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ON FRAMED TZITZEICA CURVES IN EUCLIDEAN SPACE

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Abstract. Investigations are very important for non-regular curves in differential geometry. Framed curves have been used recently to study singular curves, and they have many contributions to singularity theory. In this study, framed Tzitzeica curves are introduced with the help of framed curves. In addition, some framed special curves that satisfy the Tzitzeica condition are given. New results have been obtained among the framed curves of these curves.
Keywords: Framed curves, framed Tzitzeica curves, framed rectifying curves, framed spherical curves.

1. Introduction

One of the most famous Romanian mathematicians, Gheorghe Tzitzeica (1873-1939), introduced a type of curves that carry today his name. A Tzitzeica curve is a space curve where the ratio of the torsion τ and the square of the distance from its origin to the osculating plane d_{osc} at an arbitrary point $\gamma(s)$ of the curve γ, is constant. More precisely, this condition

$$\frac{\tau}{d_{osc}^2} = c$$

where $d_{osc} = \langle B, \gamma \rangle$ and $c \neq 0$ is a real constant, B is the binormal vector of γ. Also, A Tzitzeica surface is a spatial surface where the ratio of the Gaussian curvature

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K and the 4th power of the distance from its origin to the tangent plane d_{tan} at an arbitrary point of the surface $X(u, v)$, is constant. More precisely, this condition

$$\frac{K}{d_{tan}^4} = c$$

where $d_{tan} = \langle N, X\rangle$ and $c \neq 0$ is a real constant, N is a unit normal vector and X is the position vector of the surface.

In [1], Tzitzeica type non-null curves in Minkowski space are examined. In addition, Tzitzeica conditions are given for non-null general helices and pseudo-spherical curves. In [5], the authors determined the elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition in a Euclidean space. In this study, they showed that the Tzitzeica condition gives an ordinary differential equation of the third order that allows direct integration. In [13], the elliptic cylindrical curves verifying Tzitzeica condition were adapted to the Minkowski 3-space. In [3], they examined the Tzitzeica curve equation in the field of symmetry analysis theory and examined the Tzitzeica curve equation as a nonlinear ordinary differential equation. Moreover, these curves have the property of being centro-affine invariant [4]. The readers are referred to [1]-[7] for some interesting studies on the Tzitzeica curves and surfaces.

In the literature, the conditions for being Tzitzeica are given on the basis of a curve being regular. In this study, Tzitzeica curves will be investigated with the help of framed curve theory [9] for non-regular curves. Recently, differential geometry of curves and surfaces with singular points has extensively been investigated (for instance, see [9]-[18]). One of them is framed curves for smooth curves with singular points. A framed curve in the Euclidean space is a curve with a moving frame. Framed curves are a generalization of both regular curves with linear independent conditions and Legendre curves in unit tangent bundle. Framed curves may have singular points. Although a moving frame cannot be defined for a curve with a singular point, such a frame can be defined due to the special structure of framed curves. In addition, smooth functions such as framed curves are defined for framed curves as well as regular curves, and these functions are important for characterizing singular points.

In this study, Tzitzeica conditions of singular curves are investigated with the help of framed curve theory. First, a Tzitzeica equation is obtained of regular framed curves. Later, this equation is harmonized with the literature by using Bishop frame and Frenet frame of regular curves. Then, a Tzitzeica equation is given by using Frenet-type framed curves for non-regular curves. Later, special singular curves such as framed rectifying curve, framed spherical curve are given the conditions for being framed Tzitzeica. Differential equations related to these have been created and the solutions of these differential equations based on framed curvatures have been examined. Finally, Tzitzeica equations are given for the general position vector of a framed curve.

2. Framed Curves

In this section, a quick preliminary preparation is given to introduce the concept of the framed curve [9].

