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ON BOUNDEDNESS WITH SPEED λ IN ULTRAMETRIC FIELDS

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Abstract. In the present paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. Entries of sequences, infinite series and infinite matrices are in K. Following Kangro [2, 3, 4], we introduce the concept of boundedness with speed λ or λ-boundedness. We then obtain a characterization of the matrix class (m^λ, m^μ), where m^λ denotes the set of all λ-bounded sequences in K. We conclude the paper with a remark about the matrix class (c^λ, m^μ), where c^λ denotes the set of all λ-convergent sequences in K.
Key words: Ultrametic (or non-archimedean) field, boundedness with speed λ (or λ-boundedness), λ-bounded by the matrix A or A^λ-bounded, matrix class (m^λ, m^μ), matrix class (c^λ, m^μ).

1. Introduction and Preliminaries

Throughout this paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. Sequences, infinite series and infinite matrices have entries in K. In this paper, we suppose that indices and summation indices run from 0 to ∞ unless otherwise stated. For a given sequence $x = \{x_k\}$ in K, an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \ldots$, we define

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, n = 0, 1, 2, \ldots,$$

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where it is assumed that the series on the right converge. $A(x) = \{(Ax)_n\}$ is called the A-transform of the sequence $x = \{x_k\}$.

If X, Y are sequence spaces, we write

$$A = (a_{nk}) \in (X, Y),$$

if $\{(Ax)_n\} \in Y$, whenever $x = \{x_k\} \in X$. In the sequel, m, c respectively denote the ultrametric Banach spaces of bounded and convergent sequences.

The following results are well-known.

Theorem 1.1. *$A = (a_{nk}) \in (m, m)$ if and only if*

$$\sup_{n,k} |a_{nk}| < \infty. \tag{1.1}$$

Theorem 1.2. [5] *$A = (a_{nk}) \in (m, c)$ if and only if*

$$\lim_{k\to\infty} a_{nk} = 0, n = 0, 1, 2, \ldots; \tag{1.2}$$

and

$$\lim_{n\to\infty} \sup_{k\geq 0} |a_{n+1,k} - a_{nk}| = 0. \tag{1.3}$$

2. Boundedness with speed λ (or λ-boundedness), λ-boundedness by the matrix A (or A^λ-boundedness), characterization of the matrix class (m^λ, m^μ)

Definition 2.1. Let $\lambda = \{\lambda_n\}$ be a sequence in K such that

$$0 < |\lambda_n| \nearrow \infty, n \to \infty.$$

A sequence $x = \{x_k\}$ is said to be bounded with speed λ or λ-bounded if $x = \{x_k\} \in c$ with $\lim_{k\to\infty} x_k = s$ and $\{\lambda_n(x_n - s)\}$ is bounded.

Let m^λ denote the set of all λ-bounded sequences in K. Note that $m^\lambda \subset c$.

Definition 2.2. A sequence $x = \{x_k\}$ in K is said to be λ-bounded by the matrix A or A^λ-bounded if

$$A(x) = \{(Ax)_n\} \in m^\lambda.$$

The set of all A^λ-bounded sequences is denoted by m_A^λ. Here again, we note that

$$m_A^\lambda \subset c_A,$$

where c_A denotes the convergence field of A.

In the sequel, for each $k = 0, 1, 2, \ldots$, let

$$e_k = \{0, 0, \ldots, 0, 1, 0, \ldots\},$$

1 occurring in the kth place and 0 elsewhere, i.e., $e_k = \{e_k^j\}_{j=0}^{\infty}$, where

$$e_k^j = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k; \end{cases}$$

and

$$e = \{1, 1, 1, \ldots\}.$$

Let $\mu = \{\mu_n\}$ be a sequence in K such that

$$0 < |\mu_n| \nearrow \infty, n \to \infty.$$

We now have the following characterization of the matrix class (m^λ, m^μ).

Theorem 2.1. *Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (m^\lambda, m^\mu)$ if and only if*

$$\lim_{n\to\infty} a_{nk} = a_k, k = 0, 1, 2, \ldots; \tag{2.1}$$

$$A(e) \in m^\mu; \tag{2.2}$$

$$\lim_{k\to\infty} \frac{a_{nk}}{\lambda_k} = 0, n = 0, 1, 2, \ldots; \tag{2.3}$$

$$\lim_{n\to\infty} \left(\sup_{k\geq 0} \left| \frac{a_{n+1,k} - a_{nk}}{\lambda_k} \right| \right) = 0; \tag{2.4}$$

and

$$\sup_{n,k} \left| \frac{\mu_n (a_{nk} - a_k)}{\lambda_k} \right| < \infty. \tag{2.5}$$

Proof. Necessity. Let $A = (a_{nk}) \in (m^\lambda, m^\mu)$. Note that for $k = 0, 1, 2, \ldots$, $e_k \in m^\lambda$ and so $A(e_k) \in m^\mu$. Thus $A(e_k) \in c$.
Consequently,

$$\lim_{n\to\infty} a_{nk} = a_k,\ k = 0, 1, 2, \ldots,\ \text{i.e., (2.1) holds.}$$

We again note that $e \in m^\lambda$ and so

$$A(e) \in m^\mu, \quad \text{i.e., (2.2) holds.}$$

Let, now, $x = \{x_k\} \in m^\lambda$. Hence $x = \{x_k\} \in c$. Let $\lim_{k\to\infty} x_k = s$. Let

$$\beta_k = \lambda_k(x_k - s), k = 0, 1, 2, \ldots.$$

Then $\{\beta_k\} \in m$. Now,

$$\begin{aligned}(Ax)_n &= \sum_{k=0}^{\infty} a_{nk}x_k \\ &= \sum_{k=0}^{\infty} a_{nk}\left(\frac{\beta_k}{\lambda_k} + s\right) \\ &= \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k}\beta_k + s\sum_{k=0}^{\infty} a_{nk}. \qquad (2.6)\end{aligned}$$

