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SOME ROUGH HAUSDORFF LIMIT LAWS FOR SEQUENCES OF SETS

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Abstract. In this study, we observe the change of roughness degree for the rough Hausdorff convergence of a sequence, consisting of the product of a sequence of sets and a sequence of real numbers. Then we prove that the rough Hausdorff convergence is preserved under the operators of addition, union, Cartesian product and convex hull.
Key words: limit law, Hausdorff convergence, sequence of sets.

1. Introduction

The concept of convergence of sequences of sets was first introduced by Painleve in 1902 using the concepts of lower and upper limits of a sequence of sets. Kuratowski [6] mentioned this convergence in his book and contributed to the making of this theory more known. The practical difficulty of calculating a limit in this way has led to the need for the concept of distance between two sets. First, Hausdorff gave the definition of the Hausdorff distance by using the excess of a set A over a set B. The concept of Hausdorff convergence was defined using this distance. Then, the notion of Hausdorff convergence is expressed in different ways by using the equivalent definitions of this distance. On the other hand, in 1966, Wijsman [12] gave the concept of Wijsman convergence, which corresponds to the pointwise convergence of sequences created from distance functions. In 1998, Apreutesei [1] stated that the image of a Hausdorff convergent sequence in metric spaces under

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a uniform continuous function also preserves Hausdorff convergence. He [2] examined the basic properties for the classical Hausdorff limits of sequences of sets. In 2012, Nuray and Rhoades [7] combined the statistical convergence theory with the theories of Kuratowski, Hausdorff and Wijsman convergence, which are the most widely used in set theory.

In 2001, the idea of rough convergence of a sequence was first given by Phu [10] in normed linear spaces. We note that a convergent (or non-convergent) sequence can have different rough limits with a certain degree of roughness. In 2008, Aytar [3] extended this theory to the statistical convergence theory. Recently, the rough convergence theory has started to be applied in set theory as well. Ölmez and Aytar [8] applied the rough convergence theory to the theory of Wijsman convergence. Subramanian and Esi [11] extended the definition of rough Wijsman convergence to triple sequences. In addition, Esi and Subramanian [5] obtained a new type of convergence by combining the statistical convergence with this convergence.

In this study, we first explored the degree of rough convergence of the sequence consisting of the product of a sequence of sets and a sequence of real numbers (see Proposition 3.1). We analyzed this roughness degree by giving illustrative examples. Then we proved that the rough Hausdorff convergence is preserved under some operators such as addition, union, Cartesian product and convex hull.

2. Preliminaries

Throughout this paper, let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Let $P(X)$ and $K(X)$ be all nonempty subsets and nonempty compact subsets of X, respectively.

The product space $X \times Y$ is a normed space with the norm

$$\|\mathbf{x}\|_{X\times Y} = \max\{\|x_1\|_X, \|y_1\|_Y\} \text{ where } \mathbf{x} = (x_1, y_1).$$

Let $A \subset X$. For $x \in X$, the *distance* from x to the set A is defined by

$$d_X(x, A) = \inf_{a\in A} \|x - a\|_X .$$

Let $A \times B \subset X \times Y$. For $(x, y) \in X \times Y$, the *distance* from (x, y) to the set $A \times B$ is defined by

$$d_{X\times Y}((x, y), A\times B) = \inf_{\substack{a\in A\\ b\in B}} \max\{\|x - a\|_X, \|y - b\|_Y\} = \max\{d_X(x, A), d_Y(y, B)\}.$$

The set A is said to be *bounded* if $diam(A) < \infty$, where *diameter* $diam(A)$ of a nonempty set A in a normed space $(X, \|\cdot\|_X)$ is defined by

$$diam(A) = \sup_{a_1,a_2\in A} \|a_1 - a_2\|_X .$$

The convex hull of a set A denoted by $convA$. We know that the convex hull of a set A is the smallest convex set which includes A.

Let (x_n) be a sequence in the normed linear space X, and r be a nonnegative real number. Then the sequence (x_n) is said to be rough convergent to x with the roughness degree r, denoted by $x_n \xrightarrow{r} x$, if for each $\varepsilon > 0$ there exists an $n_0(\varepsilon) \in \mathbb{N}$ such that $\|x_n - x\|_X < r + \varepsilon$ for each $n \geq n_0$ [10].

