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THE EXTENDED SMIRNOV THEOREM FOR PSEUDONEARNESS

Dieter Leseberg[1] and Zohreh Vaziry[2]

[1]**Ernst Reuter Gesellschaft Department**
Malteserstrasse 74-100, 12249 Berlin, Germany
[2]**UAE Branch, Islamic Azad University**
Dubai, United Arab Emirates

ORCID IDs: Dieter Leseberg https://orcid.org/0000-0002-6029-0172
Zohreh Vaziry https://orcid.org/0000-0002-6496-4648

Abstract. Pseudonearness is a common extension of bornology, b-topology, pseudoproximity, and classical nearness. Furthermore, generalized contiguity, defined here as contiguous pseudonearness, can be addressed.
By employing the b-completion of a regulative contiguous pseudonear space, we obtain its b-compactification. In a special case, this represents the Hausdorff compactification of the induced Efremovic proximity space.
Keywords: pseudonearness, bornology, b-topology, pseudoproximity.

1. Basic concepts

Definition 1.1. A *pseudonearness* is defined as a pair $(\mathcal{B}^X, N)$, where $\mathcal{B}^X$ is a non-empty subset of $\underline{P}X$, the power set of a set X, and N is an operator from $\mathcal{B}^X$ into $\underline{P}(\underline{P}(\underline{P}X))$ satisfying the following conditions:

(psn$_1$) $B_1 \subset B \in \mathcal{B}^X$ implies $B_1 \in \mathcal{B}^X$;

(psn$_2$) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;

(psn$_3$) $B_1, B_2 \in \mathcal{B}^X$ implies $B_1 \cup B_2 \in \mathcal{B}^X$;

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Corresponding Author: Zohreh Vaziry
E-mail addresses: leseberg@zedat.fu-berlin.de (D. Leseberg), z_m_vaziry@yahoo.co.in (Z. Vaziry)
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We have pointed out that conditions (psn1)-(psn3) are equivalent to saying that $\mathcal{B}^X$ is a *bornology* on X, in the sense of Hogbe-Nlend [3].

(psn$_4$) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{S} \in N(B)$ implies $\{B\} \cup \mathcal{S} \in \cap\{N(F) : F \in (\mathcal{S} \cap \mathcal{B}^X) \cup \{B\}\}$ (symmetry);

(psn$_5$) $B \in \mathcal{B}^X$ implies $cl_N(B) \in \mathcal{B}^X$, where in general $cl_N(B) := \{x \in X : \{B\} \in N(\{x\})\}$ (hull-bounded);

(psn$_6$) $B \in \mathcal{B}^X$ and $\mathcal{S} \cap \mathcal{B}^X \in N(B)$, $\mathcal{S} \subset \underline{P}X$ implies $\mathcal{S} \in N(B)$ (b-absorbed);

(psn$_7$) $B \in \mathcal{B}^X$ implies $\mathcal{B}^X \notin N(B) \neq \emptyset$ (fullness);

(psn$_8$) $\mathcal{S} \in N(\emptyset)$ implies $\mathcal{S} = \emptyset$ (zero-set);

(psn$_9$) $B \in \mathcal{B}^X$ and $\mathcal{S}_1 \ll \mathcal{S} \in N(B)$ implies $\mathcal{S}_1 \in N(B)$ (corefinement), where $\mathcal{S}_1 \ll \mathcal{S}$ iff $\forall\ F_1 \in \mathcal{S}_1\ \exists\ F \in \mathcal{S}$ such that $F_1 \supset F$;

(psn$_{10}$) $B \in \mathcal{B}^X$ and $\mathcal{S}_1,\ \mathcal{S}_2 \notin N(B)$ implies $S_1 \vee S_2 \notin N(B)$ (finiteness), where $\mathcal{S}_1 \vee \mathcal{S}_2 := \{F_1 \cup F_2 : F_1 \in \mathcal{S}_1,\ F_2 \in \mathcal{S}_2\}$;

(psn$_{11}$) $x \in X$ implies $\{\{x\}\} \in N(\{x\})$ (single sets);

(psn$_{12}$) $\{cl_N(F) : F \in \mathcal{S}\} \in N(B)$, $B \in \mathcal{B}^X$ and $\mathcal{S} \subset \underline{P}X$ implies $\mathcal{S} \in N(B)$ (density).

Then we call the triple $(X, \mathcal{B}^X, N)$, where $(\mathcal{B}^X, N)$ represents a pseudonearness, *pseudonear space*.
As an intrinsic example we consider for a nearness space (X, ξ) the pseudonear space $(X, \underline{P}X, N^\xi)$, where $N^\xi : \underline{P}X \longrightarrow \underline{P}(\underline{P}(\underline{P}X))$ is defined by setting:

$$N^\xi(\emptyset) := \{\emptyset\} \text{ and for } B \in \underline{P}X \backslash \{\emptyset\};$$

$$N^\xi(B) := \{\mathcal{S} \subset \underline{P}X : \{B\} \cup \mathcal{S} \in \xi\}.$$

By **PSN** we denote the category, whose objects are the pseudonear spaces and whose morphisms are the *bibounded near maps* (in short bin-maps), where a map $f : X \longrightarrow Y$ between pseudonear spaces $(X, \mathcal{B}^X, N)$ and $(Y, \mathcal{B}^Y, M)$ is called *bin-map*, provided it fulfills the following conditions:

(b) $B \in \mathcal{B}^X$ implies $f[B] \in \mathcal{B}^Y$;

(i) $D \in \mathcal{B}^Y$ implies $f^{-1}[D] \in \mathcal{B}^X$;

(n) $B \in \mathcal{B}^X$ and $\mathcal{S} \in N(B)$ implying $\{f[F] : F \in \mathcal{S}\} =: f\mathcal{S} \in M(f[B])$.

Remark 1.1. We point it out that conditions (psn1)-(psn3)is equivalent to saying that $\mathcal{B}^X$ is a *bornology* on X, in the sense of Hogbe-Nlend [3].

Background 1.1. Firstly, already seen in [8] and [9], respectively, *sb-topology* forms a generalized symmetric topology, *pseudoproximity* forms a generalized Lodato proximity, [10] and lastly, bornology can now be interpreted as *special* case of pseudonearness. We now remind the reader of the notion of a *topoform* pseudonearness, [9] as this is closely related to some symmetric b-topology. *Symmetric b-topology* is a pair $(\mathcal{B}^X, t)$, where $\mathcal{B}^X$ is a bornology and $t : \mathcal{B}^X \longrightarrow \underline{P}X$ is an operator satisfying the following conditions:

(bt_1) $B \in \mathcal{B}^X$ implies $t(B) \in \mathcal{B}^X$;

(bt_2) $t(\emptyset) = \emptyset$;

(bt_3) $x \in X$ implies $\{x\} \cap t(\{x\}) \neq \emptyset$;

(bt_4) $B_1 \subset B \in \mathcal{B}^X$ implying $t(B_1) \subset t(B)$;

(bt_5) $B_1, B_2 \in \mathcal{B}^X$ implying $t(B_1 \cup B_2) \subset t(B_1) \cup t(B_2)$;

(bt_6) $B \in \mathcal{B}^X$ implies $t(t(B)) \subset t(B)$;

(bt_7) $x, z \in X$ and $z \in t(\{x\})$ implying $x \in t(\{z\})$.

In the case of *saturation*, meaning that $X \in \mathcal{B}^X$ and thus $\mathcal{B}^X = \underline{P}X$, the symmetric Kuratowski closure operators, [6] and its corresponding pseudonearness $(\underline{P}X, N_t)$ are *essentially* the same.
Here, $N_t : \mathcal{B}^X \longrightarrow \underline{P}(\underline{P}(\underline{P}X))$ is defined by setting:

$N_t(\emptyset) := \{\emptyset\}$, and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$

$$N_t(B) := \{\mathcal{S} \subset \underline{P}X : \cap\{t(F) : F \in (\mathcal{S} \cap \mathcal{B}^X) \cup \{B\}\} \neq \emptyset\}.$$

Then we call a pseudonearness $(\mathcal{B}^X, N)$ and its corresponding space $(X, \mathcal{B}^X, N)$ a *topoform* provided the following condition is satisfied:

(top) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{S} \in N(B)$ implying $\cap\{cl_N(F) : F \in (\mathcal{S} \cap \mathcal{B}^X) \cup \{B\})\} \neq \emptyset$.

Hence, the topoform pseudonearness and the symmetric b-topology are *essentially* the same, up to a bijection.

In what follows, we will focus on generalized nearness and contiguity, [5]. In particular, pay attention to the EF-proximity (Efremovič proximity), [11] and its corresponding *classical* counterparts like totally bounded uniformity, regular contiguity or contigual regular nearness. A *natural* generalization of those classical concepts leads to so-called RC-pseudonearness, which is simply induced by some suitable *enlarged strict topological extension.* Later we will see that the latter one is b-compact, and thus it represents the well-known "Smirnov-Theorem" for EF-proximities in the case of saturation.