Let $\gamma : I \to \mathbb{R}^3$ be a curve with singular points. The set

$$\Delta_\eta = \{\eta = (\eta_1, \eta_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle \eta_i, \eta_j \rangle = \delta_{ij}, \quad i, j = 1, 2\}$$

is an 3-dimensional smooth manifold. This type of manifold is known in the literature as the "Stiefel manifold". A unit vector is defined by $\nu = \eta_1 \times \eta_2$.

Definition 2.1. [**Framed curve**] $(\gamma, \eta) : I \to \mathbb{R}^3 \times \Delta_\eta$ is called a framed curve if $\langle \gamma^{'}(s), \eta_i(s) \rangle = 0$ for all $s \in I$ and $i = 1, 2$. Also, $\gamma : I \to \mathbb{R}^3$ is called a framed base curve if there exists $\eta : I \to \Delta_\eta$ such that (γ, η) is a framed curve [9].

The set of singular points are denoted of γ by $\Sigma(\gamma)$ where $\Sigma(\gamma) = \{s \in I | \gamma^{'}(s) = 0\}$. Let $(\gamma, \eta_1, \eta_2) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed curve and $\nu = \eta_1 \times \eta_2$. The Frenet-Serret type formula is given by

$$\begin{pmatrix} \nu^{'}(s) \\ \eta_1^{'}(s) \\ \eta_2^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & -m(s) & -n(s) \\ m(s) & 0 & l(s) \\ n(s) & -l(s) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \eta_1(s) \\ \eta_2(s) \end{pmatrix}. \tag{2.1}$$

where $l(s) = \langle \eta_1^{'}(s), \eta_2(s) \rangle$, $m(s) = \langle \eta_1^{'}(s), \nu(s) \rangle$ and $n(s) = \langle \eta_2^{'}(s), \nu(s) \rangle$.
$(l(s), m(s), n(s), \alpha(s))$ are called the framed curvature of γ. Also, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that:

$$\gamma^{'}(s) = \alpha(s)\nu(s)$$

In addition, s_0 is a singular point of γ if and only if $\alpha(s_0) = 0$. Clearly, if $\alpha(s) = 0$ for all $s \in I$, then $\gamma(s)$ is a point. Although it has a curved singular point, if a η vector can be found, a moving frame can be established. Special frames can be obtained under certain conditions for framed curves as well as for regular curves. Let's quickly examine the set up of these frames:
$(\widetilde{\eta_1}(s), \widetilde{\eta_2}(s)) \in \Delta_\eta$ is defined by

$$\begin{aligned} \widetilde{\eta_1}(s) &= \cos\varphi(s)\eta_1(s) - \sin\varphi(s)\eta_2(s), \\ \widetilde{\eta_2}(s) &= \sin\varphi(s)\eta_1(s) + \cos\varphi(s)\eta_2(s). \end{aligned}$$

Then, $(\gamma, \widetilde{\eta_1}, \widetilde{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ also a framed curve and

$$\widetilde{\nu}(s) = \nu(s).$$

Hence, according to equation (2.1), we get

(2.2)

$$
\begin{aligned}
\nu^{'}(s) &= -m(s)\eta_1(s) - n(s)\eta_2(s) \\
\widetilde{\eta_1}^{'}(s) &= -(l(s)-\varphi^{'}(s))\sin\varphi(s)\eta_1(s) \\
&\quad +(l(s)-\varphi^{'}(s))\cos\varphi(s)\eta_2(s) + (m(s)\cos\varphi(s) - n(s)\sin\varphi(s))\nu(s) \\
\widetilde{\eta_2}^{'}(s) &= -(l(s)-\varphi^{'}(s))\cos\varphi(s)\eta_1(s) \\
&\quad +(l(s)-\varphi^{'}(s))\sin\varphi(s)\eta_2(s) + (m(s)\sin\varphi(s) + n(s)\cos\varphi(s))\nu(s).
\end{aligned}
$$

i. Let's assume that $\varphi : I \to \mathbb{R}$ is a smooth function that satisfies