Throughout this paper, we assume that $A_n \subset X$ for each $n \in \mathbb{N}$. The sequence (A_n) of sets is said to be $r-$*Hausdorff convergent* (or *rough Hausdorff convergent* with the roughness degree r) to the set A if for every $\varepsilon > 0$ there exists an $n_0(\varepsilon) \in \mathbb{N}$ such that

$$H(A_n, A) = \max\{h(A_n, A), h(A, A_n)\} < r + \varepsilon \text{ for all } n \geq n_0,$$

where $h(A_n, A) = \sup_{a \in A_n} d_X(a, A)$ and $h(A, A_n) = \sup_{a \in A} d_X(a, A_n)$. In this case, we write $A_n \xrightarrow{r-H} A$ [9].

An alternative definition of the rough Hausdorff convergence can be given by the following:

$$A_n \xrightarrow{r-H} A \iff \begin{array}{l} \text{for every } \varepsilon > 0 \text{ there exists an } n_0(\varepsilon) \in \mathbb{N} \text{ such that} \\ H(A_n, A) = \sup_{x \in X} |d(x, A_n) - d(x, A)| < r + \varepsilon \text{ for all } n \geq n_0. \end{array}$$

The following proposition plays a big role in the proofs of Propositions 3.2 and 3.5.

Proposition 2.1. ([4]) *If $A, A_1, B, B_1 \in K(\mathbb{R}^n)$, then*

1. *(i) $H(tA, tB) = tH(A, B)$ for each $t \geq 0$,*
2. *(ii) $H(A + B, A_1 + B_1) \leq H(A, A_1) + H(B, B_1)$,*
3. *(iii) $H(convA, convB) \leq H(A, B)$.*

3. Main Results

Let us begin our new results with the following proposition to answer the question of whether roughness degree of a sequence consisting of the product of a sequence of sets and a sequence of real numbers can be rough Hausdorff convergent.

Proposition 3.1. *Let $A, A_n \in K(X)$ for each $n \in \mathbb{N}$. Suppose that the sequence (A_n) satisfies the conditions:*

(i) There exists an $n_1 \in \mathbb{N}$ and an $M > 0$ such that $\sup_{v \in A_n} \|v\|_X < M$ for each $n \geq n_1$.

(ii) $A_n \xrightarrow{r_1-H} A$. If $\alpha_n \xrightarrow{r_2} \alpha$ for the sequence α_n of real numbers and $\alpha \in \mathbb{R}$, then we have $\alpha_n A_n \xrightarrow{(|\alpha| r_1 + r_2 M)-H} \alpha A$.

Proof. Let $\varepsilon > 0$. Since $\alpha_n \overset{r_2}{\longrightarrow} \alpha$, there exists an $n_2(\varepsilon) \in \mathbb{N}$ such that

$$|\alpha_n - \alpha| < r_2 + \frac{\varepsilon}{2M}$$

for each $n \geq n_2$. In addition, since $A_n \overset{r_1-H}{\longrightarrow} A$, there exists an $n_3(\varepsilon) \in \mathbb{N}$ such that

$$h(A, A_n) < r_1 + \frac{\varepsilon}{2\,|\alpha|}$$

and

$$h(A_n, A) < r_1 + \frac{\varepsilon}{2\,|\alpha|}$$

for each $n \geq n_3$. Define $n_0(\varepsilon) = \max\{n_1, n_2(\varepsilon), n_3(\varepsilon)\}$. Then we get

$$\begin{aligned} h(\alpha A, \alpha_n A_n) &\leq |\alpha|\, h(A, A_n) + |\alpha - \alpha_n| \sup_{v \in A_n} \|v\|_X \\ &< |\alpha| \left(r_1 + \frac{\varepsilon}{2\,|\alpha|}\right) + \left(r_2 + \frac{\varepsilon}{2M}\right) M = |\alpha|\, r_1 + M r_2 + \varepsilon \end{aligned}$$

and

$$\begin{aligned} h(\alpha_n A_n, \alpha A) &\leq |\alpha_n - \alpha| \sup_{v \in A_n} \|v\|_X + |\alpha|\, h(A_n, A) \\ &< \left(r_2 + \frac{\varepsilon}{2M}\right) M + |\alpha| \left(r_1 + \frac{\varepsilon}{2\,|\alpha|}\right) = r_2 M + |\alpha|\, r_1 + \varepsilon \end{aligned}$$

for each $n \geq n_0$. Consequently we have

$$\begin{aligned} H(\alpha_n A_n, \alpha A) &= \max\{h(\alpha_n A_n, \alpha A), h(\alpha A, \alpha_n A_n)\} \\ &= \max\{r_2 M + |\alpha|\, r_1 + \varepsilon, |\alpha|\, r_1 + M r_2 + \varepsilon\} = |\alpha|\, r_1 + r_2 M + \varepsilon. \end{aligned}$$

for each $n \geq n_0$. Hence we write $\alpha_n A_n \overset{(|\alpha| r_1 + r_2 M)-H}{\longrightarrow} \alpha A$, which completes the proof. □

Now let's give an illustrative example that includes different cases of the roughness degree $|\alpha|\, r_1 + r_2 M$ in the Proposition 3.1.