Definition 1.2. For a nearness space (X, ξ), let $\mathcal{B}^X$ be a bornology. Then $\mathcal{B}^X$ is called *ξ-closed* provided it satisfies the following condition:

(ξ-clo) $B \in \mathcal{B}^X$ implies $cl_\xi(B) \in \mathcal{B}^X$.

Remark 1.2. For a nearness space X and bornology $\mathcal{B}^X$, we define $\mathcal{B}^X_\xi := \{D \subset X : \exists\, B \in \mathcal{B}^X, D \subset cl_\xi(B)\}$.

Proposition 1.1. *For a nearness space X, let $\mathcal{B}^X$ be a bornology. Then, $\mathcal{B}^X_\xi$ is a ξ- closed bornology such that $\mathcal{B}^X \subset \mathcal{B}^X_\xi$.*

Example 1.1. (i) For a nearness space (X, ξ), let $\underline{P}X$ be denote the power set of X, and let $\mathcal{F}^X$ represent the set of all finite subsets of X. Then $\underline{P}X$ and $\mathcal{F}^X_\xi$ define ξ-closed bornologies.

(ii) For a topological space (X, t), denote by $\mathcal{C}^X$ the set of all relatively compact subsets of X. Then $\mathcal{C}^X_{\xi^t}$ forms a ξ^t- closed bornology, where ξ^t represents the corresponding topological nearness.

(iii) For a uniform space $(X, \mathcal{U})$, denote by $\mathcal{T}^X$ the set of all totally bounded subsets of X. Then $\mathcal{T}^X_{\xi^{\mathcal{U}}}$ forms a $\xi^{\mathcal{U}}$-closed bornology, where $\xi^{\mathcal{U}}$ represents the corresponding uniform nearness.

(iv) For a bornology $\mathcal{B}^X$, let $\xi^b := \{\mathcal{S} \subset \underline{P}X : \cap\{F : F \in \mathcal{S} \cap \mathcal{B}^X\} \neq \emptyset\}$. Then ξ^b is a nearness such that $\mathcal{B}^X$ forms a ξ^b-closed bornology.

The following result provides a way of defining pseudonearness for any nearness ξ and every ξ-closed bornology on set X:

Proposition 1.2. *For a nearness space (X, ξ) let $\mathcal{B}^X$ be a ξ-closed bornology. Then the pair $(\mathcal{B}^X, N^\xi)$ forms a pseudonearness, where $N^\xi : \mathcal{B}^X \longrightarrow \underline{P}(\underline{P}(\underline{P}X))$ is defined by setting:*

$$N^\xi(\emptyset) := \{\emptyset\}, \text{ and for } B \in \mathcal{B}^X \backslash \{\emptyset\}$$

$$N^\xi(B) := \{\mathcal{S} \subset \underline{P}X : \{B\} \cup (\mathcal{S} \cap \mathcal{B}^X) \in \xi\}.$$

Proof. Here, we will only verify that $(\mathcal{B}^X, N^\xi)$ is hull-bounded. Let $B \in \mathcal{B}^X \backslash \{\emptyset\}$ with $x \in cl_{N^\xi}(B)$; hence $\{B\} \in N^\xi(\{x\})$, which implies that $\{\{x\}, B\} \in \xi$, and $x \in cl_\xi(B)$. Since $cl_\xi(B) \in \mathcal{B}^X$ and $cl_{N^\xi}(B)$ is a subset of $cl_{N^\xi}(B)$, then $cl_{N^\xi}(B) \in \mathcal{B}^X$. □

Now, since *separation concepts* play an important role in topology, we will give the following definition:

Definition 1.3. A pseudonearness $(\mathcal{B}^X, N)$ and its corresponding pseudonearspace $(X, \mathcal{B}^X, N)$ are called *separated* provided that $(\mathcal{B}^X, N)$ satisfies the following condition:

(sep) $x, z \in X$ and $\{\{z\}\} \in N(\{x\})$ implying $x = z$.

Remark 1.3. Note that for a separated pseudonearness $(\mathcal{B}^X, N)$, the underlying topological closure space (X, cl_N) is a T_1-space.

Example 1.2. For an N_1-space [2], let $\mathcal{B}^X$ be a ξ-closed bornology; then $(\mathcal{B}^X, N^\xi)$ is separated.

Definition 1.4. For a nearness space (X, ξ), let $\mathcal{B}^X$ be a bornology. Then, ξ is called $\mathcal{B}^X$*-sected* provided it satisfies the following condition:

(sec) $\mathcal{S} \subset \underline{P}X$ and $\mathcal{B}^X \cap \mathcal{S} \in \xi$ implying $\mathcal{S} \in \xi$.

Proposition 1.3. *For a nearness space* (X, ξ)*, let* $\mathcal{B}^X$ *be a bornology. By setting* $\hat{\xi} := \{\mathcal{S} \subset \underline{P}X : \mathcal{B}^X \cap \mathcal{S} \in \xi\}$*, then* $\hat{\xi}$ *defines a* $\mathcal{B}^X$*-sected nearness such that* $\xi \subset \hat{\xi}$*.*

Example 1.3. (i) For a nearness space (X, ξ), ξ is $\underline{P}X$-sected;

(ii) For a bornology $\mathcal{B}^X$, ξ^b is $\mathcal{B}^X$-sected;

(iii) For a pseudonear space $(X, \mathcal{B}^X, M)$, (X, η^M) is $\mathcal{B}^X$-sected nearness space, where $\eta^M := \{\mathcal{A} \subset \underline{P}X : \mathcal{A} \in \cap\{M(A) : A \in \mathcal{A} \cap \mathcal{B}^X\}\}$.

In this context, we will only verify the following condition:

$$\mathcal{S} \subset \underline{P}X \text{ and } \{cl_{\eta^M}(F) : F \in \mathcal{S}\} =: \mathcal{M} \in \eta^M \text{ implying } \mathcal{S} \in \eta^M.$$

Now, let $F \in \mathcal{S} \cap \mathcal{B}^X$; hence, $cl_{\eta^M}(F) \in \mathcal{B}^X$ follows, since $cl_{\eta^M}(F) \subset cl_M(F) \in \mathcal{B}^X$ are valid. Consequently, $cl_{\eta^M}(F) \in \mathcal{M}$ implies $\mathcal{M} \in M(cl_{\eta^M}(F))$. By applying the symmetry and density of $(\mathcal{B}^X, M)$ we obtain $\mathcal{M} \in M(cl_M(F)) = M(F)$. But then $\{cl_M(D) : D \in \mathcal{S} \cap \mathcal{B}^X\} << \mathcal{M}$, and $\mathcal{S} \cap \mathcal{B}^X \in M(F)$ follows, which shows $\mathcal{S} \in M(F)$, because of $(\mathcal{B}^X, M)$ is b-absorbed. Lastly, we mention that for every $A \in \underline{P}X$, the following equation is: $cl_M(A) = cl_{\eta^M}(A)$.

2. Nearbornologies

By combining the properties of the previously considered pairs, we will now provide the following definitions:

Definition 2.1. (i) Firstly, for a set X, let us call a pair $(\xi, \mathcal{B}^X)$ a *nearbornology* (on X), provided that (X, ξ) is nearness space, and $\mathcal{B}^X$ is bornology.

(ii) A nearbornology $(\xi, \mathcal{B}^X)$ is called *perfect* if ξ is $\mathcal{B}^X$-sected and $\mathcal{B}^X$ is ξ-closed.

Examples 2.1. (i) For a nearness space (X, ξ), let $\mathcal{B}^X$ be a bornology; then the pair $(\hat{\xi}, \mathcal{B}^X_\xi)$ is perfect.

(ii) For a bornology $\mathcal{B}^X$, the pair $(\xi^b, \mathcal{B}^X)$ is perfect;

(iii) For any pseudonear space $(X, \mathcal{B}^X, M)$, the pair $(\eta^M, \mathcal{B}^X)$ is perfect.

Now, in this context we will show that there exists an interesting correspondence between the class of all perfect nearbornologies on a set X and the class of all pseudonear structures (pseudonearness) on it which is onto and one-to-one.