$$l(s) - \varphi^{'}(s) = 0$$

equation. Hence, $\{\nu, \widetilde{\eta_1}, \widetilde{\eta_2}\}$ is called Bishop frame of framed curve [9],[10]. Consequently, the derivative formula is given by

$$
\begin{pmatrix} \nu^{'}(s) \\ \widetilde{\eta_1}^{'}(s) \\ \widetilde{\eta_2}^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & -\widetilde{m}(s) & -\widetilde{n}(s) \\ \widetilde{m}(s)) & 0 & 0 \\ \widetilde{n}(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \widetilde{\eta_1}(s) \\ \widetilde{\eta_2}(s) \end{pmatrix}.
$$

ii. Let's assume that $\varphi : I \to \mathbb{R}$ is a smooth function that satisfies

$$m(s)\sin\varphi(s) + n(s)\cos\varphi(s) = 0$$

equation. Therefore, this equation can be rearranged with the $m(s) = -p(s)\cos\varphi(s)$ and $n(s) = p(s)\sin\varphi(s)$ equations [15]. Since φ will change, $\widetilde{\eta_1}(s), \widetilde{\eta_2}$ vectors change and let's take the framed curve in $\{\nu, \overline{\eta_1}, \overline{\eta_2}\}$ notation. $\{\nu, \overline{\eta_1}, \overline{\eta_2}\}$ is called generalized frame or Frenet type frame of framed curve $\gamma(s)$ [9],[10],[15]. Substituting $m(s)$ and $n(s)$ equations in equation (2.2), the derivative formula is given by

$$
\begin{pmatrix} \nu^{'}(s) \\ \overline{\eta_1}^{'}(s) \\ \overline{\eta_2}^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & p(s) & 0 \\ -p(s) & 0 & q(s) \\ 0 & -q(s) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \overline{\eta_1}(s) \\ \overline{\eta_2}(s) \end{pmatrix}.
$$

The functions $(p(s), q(s), \alpha(s))$ are defined to as the framed curvature of $\gamma(s)$ [15] where

$$p(s) = \|\nu^{'}(s)\|$$

and

$$q(s) = l(s) - \varphi^{'}(s).$$

Now, another calculation method for $q(s)$ framed curvature is given for use in the following sections and the equality of these methods is shown:

Now let's show that

$$q(s) = l(s) - \varphi^{'}(s) = \frac{\langle \nu(s) \wedge \nu^{'}(s), \nu^{''}(s) \rangle}{(\|\nu'(s)\|)^2}$$

Since

$$\begin{array}{rcl} m(s) & = & -p(s)\cos(\varphi(s)) \\ n(s) & = & p(s)\sin(\varphi(s)), \\ \varphi(s) & = & \arctan\left(-\dfrac{n}{m}\right). \end{array}$$

Consequently, we get

(2.3)

$$\begin{array}{rcl} \varphi^{'}(s) & = & \dfrac{-n^{'}(s)m(s)+n(s)m^{'}(s)}{m^2(s)+n^2(s)} \\ q(s) & = & l(s)-\varphi^{'}(s)=\dfrac{l(s)m^2(s)+l(s)n^2(s)+n^{'}(s)m(s)-n(s)m^{'}(s)}{m^2(s)+n^2(s)}. \end{array}$$

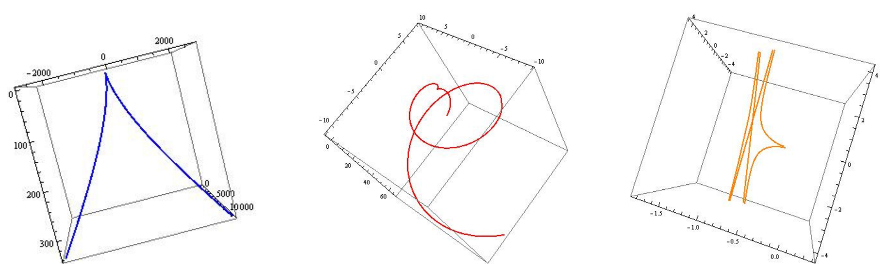
Now let's calculate the other side of the equation whose equation we want to show: According to equation (2.1), we have

$$\begin{array}{rcl} \nu^{'}(s) & = & -m(s)\eta_1(s)-n(s)\eta_2(s) \\ \nu^{''}(s) & = & (-m^2(s)-n^2(s))\nu(s)+(-m^{'}(s)+l(s)n(s))\eta_1(s) \\ & & +(-n^{'}(s)-l(s)m(s))\eta_2(s). \end{array}$$