Example 3.1. Define the sequences

$$A_n = \left[-1 + \frac{1}{n}, \frac{1}{2}\right] \text{ and } \alpha_n = 2^{1/n} - \frac{1}{2}.$$

Case 1: Let $r_1 = 2$, $r_2 = \frac{1}{4}$, $A = \left[-1, \frac{5}{2}\right]$ and $\alpha = \frac{3}{4}$. Then we write

$$A_n \overset{r_1-H}{\longrightarrow} \left[-1, \frac{5}{2}\right] \text{ and } \alpha_n \overset{r_2}{\longrightarrow} \frac{3}{4}.$$

Take $M = 1$ and $n_1 = 1$. Hence we have

$$\begin{aligned} \sup_{v\in A_n} \|v\|_X &< 1 \text{ for each } n \geq n_1, \\ \alpha_n A_n &= \left(2^{1/n} - \frac{1}{2}\right)\left[-1 + \frac{1}{n}, \frac{1}{2}\right], \\ \alpha A &= \left[-\frac{3}{4}, \frac{15}{8}\right]. \end{aligned}$$

Since

$$\overline{r} = |\alpha|\, r_1 + r_2 M = \frac{7}{4}, \tag{3.1}$$

we have $\alpha_n A_n \xrightarrow{\overline{r}-H} \alpha A$. On the other hand, we observe $\alpha_n A_n \xrightarrow{r-H} \alpha A$ for $r = \max\left\{\left|-\frac{3}{4} + \frac{1}{2}\right|, \left|\frac{15}{8} - \frac{1}{4}\right|\right\} = \frac{13}{8}$. Here we note that $\overline{r} > r$.

Case 2: Let $n_1 = 1$, $r_1 = 2$, $r_2 = \frac{1}{4}$, $A = \left[-3, \frac{1}{2}\right]$ and $\alpha = \frac{3}{4}$. Then we write

$$A_n \xrightarrow{r_1-H} \left[-3, \frac{1}{2}\right] \text{ and } \alpha_n \xrightarrow{r_2} \frac{3}{4}.$$

Hence we get

$$\alpha A = \left[-\frac{9}{4}, \frac{3}{8}\right].$$

By (3.1), we write $\alpha_n A_n \xrightarrow{\overline{r}-H} \alpha A$. In this case, since $r = \max\left\{\left|-\frac{9}{4} + \frac{1}{2}\right|, \left|\frac{3}{8} - \frac{1}{4}\right|\right\} = \frac{7}{4}$, we have $r = \overline{r}$.

It is clear from the Case 2 that the roughness degree $|\alpha|\, r_1 + r_2 M$ cannot be decreased. In other words, although the roughness degree $\overline{r}$ in Case 1 is thought to be unnecessarily high, Case 2 says that it is not.

Remark 3.1. If we do not neglect a finite number of terms while taking supremum in the Proposition 3.1 (i), then Proposition 3.1 also holds. But in this case, the roughness degree is unnecessarily high as it can be seen following example.

Define

$$A_n = \left[-1 - \frac{100}{n}, \frac{1}{2}\right] \text{ and } \alpha_n = 2^{1/n} - \frac{1}{2}.$$

Case 1: Let $r_1 = 2$, $r_2 = \frac{1}{4}$, $A = \left[-1, \frac{5}{2}\right]$ and $\alpha = \frac{3}{4}$. Then we write

$$A_n \xrightarrow{r_1-H} \left[-1, \frac{5}{2}\right] \text{ and } \alpha_n \xrightarrow{r_2} \frac{3}{4}.$$

If we take $M = 101$ and $n_1 = 1$ then

$$\begin{aligned} \sup_{v\in A_n} \|v\|_X &< 101, \text{ for each } n \geq n_1 \\ \alpha_n A_n &= \left(2^{1/n} - \frac{1}{2}\right)\left[-1 - \frac{100}{n}, \frac{1}{2}\right] \\ \alpha A &= \left[-\frac{3}{4}, \frac{15}{8}\right]. \end{aligned}$$

Since

$$\overline{r} = |\alpha| r_1 + r_2 M = \frac{107}{4},$$

we write $\alpha_n A_n \overset{\overline{r}-H}{\longrightarrow} \alpha A$. However, this number $\overline{r}$ is unnecessarily high to make the sequence rough Hausdorff convergent.