Proposition 2.1. *For a set X, let $(\xi, \mathcal{B}^X)$ be a perfect nearbornology. Then the following equation holds, i.e., $\eta^{N^\xi} = \xi$.*

Proof. For " $\subset$ ": $\mathcal{A} \notin \xi$ implies $\mathcal{A} \cap \mathcal{B}^X \notin \xi$ hence $\mathcal{A} \cap \mathcal{B}^X \neq \emptyset$. So, we can choose an $A \in \mathcal{A} \cap \mathcal{B}^X$. But then $\mathcal{A} \notin N^\xi(A)$ implies $\mathcal{A} \notin \eta^{N^\xi}$.
For " $\supset$ ": If $\mathcal{A} \in \xi$, let $A \in \mathcal{A} \cap \mathcal{B}^X$. Our goal is to show $\mathcal{A} \in N^\xi(A)$, which is equivalent to $\{A\} \cup (\mathcal{A} \cap \mathcal{B}^X) \in \xi$. But $\{A\} \cup (\mathcal{A} \cap \mathcal{B}^X)$ is a subcollection of $\mathcal{A}$, and thus the claim follows. $\square$

Proposition 2.2. *For a set X, let $(\mathcal{B}^X, M)$ be a pseudonearness, then $N^{\eta^M} = M$ holds.*

Proof. For " $\leq$ ": Let $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{S} \in N^{\eta^M}(B)$, then $\{B\} \cup (\mathcal{S} \cap \mathcal{B}^X) \in \eta^M$. If supposing $\mathcal{S} \notin M(B)$, hence $\mathcal{B}^X \cap \mathcal{S} \notin M(B)$ (b-absorbed), which implies $\{B\} \cup (\mathcal{S} \cap \mathcal{B}^X) \notin M(B)$. Since $\{B\} \cup (\mathcal{S} \cap \mathcal{B}^X) \in \eta^M$ implies $\{B\} \cup (\mathcal{S} \cap \mathcal{B}^X) \in M(B)$, $\mathcal{S} \cap \mathcal{B}^X \in M(B)$ results, which contradicts. Thus $\mathcal{S} \in M(B)$ follows.
For " $\geq$ ": Conversely, let $\mathcal{S} \in M(B)$. Our goal is $\mathcal{S} \in N^{\eta^M}(B)$, which means $\{B\} \cup (\mathcal{S} \cap \mathcal{B}^X) \in \eta^M$. So let $F \in (\{B\} \cup (\mathcal{S} \cap \mathcal{B}^X)) \cap \mathcal{B}^X = (\{B\} \cap \mathcal{B}^X) \cup ((\mathcal{S} \cap \mathcal{B}^X) \cap \mathcal{B}^X) = \{B\} \cup (\mathcal{S} \cap \mathcal{B}^X)$.
In the first case $F = B$ we obtain $\{B\} \cup \mathcal{S} \in M(F)$ by the symmetry, and $\{B\} \cup (\mathcal{S} \cap \mathcal{B}^X) \in M(F)$ follows. In the second case $F \in \mathcal{S} \cap \mathcal{B}^X$, $\{B\} \cup \mathcal{S} \in M(F)$ is valid by the symmetry, and $\{B\} \cup (\mathcal{S} \cap \mathcal{B}^X) \in M(F)$ results, which has to be shown. $\square$

Theorem 2.1. *There exists a natural correspondence between the class of all perfect nearbornologies on a set X and the class of all pseudonear structures on X which is onto and one-to-one.*

Proof. By applying the former assignments. $\square$

Definition 2.2. A pseudonearness $(\mathcal{B}^X, N)$ and its corresponding space $(X, \mathcal{B}^X, N)$ are called *saturated* provided that $X \in \mathcal{B}^X$ is holding, and thus $\mathcal{B}^X = \underline{P}X$. By **SAT-PSN** we denote the full subcategory of **PSN**, whose objects are saturated.

As an corollary we can now state the following theorem:

Theorem 2.2. *The category **NEAR** of nearness spaces and nearness preserving maps is isomorphic to **SAT-PSN**.*

Proof. Especially, we note that maps between nearness spaces are n-maps, [2] iff they are bin-maps between the corresponding saturated pseudonear spaces. $\square$

Proposition 2.3. *For a contigual nearness space* (X, ξ), *[2] let* $\mathcal{B}^X$ *be a* ξ*-closed bornology. Then the pair* $(\mathcal{B}^X, N^\xi)$ *forms a contiguous pseudonearness, [8].*

Remark 2.1. In reminding the property of being contiguous we recall its definition once more.

Definition 2.3. A pseudonearness $(\mathcal{B}^X, N)$ and its corresponding space $(X, \mathcal{B}^X, N)$ are called *contiguous* provided they are satisfying the following condition:

(ctg) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{S} \notin N(B)$ implying the existence of a finite collection $\mathcal{E} \subset \{B\} \cup \mathcal{S}$ with $\mathcal{E} \notin N(B)$.

Remark 2.2. If we denote by **C-PSN** the full subcategory of **PSN**, whose objects are contiguous we similar orientate according to a former definition by **SATC-PSN** its full subcategory, whose objects are saturated.

In this context we further note that every finite bornoform pseudonearness, [9] is contiguous and separated as well.

Proposition 2.4. *For a saturated contiguous pseudonear space* $(X, \mathcal{B}^X, M)$, (X, η^M) *is contigual nearness space.*

Proof. $\mathcal{A} \notin \eta^M$ implies $\mathcal{A} \notin M(F)$ for some $A \in \mathcal{A}$. By the supposition we can find a finite collection $\mathcal{E} \subset \{A\} \cup \mathcal{A}$ with $\mathcal{E} \notin M(A)$. Consequently, $\{A\} \cup \mathcal{E} \notin M(A)$ follows. By setting $\mathcal{E}_1 := \{A\} \cup \mathcal{E}$ we resume that $\mathcal{E}_1 \subset \mathcal{A}$ is finite with $\mathcal{E}_1 \notin \eta^M$ which concludes the proof. □

Theorem 2.3. *The category **CONT** of contiguity spaces and related maps is isomorphic to **SATC-PSN**.*

Proof. By applying former results. □

It is already known that the relation between regular extensions and the induced nearness structures is sufficiently intimate to provide a powerful tool and a considerable number of useful results, [1] and see also later. But here we will establish a more general setting in the realm of pseudonearness. At first, let us recall the definition of a nearness space for being regular, [2].

Definition 2.4. A N_1-space (X, ξ) is said to be *regular*, provided that for every $\mathcal{S} \subset \underline{P}X$, whenever the collection $\mathcal{S}^\xi := \{D \subset X : \exists\, F \in \mathcal{S} \text{ such that } F <_\xi D\} \in \xi$ then also $\mathcal{S} \in \xi$, where $F <_\xi D$ iff $\{F, X \backslash D\} \notin \xi$.

Remark 2.3. As proposed in [1], a topological space is regular in the nearness sense iff it is regular in the topological sense.

Now, a similar condition for pseudonearness will be given at next.

Definition 2.5. A separated pseudonearness $(\mathcal{B}^X, M)$ and its corresponding space $(X, \mathcal{B}^X, M)$ are said to be *regulative* provided that the following condition must hold, i.e.

(reg) $\mathcal{S} \subset \underline{P}X$ and $\mathcal{S} \notin \eta^M$ implying $\mathcal{S}^M := \{D \subset X : \exists\ F \in \mathcal{S},\ \{F, X \backslash D\} \notin \eta^M\} \notin \eta^M$.

Remark 2.4. In this context we call a nearbornology $(\xi, \mathcal{B}^X)$ *perfect regular* provided $(\xi, \mathcal{B}^X)$ is perfect and ξ is regular. Consequently, for a pseudonear space $(X, \mathcal{B}^X, M)$ the following statements are equivalent:

(i) $(\mathcal{B}^X, M)$ is regulative;

(ii) There exists a nearness ξ on X such that $(\xi, \mathcal{B}^X)$ is perfect regular with $M = N^\xi$.

Thus we conclude, a nearness space (X, ξ) is regular iff $(X, \underline{P}X, N^\xi)$ is regulative iff $(\xi, \underline{P}X)$ is perfect regular.
Now, obviously we denote by **R-PSN** the full subcategory of **PSN**, whose objects are regulative and in addition by **SATR-PSN** the full subcategory of its saturated objects.
At last we should still mention that any pseudonear space $(X, \mathcal{B}^X, M)$, whose underling nearness η_M is Hausdorff compact is already regulative.

Theorem 2.4. *The category* ***R-NEAR*** *of regular nearness spaces and n-maps is isomorphic to* ***SATR-PSN****.*

Proof. By applying former results. □

It is well-known that the category **EF-PROX** of Efremovič proximity spaces is isomorphic to **tb-UNIF**, the category of totally bounded uniform spaces, which is also isomorphic to **C-UNEAR**, the category of contigual uniform nearness spaces, [1].
Moreover, in [1] the authors have shown that the category **R-CONT** of regular contiguity spaces is isomorphic to **CR-NEAR**, the category of contigual regular nearness spaces. But the latter one is isomorphic to **EF-PROX**. These interesting results give us now the motivation for introducing the following definitions and notations, respectively.

Definition 2.6. A regulative contiguous pseudonear space is said to be an *RC-space* in short, and we denote by **RC-PSN** the corresponding full subcategory of **PSN**. Nearby, **SATRC-PSN** defines its full subcategory of the saturated objects.

Proposition 2.5. *For a regular contiguity space (X, γ) let $\mathcal{B}^X$ be a bornology such that $(\xi^\gamma, \mathcal{B}^X)$ is perfect, where ξ^γ denotes the corresponding nearness of γ. Then the triple $(X, \mathcal{B}^X, N^{\xi^\gamma})$ forms an RC-space.*

Proposition 2.6. *For a saturated RC-space $(X, \mathcal{B}^X, M)$, (X, η^M) is contigual and regular nearness space.*

Theorem 2.5. *The categories **R-CONT** and **SATRC-PSN** are isomorphic.*

Proof. By applying latter results. □

Remark 2.5. Now we point out that by formerly stated isomorphy of categories, any EF-proximity space can be considered as an saturated RC-space, and vice versa. Therefore, *ordinary* RC-spaces constitute a natural *generalization* of EF-proximity spaces.