Therefore, we have

$$\frac{\langle \nu(s)\wedge\nu^{'}(s),\nu^{''}(s)\rangle}{(\|\nu^{'}(s)\|)^2}=\frac{l(s)m^2(s)+l(s)n^2(s)+n^{'}(s)m(s)-n(s)m^{'}(s)}{m^2(s)+n^2(s)}. \tag{2.4}$$

That is, according to equations (2.3) and (2.4), the equality is seen. Now let's give some figures to better understand and visualize framed curves in space [16]-[18]:

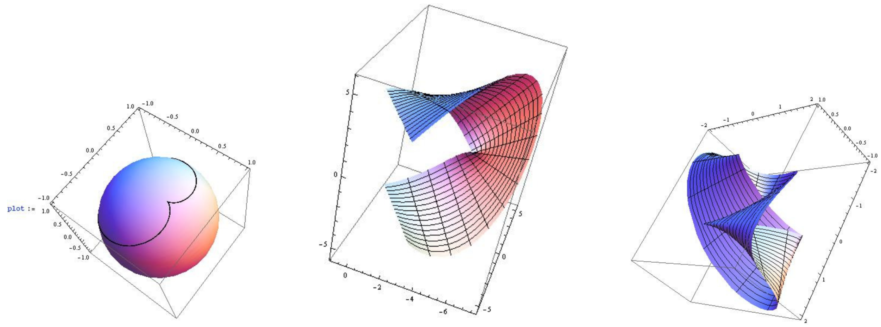


(a) The framed curve with cusp point

(b) The framed helix

(c) The framed rectifying curve

Fig. 2.1



(a) The framed spherical curve (b) The ruled surface with framed curves (c) The ruled surface with framed curves

FIG. 2.2

2.1. Regular framed Tzitzeica curves

Let $(\gamma, \eta_1, \eta_2) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a regular framed curve (i.e Frenet curve). If

$$\frac{\langle \gamma' \wedge \gamma'', \gamma''' \rangle}{\langle \gamma, \gamma' \wedge \gamma'' \rangle^2} = \frac{mn' - m'n + l(m^2 + n^2)}{\alpha(m\langle \gamma, \eta_2 \rangle - n\langle \gamma, \eta_1 \rangle)^2} = c \tag{2.5}$$

for each $s \in I$ and real constant $c \neq 0$ is called regular framed Tzitzeica curve.

Corollary 2.1. *i. [7] Let $(\gamma, \eta_1, \eta_2) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed curve with a Bishop frame $\{\nu, \widetilde{\eta}_1, \widetilde{\eta}_2\}$. Therefore, if*

$$\frac{\widetilde{m}\widetilde{n}' - \widetilde{m}'\widetilde{n}}{\alpha(\widetilde{m}\langle \gamma, \widetilde{\eta}_2 \rangle - \widetilde{n}\langle \gamma, \widetilde{\eta}_1 \rangle)^2} = c$$

for each $s \in I$ and real constant $c \neq 0$ is called framed Tzitzeica curves with Bishop frame.
ii. [2] Let $(\gamma, \eta_1, \eta_2) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed curve with a Frenet-type frame $\{\nu, \overline{\eta_1}, \overline{\eta_2}\}$. Since $\dfrac{q}{\alpha} = \tau$ [15] for regular framed curves, we get

$$\frac{q}{\alpha\langle \gamma, \overline{\eta_2} \rangle^2} = \frac{\tau}{\langle \gamma, \overline{\eta_2} \rangle^2} = c$$

for each $s \in I$ and real constant $c \neq 0$ is called framed Tzitzeica curves with Frenet-type frame.