Case 2: If we take $n_1 = 100$ then

$$\sup_{v \in A_n} \|v\|_X \leq 2, \text{ for } M = 2 \text{ and for each } n \geq 100.$$

Since

$$\overline{r}' = |\alpha| r_1 + r_2 M = 2$$

we write $\alpha_n A_n \overset{\overline{r}'-H}{\longrightarrow} \alpha A$.

If the sequence (α_n) of real numbers is ordinary convergent, that is $r_2 = 0$, in the Proposition 3.1, then we have following corollary.

Corollary 3.1. *If* $A_n \overset{r-H}{\longrightarrow} A$ *and* $\alpha_n \to \alpha$ *then we have* $\alpha_n A_n \overset{|\alpha| r-H}{\longrightarrow} \alpha A$.

Remark 3.2. We observe in Proposition 3.1 that the distance between the sequence (A_n) and the origin affects the roughness degree of the sequence $(\alpha_n A_n)$. But this distance doesn't affect the roughness degree in the Corollary 3.1.

Following proposition shows that the sum of rough Hausdorff convergent sequences is also rough Hausdorff convergent.

Proposition 3.2. *Let* $A, A_n, B, B_n \in K(X)$. *If* $A_n \overset{r-H}{\longrightarrow} A$ *and* $B_n \overset{r-H}{\longrightarrow} B$, *then* $A_n + B_n \overset{2r-H}{\longrightarrow} A + B$.

Proof. Assume that $A_n \overset{r-H}{\longrightarrow} A$ and $B_n \overset{r-H}{\longrightarrow} B$. Let $\varepsilon > 0$. Then there exist $n_1(\varepsilon), n_2(\varepsilon) \in \mathbb{N}$ such that by Proposition 2.1 (ii), we have

$$\begin{aligned} H(A_n + B_n, A + B) &\leq H(A_n, A) + H(B_n, B) \\ &\leq r + \varepsilon/2 + r + \varepsilon/r = 2r + \varepsilon \end{aligned}$$

for each $n \geq \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Hence proof is completed. □

We note that there exist sequences (A_n) and (B_n) such that $A_n \overset{r-H}{\longrightarrow} A$, $B_n \overset{r-H}{\longrightarrow} B$ and $A_n + B_n \overset{r-H}{\nrightarrow} A + B$ as can be seen following example. In other words, the roughness degree $2r$ cannot be decreased in the Proposition 3.2.

Example 3.2. Define

$$A_n = \left[2, 5 - \frac{1}{n}\right] \subset \mathbb{R} \text{ and } B_n = \left\{3 + \frac{1}{n}\right\} \subset \mathbb{R}.$$

Then we have

$$A_n \overset{r-H}{\longrightarrow} A = [2,7] \text{ and } B_n \overset{r-H}{\longrightarrow} B = \{5\}$$

for $r = 2$. On the other hand, we calculate $A_n + B_n = \left[5 + \frac{1}{n}, 8\right]$ and $A + B = [7, 12]$. Since

$$\left| d\left(15, \left[5 + \frac{1}{n}, 8\right]\right) - d(15, [7, 12]) \right| = 4$$

for $x = 15$, we get

$$H(A_n + B_n, A + B) = \sup_{x \in \mathbb{R}} |d(x, A_n + B_n) - d(x, A + B)| \not< 2 + \varepsilon, \text{ (where } 0 < \varepsilon < 2).$$

Consequently, $A_n + B_n \overset{r-H}{\nrightarrow} A + B$ for $r = 2$. In addition, we observe that $A_n + B_n \overset{2r-H}{\longrightarrow} A + B$ for $r = 2$.

Proposition 3.3. *Let $(A_n), (B_n) \subset P(X)$. If $A_n \overset{r-H}{\longrightarrow} A$ and $B_n \overset{r-H}{\longrightarrow} B$, then*

$$A_n \cup B_n \overset{r-H}{\longrightarrow} A \cup B.$$

Proof. It is clear that

$$\begin{aligned} (3.2) \qquad h(A_n \cup B_n, A \cup B) &= \sup_{x \in A_n \cup B_n} d_X(x, A \cup B) \\ &= \max\left\{ \sup_{x \in A_n} d_X(x, A \cup B), \sup_{x \in B_n} d_X(x, A \cup B) \right\} \\ &\leq \max\left\{ \sup_{x \in A_n} d_X(x, A), \sup_{x \in B_n} d_X(x, B) \right\} \\ &= \max\{h(A_n, A), h(B_n, B)\} \end{aligned}$$