In the following part, we will see that the *b-completion* of an RC-space is well-behaved in such a manner that it is especially regulative contiguous and its induced pseudonear space coincide with the *source space*, and furthermore that *strict btop-extensions* inducing RC-spaces, too, are necessary *equiform.* In this context *b-compactness* comes into play, so that in the *special* case of *saturation* Smirnov's famous theorem for EF-proximity spaces, [11] possesses a corresponding counterpart in the theory of pseudonearness.
With respect to extensions of topological spaces, their subspaces are of *fundamental* interest. So we will first consider some suitable basics.

Definition 2.7. A pseudonear space $(X, \mathcal{B}^X, N)$ is called *pseudonear subspace*, in short *psn-subspace* of a pseudonear space $(Y, \mathcal{B}^Y, M)$, provided X is subset of Y, $\mathcal{B}^X$ is subset of $\mathcal{B}^Y$ and for each $\mathcal{S} \subset \underline{P}X$ the following conditions must hold, i.e.

(sub$_1$) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $\mathcal{S} \in N(B)$ iff $\mathcal{S} \in M(B)$;

(sub$_2$) $\mathcal{B}^Y \cap \mathcal{S} \subset \mathcal{B}^X$.

Remark 2.6. If $(X, \mathcal{B}^X, N)$ is psn-subspace of $(Y, \mathcal{B}^Y, M)$ then (X, η^N) is nearness subspace of (Y, η^M). Note, for $\mathcal{S} \subset \underline{P}X$ and $\mathcal{S} \in \eta^N$ we have $\mathcal{S} \in \cap\{N(F) : F \in \mathcal{S} \cap \mathcal{B}^X\}$. Thus for $F_1 \in \mathcal{S} \cap \mathcal{B}^Y$, $F_1 \in \mathcal{B}^X$ is valid by (sub$_2$), and $\mathcal{S} \in N(F_1)$ implies $\mathcal{S} \in M(F_1)$ by (sub$_1$), hence $\mathcal{S} \in \eta^M$ follows. Conversely let $\mathcal{S} \in \eta^M$ and $F \in \mathcal{S} \cap \mathcal{B}^X$. But $F \in \mathcal{S} \cap \mathcal{B}^Y$ implies $\mathcal{S} \in M(F)$, and thus $\mathcal{S} \in N(F)$ by applying (sub$_1$), which shows the claim. Now, it is important to note, that in the saturated case with $X \in \mathcal{B}^X$ and $Y \in \mathcal{B}^Y$, (X, ξ) is nearness subspace of (Y, η) iff $(X, \mathcal{B}^X, N^\xi)$ is psn-subspace of $(Y, \mathcal{B}^Y, N^\eta)$. In fact, for $\mathcal{S} \subset \underline{P}X$ we have $\mathcal{B}^Y \cap \mathcal{S} \subset \underline{P}Y \cap \mathcal{S} = \mathcal{S} \subset \underline{P}X = \mathcal{B}^X$. All the other conditions are clear.

Proposition 2.7. *Any psn-subspace $(X, \mathcal{B}^X, S)$ of a regulative pseudonear space $(Y, \mathcal{B}^Y, M)$ is regulative.*

Proof. By using theorem 2.1, remark 2.4 and remark 2.6, respectively. □

Proposition 2.8. *Any psn-subspace of a contiguous pseudonear space is contiguous.*

Proof. Let $(X, \mathcal{B}^X, S)$ be psn-subspace of the contiguous pseudonear space $(Y, \mathcal{B}^Y, M)$. For $B \in \mathcal{B}^X \backslash \{\emptyset\}$ let $\mathcal{S} \notin S(B)$, $\mathcal{S} \subset \underline{P}X$, hence $\mathcal{S} \notin M(B)$ implies the existence of a finite collection $\mathcal{E} \subset \{B\} \cup (\mathcal{S} \cap \mathcal{B}^Y)$ with $\mathcal{E} \notin M(B)$. $\{B\} \cup (\mathcal{S} \cap \mathcal{B}^Y) \subset \{B\} \cup (\mathcal{S} \cap \mathcal{B}^X)$ by using (sub2). But $\mathcal{E} \notin N(B)$ by (sub$_1$) shows the claim. □

Corollary 2.1. *Any psn-subspace of an RC-space is RC-space.*

Remark 2.7. Evidently, any psn-subspace of a separated pseudonear space is separated as well.

Another important separation axiom comes into play whenever one is considering Hausdorff spaces.

Definition 2.8. We call a separated pseudonearness $(\mathcal{B}^X, M)$ and its corresponding space $(X, \mathcal{B}^X, M)$ *star-separated* in short S_2*-space* provided they are satisfying the following condition:

(S_2) For every $\mathcal{S} \subset \underline{P}X$, whenever $\mathcal{S}, \mathfrak{sec}\mathcal{S} \in \eta^M$, the collection $\mathcal{S}^{(M)} := \{D \subset X : \{D\} \cup \mathcal{S} \in \eta^M\} \in \eta^M$, where $\mathfrak{sec}\mathcal{S} := \{A \subset X : \forall\ F \in \mathcal{S}\ A \cap F \neq \emptyset\}$.

Proposition 2.9. *For an Hausdorff nearness space (X, ξ), [1] and a bornology $\mathcal{B}^X$ let $(\xi, \mathcal{B}^X)$ be perfect. Then the pseudonearness $(\mathcal{B}^X, N^\xi)$ is star-separated.*

Proof. $(\mathcal{B}^X, N^\xi)$ is separated, since (X, ξ) is N_1-space by the hypothesis. Now, let for $\mathcal{S} \subset \underline{P}X$, $\mathcal{S}, \mathfrak{sec}\mathcal{S} \in \eta^{N^\xi}$, hence by proposition 2.1. $\mathcal{S}, \mathfrak{sec}\mathcal{S} \in \xi$ follows and the collection $\mathcal{V} := \{B \subset X : \{B\} \cup \mathcal{S} \in \xi\}$ is ξ-near, because by the hypothesis (X, ξ) is Hausdorff nearness space. But then $\mathcal{V} \in \eta^{N^\xi}$ is valid with $\mathcal{S}^{(N^\xi)} \subset \mathcal{V}$. Consequently $\mathcal{S}^{N^\xi} \in \eta^{N^\xi}$ results, which is proving the claim. □

Remark 2.8. In the *opposite* direction, (X, η^M) is Hausdorff nearness space, whenever $(X, \mathcal{B}^X, M)$ is saturated $\mathcal{S}_2$-space. Thus, a topological space is an Hausdorff space in the topological sense iff its associated saturated topoform pseudonear space is an S_2-space (Compare with, [9] and background 1.1.).

Proposition 2.10. *Any regulative pseudonear space is an S_2-space.*

Proof. Let for $\mathcal{S} \subset \underline{P}X$ $\mathcal{S}, \mathfrak{sec}\mathcal{S}$ being elements of η^M. To show that $\mathcal{S}^{(M)} \in \eta^M$ it suffices, by using the property regulative, to verify that $\mathcal{A} := \{A \subset X : \exists\ D \in \mathcal{S}^{(M)}\ \{D, X\backslash A\} \notin \eta^M\}$ is an element of η^M. Therefore it is enough to prove $\mathcal{A} \cap \mathcal{B}^X \in \eta^M$. But this follows from the fact that $\mathcal{A} \cap \mathcal{B}^X$ is subset of $\mathfrak{sec}\mathcal{S} \cap \mathcal{B}^X$. Indeed, for $A \in \mathcal{A} \cap \mathcal{B}^X$ and $F \in \mathcal{S}$ we have $A \in \mathcal{B}^X$ and $\{D, X\backslash A\} \notin \eta^M$ for some $D \in \mathcal{S}^{(M)}$. Hence $\{D\} \cup \mathcal{S} \in \eta^M$ follows by the definition. If supposing $A \cap F = \emptyset$, $X\backslash A \supset F$ implying $\{D, X\backslash A\} << \{D, F\} \subset \{D\} \cup \mathcal{S}$, and consequently $\{D, X\backslash A\} \in \eta^M$ results, which contradicts. But then, by taking all into account, the claim immediately follows. □

Proposition 2.11. *Any psn-subspace of an S_2- space is S_2-space.*

Proof. Let $(X, \mathcal{B}^X, S)$ be psn-subspace of the S_2-space $(Y, \mathcal{B}^Y, M)$ and let $\mathcal{S}, \mathfrak{sec}\mathcal{S} \in \eta^S$. We must show $\mathcal{S}^{(S)} := \{D \subset X : \{D\} \cup \mathcal{S} \in \eta^S\} \in \eta^S$.
$\mathcal{S}, \mathfrak{sec}\mathcal{S} \in \eta^M$ by applying remark 2.6., and $\mathcal{S}^{(M)} := \{A \subset Y : \{A\} \cup \mathcal{S} \in \eta^M\} \in \eta^M$ follows by the hypothesis.
Our goal is $\mathcal{S}^{(S)} \subset \mathcal{S}^{(M)}$. $D \in \mathcal{S}^{(S)}$ implies $\{D\} \cup \mathcal{S} \in \eta^S$, hence $\{D\} \cup \mathcal{S} \in \eta^M$, and $D \in \mathcal{S}^{(M)}$ follows. Thus $\mathcal{S}^{(S)} \in \eta^M$ implies $\mathcal{S}^{(S)} \in \eta^S$, which has to be shown. □

Next, let us focus our attention to the extension of certain topological structures. An intrinsic example of that is the Hausdorff-completion of a separated uniform space. We will consider a similar concept for pseudonearness, where in a special case the *Herrlich-completion* of a nearness space as well as the above mentioned Hausdorff-completion can be recovered. As already known, minimal Cauchy filters, and near clusters are closely related to each other and both were used for constructing the prevailing completion. But here we will consider a more general framework for pseudonearness, which in the saturated case coincide with the classical ones.