2.2. Framed Tzitzeica curves with singular points

Singular curves γ' and γ'' can be linearly dependent or $\alpha(s) = 0$ for some s. Therefore, equation (2.5) is only used for regular framed curves. For framed curves with singular points, we can use Frenet-type framed curves. If a regular spherical ν curve is found for the γ framed curve, we can talk about the Frenet-type framed base curve.

Definition 2.2. [**Frenet-type framed Tzitzeica curves**] Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ Frenet-type framed curve with at least one non-singular point. If γ is a Frenet-type framed Tzitzeica curves, then the equation

$$\frac{\langle \nu \wedge \nu', \nu'' \rangle}{\langle \gamma, \nu \wedge \nu' \rangle^2} = \frac{q(s)}{\langle \gamma(s), \overline{\eta_2}(s) \rangle^2} = c\alpha(s)$$

for each $s \in I$ and real constant $c \neq 0$ is called Frenet-type framed Tzitzeica curves.

Remark 2.1. If γ is regular framed base curve, since $\dfrac{q(s)}{\alpha(s)} = \tau(s)$ and $\overline{\eta_2}(s) = B(s)$, we get $\dfrac{\tau}{\langle \gamma, B \rangle^2} = c$=constant.

Proposition 2.1. *Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ Frenet-type framed curve. If γ is a framed Tzitzeica curves, then the equation*

$$q'(s)\langle \gamma(s), \overline{\eta_2}(s) \rangle + 2q^2(s)\langle \gamma(s), \overline{\eta_1}(s) \rangle = c\alpha'(s)\langle \gamma(s), \overline{\eta_2}(s) \rangle^3 \tag{2.6}$$

holds.

Let us now examine under what conditions they are Tzitzeica curves for some special framed curves:

2.3. Framed rectifying curves satisfying the Tzitzeica condition

In [15], framed rectifying curves are defined by Pei, Wang and Gao. Many characterizations are given in this study, which includes both regular rectifying curves with linear independent conditions and non-regular rectifying curves.

Definition 2.3. Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ framed curve. If the position vector of the γ framed base curve satisfies the $\gamma(s) = \delta(s)\nu(s) + \varepsilon(s)\overline{\eta_2}(s)$ equation for some functions $\delta(s)$ and $\varepsilon(s)$, it is called a framed rectifying curve [15].

Lemma 2.1. *Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed curve with at least one non-singular point and $q(s) = k\alpha(s)$ where k is a non-zero constant. $\gamma(s)$ is a framed Tzitzeica curve if and only if $\gamma(s)$ framed rectifying curve.*

Proof. Assume that $\gamma : I \to \mathbb{R}^3$ framed Tzitzeica curve with at least one non-singular point and $q(s) = k\alpha(s)$ where k is a non-zero constant. Therefore, we have $\dfrac{q(s)}{\langle \gamma(s), \overline{\eta_2}(s) \rangle^2} = c\alpha(s)$ where c is a non-zero constant. Since $q(s) = k\alpha(s)$, then

$\langle \gamma(s), \overline{\eta_2}(s)\rangle = constant$. If the derivative of both sides of the equation is taken, we have

$$\langle \gamma^{'}(s), \overline{\eta_2}(s)\rangle + \langle \gamma(s), \overline{\eta_2}^{'}(s)\rangle = 0,$$

$$\alpha(s)\langle \nu(s), \overline{\eta_2}(s)\rangle - q(s)\langle \gamma(s), \overline{\eta_1}(s)\rangle = 0,$$

