq q_n \text{ and } t_m < j \leq r_m \ \ |d(x, A_{kj}) - d(x, A)| < \frac{1}{3}\right\}\right| > \frac{2}{3}.$$

We continue this process and construct by induction a sequence $(n_1, m_1) < (n_2, m_2) < (n_3, m_3) \cdots (n_j, m_j) < \cdots$ such that $(n_j, m_j) \in K^j$ with

$$\frac{1}{\psi_n \omega_m}\left|\left\{(k,j) \in \mathbb{N}\times\mathbb{N} : p_n < k \leq q_n \text{ and } t_m < j \leq r_m \ \ |d(x, A_{kj}) - d(x, A)| < \frac{1}{j}\right\}\right| > \frac{j-1}{j}$$

for all $(n, m) \geq (n_j, m_j)$.

Let us consider $K = \Big([1, n_1) \times [1, m_1)\Big) \bigcup_j \Bigg(\Big([\ n_j, n_{j+1}) \times [m_j, m_{j+1})\Big) \bigcap K^j\Bigg)$.

Now for each (n, m) such that $(n_j, m_j) < (n, m) < (n_{j+1}, m_{j+1})$, we get

$$\begin{aligned} & \frac{1}{\psi_n \omega_m}\left|\{(k,j) \in K : p_n < k \leq q_n \text{ and } t_m < j \leq r_m\}\right| \\ \geq \quad & \frac{1}{\psi_n \omega_m}\left|\left\{(k,j) \in \mathbb{N}\times\mathbb{N} : p_n < k \leq q_n \text{ and } t_m < j \leq r_m \ \ |d(x, A_{kj}) - d(x, A)| < \frac{1}{j}\right\}\right| \\ > \quad & \frac{j-1}{j} \text{ for each j} \in \mathbb{N}. \end{aligned}$$

From this, we have $\delta_D(K) = 1$. Let $\varepsilon > 0$. Choose j such that $\frac{1}{j} < \varepsilon$. Now for all $(n, m) \geq (n_j, m_j)$ and $(n, m) \in K$, choose $p \geq j$ such that
$(n_p, m_p) \leq (n, m) \leq (n_{p+1}, m_{p+1})$, we get $(n, m) \in K^p$ which in turn yields

$$|d(x, A_{kj}) - d(x, A)| < \frac{1}{p} \leq \frac{1}{j} < \varepsilon.$$

Conversely, suppose there exists a set $K \subset \mathbb{N}\times\mathbb{N}$ such that $\delta_D(K) = 1$ and $\lim_{\substack{k,j\to\infty \\ (k,j)\in K}} A_{kj} = A$. For $\varepsilon > 0$, there exists $(n_0, m_0) \in \mathbb{N}\times\mathbb{N}$ such that

$|d(x, A_{kj}) - d(x, A)| < \varepsilon$ for all $(k, j) \geq (n_0, m_0)$ and $(k, j) \in K$.
Taking $K_\varepsilon = \{(k, j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}$, the result follows in view of the facts that $K_\varepsilon \subset (\mathbb{N} \times \mathbb{N}) - K$. $\square$

Before proceeding further first we introduce the following notation:

If $A = (A_{kj})$ is a double sequence of sets such that (A_{kj}) satisfies property P for all (k, j), except a set of deferred double natural density zero, then we say $A = (A_{kj})$ satisfies P for "almost all (k, j) deferred double with respect to $D = (p, q : r, t)$" and we abbreviate this by "a.a. (k, j) deferred double w.r.t.D" where $p = (p_n)$, $q = (q_n), r = (r_m)$ and $t = (t_m)$ be sequences of non-negative integers satisfying the conditions (2.1) *and* (2.2).

Finally we establish that the terms of a Wijsman deferred statistically convergent double sequence (A_{kj}) are coincident with those of a Wijsman convergent sequence for a. a. (k, j) deferred double w.r.t. D.

Theorem 2.7. *A double sequence (A_{kj}) of sets is Wijsman deferred statistically convergent if and only if there exists a Wijsman convergent double sequence (B_{kj}) of sets such that $B_{kj} = A_{kj}$ a. a. (k, j) deferred double w.r.t. D.*

Proof. Let (A_{kj}) is Wijsman deferred statistically convergent to A. So for each $\varepsilon > 0$ we have $\delta_D(K) = 0$ where $K = \{(k, j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}$.
Consider

$$B_{kj} = \begin{cases} A_{kj}, & \text{if } (k, j) \in (\mathbb{N} \times \mathbb{N}) - K \\ A, & \text{otherwise} \end{cases}.$$

Then (B_{kj}) is a Wijsman convergent double sequence of sets converging to A such that $B_{kj} = A_{kj}$ a. a. (k, j) deferred double w.r.t. D.
Conversely, let (B_{kj}) is a Wijsman convergent double sequence of sets converging to A such that $B_{kj} = A_{kj}$ a. a. (k, j) deferred double w.r.t. D. Then for $\varepsilon > 0$ there exists $k_0, j_0 \in \mathbb{N}$ such that $|d(x, B_{kj}) - d(x, A)| < \varepsilon$ for all $(k, j) \geq (k_0, j_0)$.
Let $K = \{(k, j) \in \mathbb{N} \times \mathbb{N} : B_{kj} \neq A_{kj}\}$. Now $\{(k, j) : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \subset K \cup [1, k_0) \times [1, j_0)$, yields the result. $\square$

3. Acknowledgments

The authors acknowledge that some of the results were presented at the 8th International Conference on Recent Advances in Pure and Applied Mathematics (ICRAPAM 2021) 24-27 September 2021, Bodrum/Muğla, Turkey.

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