3. The b-completion

Definition 3.1. For a pseudonear space $(X, \mathcal{B}^X, N)$, $\tau \subset \underline{P}X$ is called *N-tape in* $\mathcal{B}^X$ provided it satisfies the following conditions:

(tp_1) $\tau \in \underline{P}\mathcal{B}^X \cap N(B)\backslash\{\emptyset\}$ for some $B \in \mathcal{B}^X\backslash\{\emptyset\}$;

(tp_2) $A \in \mathcal{B}^X$ and $\{A\} \cup \tau \in N(D)$, $D \in \mathcal{B}^X\backslash\{\emptyset\}$ implying $A \in \tau$.

Remark 3.1. As already seen in [8], for a nearness space (X, ξ) and a collection $\mathcal{C} \subset \underline{P}X$ the statements of being $\mathcal{C}$ is ξ-cluster and $\mathcal{C}$ is N^{ξ}-tape in $\underline{P}X$ are equivalent. Moreover, for a pseudonear space $(X, \mathcal{B}^X, N)$ and for each $x \in X$ $\tau_x^N := \{A \in \mathcal{B}^X : \{A\} \in N(\{x\})\}$ is an N-tape in $\mathcal{B}^X$.

Definition 3.2. A pseudonear space $(X, \mathcal{B}^X, N)$ is called *b-complete*, provided $(\mathcal{B}^X, N)$ satisfies the following condition:

(b-cpl) $\forall\ \tau \subset \underline{P}X$ N-tape in $\mathcal{B}^X$ $\exists\ x \in X$ such that $\{x\} \in \tau$.

Remark 3.2. According to the definition of completeness in a nearness space, [2] we point out that in the saturated case the terms b-complete and complete coincide. Further, we note that every non-empty finite pseudonear space is already b-complete. In addition we infer that a uniform space is complete as uniform space iff its associated saturated pseudonear space is b-complete [2]. At last, we note that every bornoform pseudonear space as well as every topoform pseudonear space is b-complete.

Theorem 3.1. *Let $(X, \mathcal{B}^X, N)$ be a pseudonear space. Then we consider the triple $(X^*, \mathcal{B}^{X^*}, N^*)$, where $X^* := \{\tau \subset \underline{P}X : \tau$ is N-tape in $\mathcal{B}^X\}$, $\mathcal{B}^{X^*} := \{B^* \subset X^* : \exists\ D \in \mathcal{B}^X\ \forall\ \tau \in B^*\ \tau \in N(D)\}$, and $N^* : \mathcal{B}^{X^*} \longrightarrow \underline{P}(\underline{P}(\underline{P}X^*))$ is defined by setting $N^*(\emptyset) := \{\emptyset\}$, and for each $B^* \in \mathcal{B}^{X^*}\backslash\{\emptyset\}$ we put: $N^*(B^*) := \{\mathcal{A}^* \subset \underline{P}X^* : \exists\ B \in \mathcal{B}^X\backslash\{\emptyset\}\ \{F \in \mathcal{B}^X : \exists\ A^* \in (\mathcal{A}^* \cap \mathcal{B}^{X^*}) \cup \{B^*\}\ F \in \Delta A^*\} \in N(B)\}$, where for $A^* \subset X^* \Delta A^* := \{A \in \mathcal{B}^X : \forall\ \tau \in A^*, A \in \tau\}$. Then $(X^*, \mathcal{B}^{X^*}, N^*)$ is a separated b-complete pseudonear space such that $cl_{N^*}(j[X]) = X^*$, where $j : X \longrightarrow X^*$ denotes that function which assigns the N-tape τ_x^N to each $x \in X$. $j : (X, \mathcal{B}^X, N) \longrightarrow (X^*, \mathcal{B}^{X^*}, N^*)$ is bin-map, and for each $B \in \mathcal{B}^X\backslash\{\emptyset\}$ and $\mathcal{A} \subset \underline{P}X$ the statements $\mathcal{A} \in N(B)$ and $j\mathcal{A} \in N^*(j[B])$ are equivalent.*

Proof. For the argumentation the reader is referred to [8]. □

Lemma 3.1. *For a pseudonear space $(X, \mathcal{B}^X, N)$ each successive pair of conditions are equivalent:*

(i) *j is injective;*

(ii) *$(X, \mathcal{B}^X, N)$ is separated.*

(iii) *j is surjective;*

(iv) *$(X, \mathcal{B}^X, N)$ is b-complete.*

Remark 3.3. Now, it is interesting to note that in the case whenever $(X, \mathcal{B}^X, N)$ is a saturated pseudonear space, in other words representing a nearness space, the space $(X^*, \mathcal{B}^{X^*}, N^*)$ is saturated, too. Thus, Herrlich's completion of a nearness space can be interpreted as *special* case. Up to now the spaces in question will be separated or T_1-spaces or N_1-spaces respectively with non-empty underlying carrier set. As a consequence we note that there is no need to distinguish, for any subset A of X, between A and $j[A]$. Therefore, $(X, \mathcal{B}^X, N)$ can be considered as psn-subspace of $(X^*, \mathcal{B}^{X^*}, N^*)$. Indeed, it remains to verify that for $\mathcal{A} \subset \underline{P}X$, $\mathcal{B}^{X^*} \cap j\mathcal{A} \subset \mathcal{B}_{X^*}$, where $\mathcal{B}_{X^*} := \{j[B] : B \in \mathcal{B}^X\}$. $B^* \in \mathcal{B}^{X^*} \cap j\mathcal{A}$ implies $B^* = j[A]$ for some $A \in \mathcal{A}$, and there exists $D \in \mathcal{B}^X$ $\forall\ \tau \in B^*, \tau \in N(D)$. Our goal is $B^* \subset j[cl_N(D)]$. $\tau \in B^*$ implies $\tau \in j[A]$ for some $A \in \mathcal{A}$. Hence $\tau = j(x)$ for some $x \in A$, and $j(x) \in N(D)$ follows. By the symmetry $\{D\} \cup j(x) \in N(D)$ is valid, and $D \in j(x)$, since $j(x)$ is N-tape in $\mathcal{B}^X$. Consequently, $\{D\} \in N(\{x\})$ implies $x \in cl_N(D)$, and $\tau = j(x) \in j[cl_N(D)]$ results, which shows the claim.

Proposition 3.1. *For a pseudonear space $(X, \mathcal{B}^X, N)$ the following statements are equivalent:*

(i) *$(X, \mathcal{B}^X, N)$ is contiguous;*

(ii) *$(X^*, \mathcal{B}^{X^*}, N^*)$ is contiguous.*

Proof. See [8]. □

Theorem 3.2. *The b-completion $(X^*, \mathcal{B}^{X^*}, M^*)$ of a regulative pseudonear space $(X, \mathcal{B}^X, M)$ is regulative.*

Proof. First observe, that whenever F, D are subsets of X then $\{F, X\backslash D\} \notin \eta^M$ implies $\{cl_{\eta^{M^*}}(F), X^*\backslash cl_{\eta^{M^*}}(D)\} \notin \eta^{M^*}$.
Now let $\mathcal{A}^* \subset \underline{P}X^*$ such that ${\mathcal{A}^*}^{\mathcal{M}^*} := \{D^* \subset X^* : \exists\ A^* \in \mathcal{A}^*\ \{A^*, X\backslash D^*\} \notin \eta^{M^*}\} \in \eta^{M^*}$.
Our goal is $\mathcal{A}^* \in \eta^{M^*}$. We put $\mathcal{S}_{\mathcal{A}^*} := \{G \subset X : \exists\ A^* \in \mathcal{A}^*, A^* \subset cl_{\eta^{M^*}}(G)\}$, $\mathcal{S}_{{\mathcal{A}^*}^{M^*}} := \{G \subset X : \exists\ D^* \in {\mathcal{A}^*}^{M^*}, D^* \subset cl_{\eta^{M^*}}(G)\}$ and define $\mathcal{A} := \{A \subset X : \exists\ G \in \mathcal{S}_{\mathcal{A}^*}\{G, X\backslash A\} \notin \eta^M\}$. Thus $\mathcal{S}_{{\mathcal{A}^*}^{M^*}} \in \eta^M$ by the definition, and $\mathcal{A} \in \eta^M$, since by the above observation $\mathcal{A} \subset \mathcal{S}_{{\mathcal{A}^*}^{M^*}}$ is valid. By the regulativity, $\mathcal{S}_{\mathcal{A}^*} \in \eta^M$ follows, and so $\mathcal{A}^* \in \eta^{M^*}$ results, which shows the claim. □