Consequently, we get

$$q(s)\langle \gamma(s), \overline{\eta_1}(s)\rangle = 0$$

Since $q(s) = k\alpha(s)$ where k is a non-zero constant and $\gamma : I \to \mathbb{R}^3$ has at least one non-singular point , we have

$$\langle \gamma(s), \overline{\eta_1}(s)\rangle = 0$$

That is, $\gamma(s)$ is framed rectifying curves. Conversely, let us assume that $\gamma(s)$ is framed rectifying curve with $q(s) = k\alpha(s)$ where k is a non-zero constant. Therefore, we have $\gamma(s) = \delta(s)\nu(s) + \varepsilon(s)\overline{\eta_2}(s)$ equation for some functions $\delta(s)$ and $\varepsilon(s)$. Since $q(s) = k\alpha(s)$, it must be shown that $\langle \gamma(s), \overline{\eta_2}(s)\rangle = constant$ so that the curve is a framed Tzitzeica curve. Consequently, we have

$$\langle \gamma(s), \overline{\eta_2}(s)\rangle^{'} = \alpha(s)\langle \nu(s), \overline{\eta_2}(s)\rangle - q(s)\langle \gamma(s), \overline{\eta_1}(s)\rangle$$

Since the $\gamma(s)$ framed rectifying curve is $\langle \gamma(s), \overline{\eta_1}(s)\rangle = 0$, it is clear that $\langle \gamma(s), \overline{\eta_2}(s)\rangle^{'} = 0$. Conseguently, we have $\langle \gamma(s), \overline{\eta_2}(s)\rangle = constant$ and $\dfrac{q(s)}{\langle \gamma(s), \overline{\eta_2}(s)\rangle^2} = k\alpha(s)$ where k is a non-zero constant. That is, $\gamma(s)$ is a framed Tzitzeica curve. $\square$

Corollary 2.2. *Framed rectifying curves is framed Tzitzeica curve with condition $q(s) = k\alpha(s)$ where k is a non-zero constant.*

Corollary 2.3. *Regular framed rectifying curves is framed Tzitzeica curve with condition $\tau(s) = k$ where k is a non-zero constant.*

Also, a classification for the curvatures of framed rectifying curves is given as follows:

Theorem 2.1. *Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve with $p(s) > 0$. If γ is a framed rectifying curve, then*

$$\frac{q(s)}{p(s)} = c_1 \int \alpha(s)ds + c_2$$

for some constants $c_1 \neq 0$ and c_2 [15].

Corollary 2.4. *Since framed rectifying curves are framed Tzitzeica curve $q(s) = k\alpha(s)$, according to Theorem 2.1, the condition of being framed Tzitzeica curve can be given by*

$$\left(\frac{\alpha(s)}{p(s)}\right)^{'} = \frac{c_1}{k}\alpha(s)$$

where $k, c_1 \neq 0$.

Corollary 2.5. *For a framed curve γ, the curve defined by the vector*

$$d = q(s)\nu(s) + p(s)\overline{\eta_2}(s)$$

which is called the centrode of framed curve γ. According to [15], "The centrode of a framed curve with nonzero constant framed curvature function $q(s)$ and nonconstant framed curvature function $p(s)$ is a framed rectifying curve." Therefore, if $\alpha(s)$ nonzero constant the centrode of framed curve γ is framed Tzitzeica curve where nonzero constant framed curvature function $q(s)$.

2.4. Framed spherical curves satisfying the Tzitzeica condition

The framed base curves on the S^2 sphere are called framed spherical curves. There are many studies in the literature about framed spherical curves including both regular and non-regular curves. Honda and Takahashi studied evolutes and focal surfaces with the help of framed immersions and gave some characterizations of framed spherical curves [12]. In this section, framed spherical curves, which meet the condition of being framed Tzitzeica curve, have been examined and important results have been obtained:

Theorem 2.2. *Let $\gamma(s)$ framed spherical curve in $\mathbb{R}^3$ with $q(s) \neq 0$ for every $s \in I$. If γ is a Tzitzeica curve, we have*

$$-\frac{q^{'}(s)}{q^2(s)}\left(\frac{\alpha(s)}{p(s)}\right)^{'} + \frac{c\alpha^{'}(s)}{q^4(s)}\left(\left(\frac{\alpha(s)}{p(s)}\right)^{'}\right)^3 = 2q(s)\left(\frac{\alpha(s)}{p(s)}\right). \tag{2.7}$$