and

$$(3.3) \qquad h(A \cup B, A_n \cup B_n) \leq \max\{h(A, A_n), h(B, B_n)\}$$

for each $n \in \mathbb{N}$. Let $\varepsilon > 0$. Since $A_n \overset{r-H}{\longrightarrow} A$ and $B_n \overset{r-H}{\longrightarrow} B$, there exist $n_1(\varepsilon), n_2(\varepsilon) \in \mathbb{N}$ such that

$$(3.4) \qquad h(A_n, A) < r + \varepsilon \text{ and } h(A, A_n) < r + \varepsilon \text{ for each } n \geq n_1$$

and

$$(3.5) \qquad h(B_n, B) < r + \varepsilon \text{ and } h(B, B_n) < r + \varepsilon \text{ for each } n \geq n_2$$

Define $n_0(\varepsilon) = \max\{n_1, n_2\}$. By (3.2)-(3.5), we get

$$h(A_n \cup B_n, A \cup B) \leq \max\{h(A_n, A), h(B_n, B)\} < r + \varepsilon$$

and

$$h(A \cup B, A_n \cup B_n) \leq \max\{h(A, A_n), h(B, B_n)\} < r + \varepsilon$$

for each $n \geq n_0$. Therefore we have

$$H(A_n \cup B_n, A \cup B) = \max\{h(A_n \cup B_n, A \cup B), h(A \cup B, A_n \cup B_n)\} < r + \varepsilon$$

for each $n \geq n_0$. Consequently we write $A_n \cup B_n \overset{r-H}{\longrightarrow} A \cup B$. □

Proposition 3.4. *Let $A, A_n \subset X$ and $B, B_n \subset Y$ for each $n \in \mathbb{N}$. If $A_n \xrightarrow{r-H} A$ and $B_n \xrightarrow{r-H} B$ then*

$$A_n \times B_n \xrightarrow{r-H} A \times B.$$

Proof. It is clear that

$$\begin{aligned} (3.6) \qquad h(A_n \times B_n, A \times B) &= \sup_{(a_n,b_n)\in A_n \times B_n} d_{X\times Y}((a_n,b_n), A \times B) \\ &= \max\left\{ \sup_{a_n \in A_n} d_X(a_n, A), \sup_{b_n \in B_n} d_Y(b_n, B) \right\} \\ &= \max\left\{ h(A_n, A), h(B_n, B) \right\} \end{aligned}$$

and

$$(3.7) \qquad h(A \times B, A_n \times B_n) = \max\left\{ h(A, A_n), h(B, B_n) \right\}$$

for each $n \in \mathbb{N}$. Let $\varepsilon > 0$. Since the sequences (A_n) and (B_n) are $r-$Hausdorff convergent to the sets A and B, respectively, there exist $n_1(\varepsilon), n_2(\varepsilon) \in \mathbb{N}$ such that we have

$$(3.8) \qquad h(A_n, A) < r + \varepsilon \text{ and } h(A, A_n) < r + \varepsilon \text{ for each } n \geq n_1$$

and

$$(3.9) \qquad h(B_n, B) < r + \varepsilon \text{ and } h(B, B_n) < r + \varepsilon \text{ for each } n \geq n_2.$$

Define $n_0(\varepsilon) = \max\{n_1, n_2\}$. By the facts (3.6)-(3.9), we have

$$h(A_n \times B_n, A \times B) = \max\left\{ h(A_n, A), h(B_n, B) \right\} < r + \varepsilon$$

and

$$h(A \times B, A_n \times B_n) = \max\left\{ h(A, A_n), h(B, B_n) \right\} < r + \varepsilon$$

for each $n \geq n_0$. Hence we get

$$H(A_n \times B_n, A \times B) = \max\{h(A_n \times B_n, A \times B), h(A \times B, A_n \times B_n)\} < r + \varepsilon$$

for each $n \geq n_0$, which proves that $A_n \times B_n \xrightarrow{r-H} A \times B$. □

Now, let's put an end to our work by giving the following proposition, which states that the sequences consisting of convex hulls of a rough Hausdorff convergent sequence is also rough Hausdorff convergent.

Proposition 3.5. *Let $A, A_n \in K(X)$. If $A_n \xrightarrow{r-H} A$ then $convA_n \xrightarrow{r-H} convA$.*

Proof is obvious from Proposition 2.1 (iii).

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