Corollary 3.1. *The b-completion of an RC-space is an RC-space.*

Proof. According to proposition 2.8., proposition 3.1. and theorem 3.2., respectively. □

Theorem 3.3. *The b-completion of an S_2-space $(X, \mathcal{B}^X, M)$ is an S_2-space.*

Proof. Let $\mathcal{A}^*$, $\mathfrak{sec}\mathcal{A}^* \in \eta^{M^*}$. Our goal is $\mathcal{A}^{*(M^*)} := \{D^* \subset X^* : \{D^*\} \cup \mathcal{A}^* \in \eta^{M^*}\} \in \eta^{M^*}$. By the hypothesis $\mathcal{A} := \{A \subset X : \exists\ A^* \in \mathcal{A}^*$ such that $A^* \subset cl_{\eta^{M^*}}(j[A])\} \in \eta^M$, and for $\mathcal{S} := \{F \subset X : \exists\, A^* \in \mathcal{A}^*$ such that $A^* \cap cl_{\eta^{M^*}}(j[X \backslash F]\} = \emptyset\}$, $\mathfrak{sec}\mathcal{S} \in \eta^M$ follows. Obviously $\mathcal{S} \subset \mathcal{A}$ implies $\mathfrak{sec}\mathcal{A} \in \eta^M$. Since $(X, \mathcal{B}^X, M)$ is S_2-space the collection $\mathcal{D} := \{D \subset X : \{D\} \cup \mathcal{A} \in \eta^M\}$ is η^M-near. Now, it is sufficient to show that $\mathcal{H} := \{H \subset X : \exists\ D^* \in \mathcal{A}^{*(M^*)}, D^* \subset cl_{\eta^{M^*}}(j[H])\} \in \eta^M$. But this follows immediately from $\mathcal{H} \subset \mathcal{D}$. □

Now, we will turn out focus on the *compactness* of spaces. At first, for a pseudonearness, we introduce the *fundamental* property of being b-hullsected.

Definition 3.3. We call a pseudonearness $(\mathcal{B}^X, N)$ and its corresponding space $(X, \mathcal{B}^X, N)$ *b-hullsected* provided they are satisfying the following condition:

(bhsc) $\forall\, \mathcal{S} \in \underline{P}\mathcal{B}^X$ with $\cap\{cl_N(F) : F \in \mathcal{S}\} = \emptyset\ \exists\, \mathcal{S}_\circ \subset \mathcal{S}$ finite such that $\cap\{cl_N(A) : A \in \mathcal{S}_\circ\} = \emptyset$.

Remark 3.4. Evidently, every finite pseudonear space is b-hullsected.

Proposition 3.2. *For a topoform pseudonear space $(X, \mathcal{B}^X, N)$ the following statements are equivalent:*

(i) $(\mathcal{B}^X, N)$ *is b-hullsected;*

(ii) $(\mathcal{B}^X, N)$ *is contiguous.*

Proof. See [8]. □

By transforming this result to nearness spaces we can now infer that for a topological nearness space the properties of being contigual and compact are essentially the same.

Motivated by the obtained result, we are giving the following *intrinsic* definition:

Definition 3.4. We call a pseudonearness $(\mathcal{B}^X, N)$ and the corresponding space $(X, \mathcal{B}^X, N)$ *b-compact* provided $(\mathcal{B}^X, N)$ is topoform and b-hullsected.

Remark 3.5. According to remark 3.4. we note that any finite topoform pseudonear space is b-compact. Furthermore, a nearness space (X, ξ) is compact iff the pseudonear space $(X, \underline{P}X, N^\xi)$ is b-compact.

In this context another property comes into play.

Definition 3.5. We call a pseudonearness $(\mathcal{B}^X, N)$ and the corresponding space $(X, \mathcal{B}^X, N)$ *precede* provided it satisfies the following condition:

(pc) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{S} \in N(B)$ with $\mathcal{S} \cap \mathcal{B}^X \neq \emptyset$ implying the existence of an N-tape τ in $\mathcal{B}^X$ such that $\mathcal{S} \cap \mathcal{B}^X \subset \tau$.

Remark 3.6. Here we point out that, in the case of saturation, precede pseudonear spaces and concrete nearness spaces are *essentially* the same.

Example 3.1. Every contiguous pseudonear space is precede.

Proof. See in particular [9]. □

Lemma 3.2. *For a precede pseudonear space* $(X, \mathcal{B}^X, M)$ *its b-completion* $(X^*, \mathcal{B}^{X^*}, M^*)$ *is topoform.*

Proof. For $B^* \in \mathcal{B}^{X^*} \backslash \{\emptyset\}$ let $A^* \in M^*(B^*)$. Hence, there exists $B \in \mathcal{B}^X \backslash \{\emptyset\}$ such that $\mathcal{V} := \{F \in \mathcal{B}^X : F \in \Delta A^* \text{ for some } A^* \in (\mathcal{A}^* \cap \mathcal{B}^{X^*}) \cup \{B^*\}\} \in M(B)$. By the symmetry, $\{B\} \cup \mathcal{V} \in M(B)$ implies $(\{B\} \cup \mathcal{V}) \cap \mathcal{B}^X \neq \emptyset$. Hence, by the hypothesis, we can find an N-tape τ in $\mathcal{B}^X$ with $(\{B\} \cup \mathcal{V}) \cap \mathcal{B}^X \subset \tau$. Since $\Delta B^* \cup \tau \subset \tau$ and $\Delta A^* \cup \tau \subset \tau$ for $A^* \in \mathcal{A}^* \cap \mathcal{B}^{X^*}$ are valid, we obtain the desired result. □

Definition 3.6. (i) A pseudonear space $(Y, \mathcal{B}^Y, M)$ is called *b-compactification* of a pseudonear space $(X, \mathcal{B}^X, N)$, provided that $(Y, \mathcal{B}^Y, M)$ is b-compact and $(X, \mathcal{B}^X, N)$ is pseudonear subspace of $(Y, \mathcal{B}^Y, M)$ with $cl_M(X) = Y$.
(ii) A b-compactification $(Y, \mathcal{B}^Y, M)$ of a pseudonear space $(X, \mathcal{B}^X, N)$ is called *strict*, provided $\forall\, D \subset Y, D = cl_M(D)$ and $\forall\, y \notin D, \exists\, F \in \mathcal{B}^X$ such that $y \notin cl_M(F)$ and $D \subset cl_M(F)$.
(iii) For a pseudonear space $(X, \mathcal{B}^X, N)$, $\mathcal{C} \subset \underline{P}X$ is called an *unit*(in $(\mathcal{B}^X, N)$), provided $\mathcal{C}$ satisfies the following conditions:

(ut_1) $\mathcal{C} \in \underline{P}\mathcal{B}^X \cap N(B)$ for some $B \in \mathcal{B}^X \backslash \{\emptyset\}$;

(ut_2) $B \in \mathcal{C}$;

(ut_3) $B_1 \supset D \in \mathcal{C}$, $B_1 \in \mathcal{B}^X$ implies $B_1 \in \mathcal{C}$;

(ut_4) $B_1, B_2 \notin \mathcal{C}$, $B_1, B_2 \in \mathcal{B}^X$ implying $B_1 \cup B_2 \notin \mathcal{C}$;

(ut_5) $cl_N(D) \in \mathcal{C}$, $D \in \mathcal{B}^X$ implies $D \in \mathcal{C}$.

Remark 3.7. First, we note that every N-tape in $\mathcal{B}^X$ forms an unit in $(\mathcal{B}^X, N)$. And for a nearness ξ, the following statements are equivalent:

(i) $\mathcal{C} \subset \underline{P}X$ is a ξ-bunch;
(ii) $\mathcal{C} \subset \underline{P}X$ is an unit in $(\underline{P}X, N^{\xi})$.

Theorem 3.4. *Every contiguous pseudonear space has a strict b-compactification.*

Proof. Let $(X, \mathcal{B}^X, M)$ be a contiguous pseudonear space. Then, by example 3.1., it is precede. Thus, the b-completion $(X^*, \mathcal{B}^{X^*}, M^*)$ is topoform by applying lemma 3.2. Furthermore, it is contiguous by using proposition 3.1. But then, the b-completion is b-compact according to proposition 3.2. with $(X, \mathcal{B}^X, M)$ being a pseudonear subspace of $(X^*, \mathcal{B}^{X^*}, M^*)$. Note that there is no need to distinguish, for a subset $A \subset X$, between A and $j[A]$. It remains to prove that $(X^*, \mathcal{B}^{X^*}, M^*)$ is strict. Now, consider $A^* \subset X^*$ being closed with $\tau \notin A^*$. Then $\tau \notin cl_{M^*}(A^*)$ implies $\{A^*\} \notin M^*(\{\tau\})$. On the other hand, $\tau \in \underline{P}\mathcal{B}^X \cap M(B)$ for some $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $\mathcal{V} := \{A \in \mathcal{B}^X : \exists\ D^* \in (A^* \cap \mathcal{B}^{X^*}) \cup \{\{\tau\}\}, A \in \Delta A^*\} \notin M(B)$, hence $\Delta A^* \cup \tau \nsubseteq \tau$. Otherwise, since $\mathcal{V} \subset \Delta A^* \cup \tau$ is valid, we get a contradiction. Consequently, we can find $F \in \Delta A^* \cup \tau$ with $F \notin \tau$, hence $F \in \Delta A^*$ follows. Our goal is to verify

(1) $\tau \notin cl_{M^*}(j[F])$ and

(2) $A^* \subset cl_{M^*}(j[F])$.