Proof. Let $\gamma(s)$ framed spherical curve in $\mathbb{R}^3$. According to [12], then we have

$$\gamma(s) - c = u(s)\overline{\eta_1}(s) + v(s)\overline{\eta_2}(s)$$

That is

$$\langle \gamma(s), \nu(s) \rangle = 0. \tag{2.8}$$

Taking the derivative of equation (2.8), we have

$$u(s) = \langle \gamma, \overline{\eta_1}(s) \rangle = -\frac{\alpha(s)}{p(s)} \tag{2.9}$$

Finally, by taking the derivative in equation (2.9), we get

$$v(s) = \langle \gamma, \overline{\eta_2}(s) \rangle = \frac{1}{q(s)}\left(-\frac{\alpha(s)}{p(s)}\right)^{'} \tag{2.10}$$

If equations (2.9) and (2.10) are substituted in the equation (2.6), the proof is completed. $\square$

Remark 2.2. Since framed spherical curves are defined with the condition $q(s) \neq 0$ for each $s \in I$, $\alpha(s) \neq 0$ for each $s \in I$ must be. Thus, framed spherical Tzitzeica curves are regular curves.

Proposition 2.2. *[12] Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed spherical curve with $q(s) \neq 0$ and $p(s) > 0$. Then, the framed curvature of $\gamma(s)$ satisfies the equation*

$$\frac{\alpha(s)q(s)}{p(s)} + \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)^{'}\right)^{'} = 0 \tag{2.11}$$

Corollary 2.6. *Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed spherical Tzitzeica curve with $q(s) \neq 0$ (i.e $\alpha(s) \neq 0$ for every $s \in I$). Then, the framed curvature of $\gamma(s)$ satisfies the equation*

$$\frac{1}{q^2(s)}\left(\frac{\alpha(s)}{p(s)}\right)^{''} - \frac{c\alpha^{'}(s)}{q^5(s)}\left(\left(\frac{\alpha(s)}{p(s)}\right)^{'}\right)^3 + 3\left(\frac{\alpha(s)}{p(s)}\right) = 0 \tag{2.12}$$

Proof. According to equation (2.11), we have

$$\frac{\alpha(s)q(s)}{p(s)} - q^{'}(s)\left(\frac{\alpha(s)}{p(s)}\right)^{'}\frac{1}{q^2(s)} + \frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)^{''} = 0. \tag{2.13}$$

Substituting (2.7) into (2.13), we get result. □

The differential equation (2.12) is a non-linear differential equation. According to some special cases of curvatures, their solutions can be studied as a linear differential equation. Now some results will be given depending on whether the framed curvatures are constant or not:

Corollary 2.7. *Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed spherical Tzitzeica curve with non-zero constants $q(s)$ and $\alpha(s)$. Then, we get*

$$\frac{\alpha(s)}{p(s)} = \left(c_1 \sin(\sqrt{3}qs) + c_2 \cos(\sqrt{3}qs)\right) \tag{2.14}$$

where c_1, c_2 are integral constants.

Proof. Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed spherical Tzitzeica curve with non-zero constant $q(s)$, $\alpha(s)$ and non-constant $p(s)$. According to equation (2.12), we have the differential equation

$$\frac{1}{q^2(s)}\left(\frac{\alpha(s)}{p(s)}\right)^{''} + 3\left(\frac{\alpha(s)}{p(s)}\right) = 0$$

On the other hand, since framed curvature $q(s)$ is constant. Therefore, we have second-order linear ordinary differential equation. The solution of this differential equation gives the equation (2.14). □