For (1): If $\tau \in cl_{M^*}(j[F])$, then $\{j[F]\} \in M^*(\{\tau\})$ implies the existence of $D \in \mathcal{B}^X \backslash \{\emptyset\}$ such that $m := \{M \in \mathcal{B}^X : \exists\ D^* \in \{j[F]\} \cup \{\{\tau\}\}, M \in \Delta D^*\} \in M(D)$. Hence $\Delta j[F] \cup \tau \subset m$ follows because $A \in \Delta j[F] \cup \tau$ implies $A \in \Delta j[F]$ or $A \in \tau = \Delta\{\tau\}$. In both cases $A \in m$ results, thus $\Delta j[F] \cup \tau \in M(D)$ is valid. But $F \in \Delta j[F] \cup \tau$ implies $\{F\} \cup \tau \in M(D)$ and $F \in \tau$ results, since τ satisfies (tp_2). But this contradicts.
For (2): For $\mathcal{D} \in A^*$ we have $F \in \mathcal{D}$, hence $\Delta j[F] \subset \mathcal{D}$. Note that $A \in \Delta j[F]$ implies $F \subset cl_M(A)$, and $cl_M(A) \in \mathcal{D}$ implies $A \in \mathcal{D}$. Observe that $\mathcal{D}$ is an unit in $(\mathcal{B}^X, M)$ according to remark 3.7. Thus, $\mathcal{D} \in cl_{M^*}(j[F])$. □

Corollary 3.2. *Any RC-space has a strict b-compactification which in addition is regulative.*

Proof. By applying Corollary 3.1. and Theorem 3.4., respectively. □

Remark 3.8. In the case of *saturation* the above considered b-compactification represents the Hausdorff compactification of its corresponding EF-proximity space.

By this famous theorem of Smirnov, [11], every Hausdorff compactification can be recovered from its induced EF-proximity relation. Hence, with any EF-proximity relation one can associate, via the corresponding Hausdorff compactification, a regular contiguity space which constitutes a saturated RC-space in our sense up to isomorphism. In this context we also note that for a saturated b-compact pseudonear space the properties of being regulative and being an S_2-space are equivalent by applying remark 2.8. Moreover, we can state that the above mentioned strict construct is the *representative* of a certain *equivalence class of extensions*. But in the following section this will be more precisely explained .

In relation to the above, we should mention that S. Leader in 1967, [7] has introduced the concept of local proximity spaces. He defines a *local proximity space* by using proximity and boundedness as primitive terms. It turns out that EF-proximity relations are the special case for which all subsets are bounded. In this context, a locally compact Hausdorff space Y can be reconstructed from a dense subspace X if one is knowing not only which pairs of sets are *close* but also which subsets are *bounded*, that is, having compact closures in Y. Then a local compactification can be constructed for a local proximity space X generalizing the Smirnov compactification for EF-proximity spaces.

4. Strict bornotopological extensions

Closely related to the *canonical* construction which embeds each pseudonear space into a b-complete pseudonear space, we introduce the notion of a so-called *bornotopological extension*. It turns out that this concept is convenient for studying strict topological extensions. A main result is that we obtain a natural *correspondence* between equivalence classes of *strict bornotopological extensions* and precede pseudonear spaces which is onto and one- to-one. In the case of saturated contiguous pseudonear spaces, we recover a *classical* result obtained by Ivanova and Ivanov, [5] in the past.

Definition 4.1. A *bornotopological extension* (in short *btop-extension*) consists of a triple $(e, \mathcal{B}^X, Y)$, where $X := (X, t_X)$, $Y := (Y, t_Y)$ are topological spaces (given by closure operators t_X and t_Y respectively), $\mathcal{B}^X$ is bornology such that $B \in \mathcal{B}^X$ implies $t_X(B) \in \mathcal{B}^X$ and $e : X \longrightarrow Y$ is an injective map satisfying the following conditions:

(btx$_1$) $B \in \mathcal{B}^X$ implies $t_X(B) = e^{-1}[t_Y(e[B])$, where e^{-1} denoted the *inverse* image under e;

(btx$_2$) $t_Y(e[X]) = Y$, which means the image of X under e is *dense* in Y.

Remark 4.1. Note, that if $\mathcal{B}^X$ is saturated, the above description and that of a topological extension in the usual sense coincide, [1].

Lemma 4.1. *For a bornotopological extension $(e, \mathcal{B}^X, Y)$, $(\mathcal{B}^X, N_e)$ is separated pseudonearness, where $N_e(\emptyset) := \{\emptyset\}$ and $N_e(B) := \{\mathcal{S} \subset \underline{P}X : \cap\{t_Y(e(F])), F \in (\mathcal{S} \cap \mathcal{B}^X) \cup \{B\} \neq \emptyset\}$ if $B \in \mathcal{B}^X \backslash \{\emptyset\}$, such that the triple $(X, \mathcal{B}^X, N_e)$ defines a separated pseudonear space with $cl_{N_e}(B) = t_X(B)\ \forall\ B \in \mathcal{B}^X$.*

Definition 4.2. (i) btop-extensions $(e, \mathcal{B}^X, Y)$, $(e', \mathcal{B}^X, Y')$ are called *isovalent* provided that there exists a bijective map $h : Y \longrightarrow Y'$ with $h \circ e = e'$ such that $\forall\ D \in \mathcal{B}^X\ \forall\ y \in Y, y \in t_Y(e[D])$ iff $h(y) \in t_{Y'}(e'[D])$;
(ii) *equiform*, provided that $N_e = N_{e'}$ is holding;
(iii) $(e, \mathcal{B}^X, Y)$ is called *strict*, provided $\forall\ D \subset Y$, $D = t_Y(D)$, $\forall\ y \notin D\ \exists\ F \in \mathcal{B}^X$ such that $y \notin t_Y(e[F])$ and $D \subset t_Y(e[F])$, (compare with definition 3.6.(ii)).

(iv) For a pseudonear space $(X, \mathcal{B}^X, M)$ we say that the pseudonearness $(\mathcal{B}^X, M)$ is *induced by a btop-extension*, provided that there exists a btop-extension $(e, \mathcal{B}^X, Y)$ such that $M = N_e$. In that case we also cited that $(e, \mathcal{B}^X, Y)$ is inducing $(\mathcal{B}^X, M)$.

Remark 4.2. Here, we note that if $\mathcal{B}^X$ is saturated, strict topological extensions and strict btop-extensions are *essentially* the same. Furthermore, we note that any separated topoform pseudonearness $(\mathcal{B}^X, M)$ is induced by $(id_X, \mathcal{B}^X, X)$ with $id_X : X \longrightarrow X$ denoting the identity and $X := (X, cl_M)$. Additionally, for a separated topoform pseudonearness $(\mathcal{B}^X, M)$ we conclude that the btop-extensions $(id_X, \mathcal{B}^X, X)$ and $(j, \mathcal{B}^X, X^*)$ are isovalent by applying remark 3.2. and lemma 3.1., respectively. And finally, we infer that isovalent btop-extensions are equiform. In fact, let $(e, \mathcal{B}^X, Y)$, $(e', \mathcal{B}^X, Y')$ being isovalent btop-extensions. We denote by $h : Y \longrightarrow Y'$ the existing bijective map with its corresponding property.
For $B \in \mathcal{B}^X \setminus \{\emptyset\}$, let $\mathcal{S} \in N_e(B)$ then by the definition of N_e, $\cap\{t_Y(e[F]) | F \in (\mathcal{S} \cap \mathcal{B}^X) \cup \{B\}\} \neq \emptyset$. Choose $y \in Y$ such that for $A \in (\mathcal{S} \cap \mathcal{B}^X) \cup \{B\}$, $y \in t_Y(e[A])$. Hence, $h(y) \in t_{Y'}(e'[A])$ follows by applying the hypothesis. Consequently, $\mathcal{S} \in N_{e'}(B)$ results immediately. Conversely, we use the inverse function of h.