Corollary 2.8. *Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed spherical Tzitzeica curve with non-zero constant $p(s)$ and non-constant $q(s)$ and $\alpha(s)$. Then, we have a second order differential equation with variable coefficients*

$$\alpha^{''}(s)q^3(s) - c(\alpha^{'}(s))^2 + 3q^5(s)\alpha(s) = 0$$

Now, with the help of the position vectors of framed curves, the framed Tzitzeica curve are examined and the characterizations are given according to the state of the framed curves:

Proposition 2.3. *Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed curve. The position vector of γ satisfies the equation*

$$\gamma(s) = \lambda_0(s)\nu(s) + \lambda_1(s)\overline{\eta_1}(s) + \lambda_2(s)\overline{\eta_2}(s) \tag{2.15}$$

where smooth functions $\lambda_0(s), \lambda_1(s), \lambda_2(s)$. If γ is a framed Tzitzeica curve then we have

$$\begin{array}{lcl} \lambda_0^{'}(s) - p(s)\lambda_1(s) & = & \alpha(s), \\ \lambda_1^{'}(s) + p(s)\lambda_0(s) - q(s)\lambda_2(s) & = & 0, \\ \lambda_2^{'}(s) + q(s)\lambda_1(s) & = & 0, \\ q(s) & = & c\alpha(s)\lambda_2(s)^2 \end{array} \tag{2.16}$$

Proof. Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed curve. Taking the derivative of equation (2.15), since γ framed curve we get

$$\begin{array}{rcl} \gamma^{'}(s) = \alpha(s)\nu(s) & = & (\lambda_0^{'}(s) - p(s)\lambda_1(s))\nu(s) \\ & + & (\lambda_1^{'}(s) + p(s)\lambda_0(s) - q(s)\lambda_2(s))\overline{\eta_1}(s) \\ & + & (\lambda_2^{'}(s) + q(s)\lambda_1(s))\overline{\eta_2}(s). \end{array}$$

Also, since γ is the framed Tzitzeica curve, considering the equation (2.5), we get result. □

Corollary 2.9. *Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed Tzitzeica curve with $q(s) = k\alpha(s)$ and non-constant $p(s)$ where k is non-zero constant. Then γ is a framed rectifying curve with*

$$\gamma(s) = \left(\int \alpha(s)ds + c_1 \right) \nu(s) + c_2\overline{\eta_2}(s)$$

where c_1 and c_2 constants.

Proof. By using equation (2.16) and since non-zero constant $q(s) = k\alpha(s)$ where k is non-zero constant, we get

$$\begin{array}{lcl} \lambda_0(s) & = & \int \alpha(s)ds + c_1, \\ \lambda_1(s) & = & 0, \\ \lambda_2(s) & = & c_2 = constant. \end{array} \tag{2.17}$$

where c_1 and c_2 are integral constants. Therefore, by combining the equations (2.15) and (2.17), we get the result. □

Corollary 2.10. *If $q(s) = k\alpha(s)$ where k non-zero constant and $p(s)$ non-zero constant, then we get $\alpha(s) = k\alpha^{'}(s)$.*

Proof. Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed Tzitzeica curve with $q(s) = k\alpha(s)$ where k non-zero constant and $p(s)$ non-zero constant. By using equation (2.16) and (2.17), we get

$$\frac{q(s)}{p(s)} = \frac{\lambda_0}{\lambda_2} = \frac{\int \alpha(s)ds + c_1}{c_2} = \frac{k\alpha(s)}{c_2}. \tag{2.18}$$

If the derivative of both sides is taken, the desired result is obtained. □

Corollary 2.11. *From equation (2.18), $\dfrac{q(s)}{p(s)}$ is non-constant. If γ is a regular framed curve, we have $\dfrac{q(s)}{p(s)} = \dfrac{\tau(s)\alpha(s)}{\kappa(s)\alpha(s)} = \dfrac{\tau(s)}{\kappa(s)}$. If $\tau(s)$ and $\kappa(s)$ are non-zero constantsi then this is a contradiction. Consequently, there is no regular framed Tzitzeica curve with non-zero constant $\tau(s)$ and $\kappa(s)$.*

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