Lemma 4.2. *Let $(X, \mathcal{B}^X, M)$ be a pseudonearness space induced by a strict btop-extension. Then, $(X, \mathcal{B}^X, M)$ is precede (compare with Definition 3.5.)*

Proof. See [8]. □

Proposition 4.1. *For a pseudonear space $(X, \mathcal{B}^X, M)$, let us denote by $(X^*, \mathcal{B}^{X^*}, M^*)$ its corresponding b-completion. Then, for every $\tau \in X^*$ and $D \in \mathcal{B}^X$ the following statements are equivalent:*

(i) $\tau \in cl_{M^*}(j[D])$;

(ii) $D \in \tau$.

Proof. See [8]. □

Lemma 4.3. *For a pseudonear space $(X, \mathcal{B}^X, M)$ let its b-completion $(X^*, \mathcal{B}^{X^*}, M^*)$ be topoform. Then, $(j, \mathcal{B}^X, X^*) =: E$ is a strict btop-extension such that $(X, \mathcal{B}^X, M)$ is induced by E.*

Proof. Here, E consists of $X := (X, cl_M)$, $X^* := (X^*, cl_{M^*})$ and $j : X \longrightarrow X^*$ as the canonical embedding. For the strictness condition see proof of the theorem 3.4. Evidently, E satisfies the conditions (btx_1) and (btx_2) in definition 4.1. Thus, we have to verify that M equals N_j (see lemma 4.1.).
For "$M \leq N_j$": For $B \in \mathcal{B}^X \setminus \{\emptyset\}$ let $\mathcal{S} \in M(B)$, hence by theorem 3.1. $j\mathcal{S} \in M^*(j[B])$. By the hypothesis we can find $\tau \in cl_{M^*}(j[B])$, $\tau \in \cap cl_{M^*}(A) : A \in j\mathcal{S} \cap \mathcal{B}^{X^*}\}$. Now, let $F \in \mathcal{S} \cap \mathcal{B}^X$, then $j[F] \in j\mathcal{S} \cap \mathcal{B}^{X^*}$ follows, and $\tau \in cl_{M^*}(j[F])$ results. On the other hand, $\tau \in cl_{M^*}(j[B])$ closes this part of the proof.
For "$N_j \leq M$": Conversely, let $\mathcal{S} \in N_j(B)$, hence we can find $\tau \in cl_{M^*}(j[B])$

with $\tau \in \cap\{cl_{M^*}(j[F]) : F \in \mathcal{S} \cap \mathcal{B}^X\}$. By applying proposition 4.1., $B \in \tau$ and $\tau \in N(D)$ are valid for some $D \in \mathcal{B}^X \backslash \{\emptyset\}$. By the symmetry, $\{D\} \cup \tau \in M(B)$ implies $\tau \in M(B)$. But $F \in \mathcal{S} \cap \mathcal{B}^X$ implies $\tau \in cl_{M^*}(j[F])$ and, by proposition 4.1., $F \in \tau$ results, showing that $\mathcal{S} \cap \mathcal{B}^X \in M(B)$ is valid. Consequently, $\mathcal{S} \in M(B)$ since $(\mathcal{B}^X, M)$ is b-absorbed. □

Theorem 4.1. *For any pseudonear space $(X, \mathcal{B}^X, M)$ the following conditions are equivalent:*

(i) $(\mathcal{B}^X, M)$ is a pseudonearness induced by a strict btop-extension;

(ii) The b-completion $(X^*, \mathcal{B}^{X^*}, M^*)$ of $(X, \mathcal{B}^X, M)$ is topoform;

(iii) $(X, \mathcal{B}^X, M)$ is a precede pseudonear space.

Proof. By applying the previous results in corollary 3.1., lemma 4.2. and lemma 4.3., respectively. □

Corollary 4.1. *If $(\mathcal{B}^X, M)$ is the pseudonearness induced by a strict btop-extension $(e, \mathcal{B}^X, Y)$, then $(e, \mathcal{B}^X, Y)$ and $(j, \mathcal{B}^X, X^*)$ are equiform.*

Proof. By the hypothesis, we get $N_e = M = N_j$. □

Proposition 4.2. *Let strict btop-extensions $(e, \mathcal{B}^X, Y)$, $(e', \mathcal{B}^X, Y')$ be equiform such that $\mathcal{B}^X$ is saturated. Then, $(e, \mathcal{B}^X, Y)$, $(e', \mathcal{B}^X, Y')$ are isovalent.*

Proof. By the hypothesis, $N_e = N_{e'}$. Hence $N_e = N_j$ with $(j, \mathcal{B}^X, X^*)$, where $(X^*, \mathcal{B}^{X^*}, X^*)$ denotes the b-completion of $(X, \mathcal{B}^X, N_e)$. We define a map $h : Y \longrightarrow X^*$ by setting for each $y \in Y$, $h(y) := \tau^y := \{D \in \mathcal{B}^X : y \in t_Y(e[D])\}$. τ^y is an N_e-tape in $\mathcal{B}^X$ since $\tau \in \underline{P}X \cap N_e(X) \backslash \{\emptyset\}$ is valid and $\{y\} \in \tau^y$ by applying strictness.
Further note that $\mathcal{B}^X$ is saturated. Now, let $\{A\} \cup \tau^y \in N_e(B)$ for some $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $A \in \mathcal{B}^X$. Then $\{A, \{y\}\} \cup \tau^y \in N_e(B)$ holds, and by applying the symmetry, we get $\{B\} \cup (\{A, \{y\}\} \cup \tau^y) \in N_e(\{y\})$. Hence, $\{A\} \in N_e(\{y\})$ implies $y \in t_Y(e[A])$, and thus $A \in \tau^y$. Moreover, h is bijective and satisfies the condition in definition 4.2.(i). Thus, $h : Y \longrightarrow X^*$ is a homeomorphism and $Y \widetilde{h} X^*$ results. Analogously, we obtain $Y' \widetilde{h} X^*$. Hence, $Y \widetilde{h} Y'$ is valid, and the claim follows. □

Corollary 4.2. *For strict btop-extensions $(e, \mathcal{B}^X, Y)$, $(e', \mathcal{B}^X, Y')$ such that $\mathcal{B}^X$ is saturated, the following statements are equivalent:*

(i) There exists a homeomorphism $h : Y \longrightarrow Y'$ with $h \circ e = e'$;

(ii) $(e, \mathcal{B}^X, Y)$, $(e', \mathcal{B}^X, Y')$ are equiform.

Remark 4.3. By applying theorem 3.4. and corollary 4.1., respectively, we can now state that every separated contiguous pseudonear space has a strict b-compactification. Vice versa, each strict btop-extension inducing such a kind of space implies that this one and that of its strict b-compactification are equiform. That immediately implies a natural correspondence between the class of all separated contiguous pseudonear spaces and the class of equivalence classes of equiform strict btop-extensions inducing such a kind of spaces which is onto and one-to-one. Here, we point out that, in the *saturated* case, a corresponding result has already been published by Bentley and Herrlich, [1]. But according to theorem 3.4., our result also represents a generalization of a corresponding Theorem for contiguity spaces and bicompact extensions in [5]. In addition take into account that, in this case, a strict b-compactification consists of a strict btop-extension $(e, \mathcal{B}^X, Y)$ such that Y is a compact topological space, see also remark 4.2.

Remark 4.4. By applying corollary 3.2., and corollary 4.1., respectively, we can now state that every RC-space has a strict b-compactification. Vice versa, each strict btop-extension inducing such a kind of space implies that this one and that of its strict b-compactification are equiform. This immediately implies a *natural* correspondence between the class of all RC-spaces and the class of equivalence classes of equiform strict btop-extensions inducing such a kind of spaces which is onto and one-to-one. Thus our result also represents a *generalization* of the famous Theorem of Smirnov, compared to remark 3.8. With respect to a comparative Theorem presented by Lodato, [10] the reader is referred to [9].

REFERENCES

1. H. L. Bentley, H. Herrlich: *Extensions of topological spaces.* Topol. Proc. Memphis State Univ. Conf. **1975**(1976), 129-184.
2. H. Herrlich: *A concept of nearness.* Gen. Topol. Appl. **4**(1974), 191-212.
3. H. Hogbe-Nlend: *Bornologies and Functional Analysis.* North- Holland, Amsterdam (1977).
4. M. Hušek: *Categorical connections between generalized proximity spaces and compactifications.* Contr. Extension Theory, Symp. Berlin **1967**(1969), 127-132.
5. V. M. Ivanova, A. A. Ivanov: *Contiguity spaces and bicompact extensions.* Dokl. Akad. Nauk SSSR **127**(1959), 20-22.
6. C. Kuratowski: *Sur l' opération á de l' analysis situs.* Fund. Math. **3**(1922), 182-199.
7. S. Leader: *Local proximity spaces.* Math. Annalen **169**(1967), 275-281.
8. D. Leseberg and Z. Vaziry: *Bounded Topology.* LAP Acad. Pub. (2019).
9. D. Leseberg and Z. Vaziry: *On bornological induced pseudonearness.* Math. Appl. **9**(2020), 95-121.
10. M. W. Lodato: *On topological induced generalized proximity relations II.* Pacific journal of Math Vol. **17**, No. 1(1966), 131-135.
11. Y. M. Smirnov: *On completeness of proximity spaces.* Dokl. Acad. Nauk. SSSR **88**(1953), 761-764.