In view of (2.2),

$$\left\{\sum_{k=0}^{\infty} a_{nk}\right\}_{n=0}^{\infty} \in m^\mu$$

and so

$$\left\{\sum_{k=0}^{\infty} a_{nk}\right\}_{n=0}^{\infty} \in c.$$

Thus

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = a \text{ (say)}. \qquad (2.7)$$

Since $\{(Ax)_n\} \in c$ and $\{\beta_k\} \in m$, using (2.6) and (2.7), it follows that the infinite matrix

$$\left(\frac{a_{nk}}{\lambda_k}\right) \in (m, c).$$

Consequently, (2.3) and (2.4) hold, using Theorem 1.2. By hypothesis, $\{(Ax)_n\} \in m^\mu$ and so $\{(Ax)_n\} \in c$. Let $\lim_{n\to\infty}(Ax)_n = y$. Now,

$$\begin{aligned}y &= \lim_{n\to\infty}(Ax)_n \\ &= \lim_{n\to\infty}\left(\sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k}\beta_k + s\sum_{k=0}^{\infty} a_{nk}\right) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k}\beta_k + sa, \text{ using (2.4) and (2.7).} \qquad (2.8)\end{aligned}$$

In view of (2.6) and (2.8), we have,

$$(Ax)_n - y = \sum_{k=0}^{\infty} \frac{a_{nk} - a_k}{\lambda_k}\beta_k + s\left(\sum_{k=0}^{\infty} a_{nk} - a\right).$$

Hence

$$\mu_n[(Ax)_n - y] = \sum_{k=0}^{\infty} \frac{\mu_n(a_{nk}-a_k)}{\lambda_k}\beta_k + s\mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right). \tag{2.9}$$

Since $\{(Ax)_n\}$, $A(e) \in m^{\mu}$,

$$\{\mu_n[(Ax)_n - y]\}, \left\{\mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right)\right\} \in m.$$

Already $\{\beta_k\} \in m$. Thus, the infinite matrix

$$\left(\frac{\mu_n(a_{nk}-a_k)}{\lambda_k}\right) \in (m, m).$$

Using Theorem 1.1,

$$\sup_{n,k}\left|\frac{\mu_n(a_{nk}-a_k)}{\lambda_k}\right| < \infty, \ i.e., (2.5) \text{ holds.}$$

Sufficiency. Let (2.1) - (2.5) hold. Then, using (2.2), (2.7) holds. Let $x = \{x_k\} \in m^{\lambda}$, $\lim_{k\to\infty} x_k = s$, $\beta_k = \lambda_k(x_k - s)$. Then $\{\beta_k\} \in m$. Using (2.3) and (2.4), the infinite matrix

$$\left(\frac{a_{nk}}{\lambda_k}\right) \in (m, c).$$

Using (2.6) and (2.7), it now follows that $\{(Ax)_n\} \in c$. Let

$$\lim_{n\to\infty}(Ax)_n = y.$$

So (2.8) and (2.9) hold.
In view of (2.5), the infinite matrix

$$\left(\frac{\mu_n(a_{nk}-a_k)}{\lambda_k}\right) \in (m, m).$$

Since $\{\beta_k\} \in m$,

$$\left\{\sum_{k=0}^{\infty} \frac{\mu_n(a_{nk}-a_k)}{\lambda_k}\beta_k\right\}_{n=0}^{\infty} \in m.$$

Using (2.2),

$$\left\{\mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right)\right\}_{n=0}^{\infty} \in m.$$

In view of (2.9),

$$\{\mu_n[(Ax)_n - y]\}_{n=0}^{\infty} \in m.$$

Consequently,

$$\{(Ax)_n\} \in m^{\mu}.$$

This completes the proof of the theorem. □

Definition 2.3. We say that an infinite matrix $A = (a_{nk})$ preserves λ-boundedness if $A \in (m^{\lambda}, m^{\lambda})$.

Definition 2.4. An infinite matrix $A = (a_{nk})$ is said to be regular if $A \in (c, c)$ and $\lim_{n\to\infty} (Ax)_n = \lim_{k\to\infty} x_k$, $x = \{x_k\} \in c$.

The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let $A = (a_{nk})$ be a regular matrix. Then A preserves λ-boundedness if and only if*

$$\sup_{n,k} \left| \frac{\lambda_n a_{nk}}{\lambda_k} \right| < \infty. \tag{2.10}$$

Definition 2.5. [8] A sequence $\{x_k\}$ in $K = Q_p$, the p-adic field for a prime p, is said to be Y-summable to ℓ if

$$\frac{x_n + x_{n-1}}{2} \to \ell, n \to \infty.$$

Note that the Y-method is defined by the infinite matrix $A = (a_{nk})$, where,

$$a_{nk} = \begin{cases} \frac{1}{2}, & \text{if } k = n-1, n; \\ = 0, & \text{otherwise.} \end{cases}$$

It is easy to check that the Y-method is regular. In addition, using (2.10), we can easily check that the Y-method preserves λ-boundedness if and only if

$$\left\{ \frac{\lambda_n}{\lambda_{n-1}} \right\} \in m.$$

For instance, choose $\lambda_n = \frac{1}{p^n}$, $n = 0, 1, 2, \ldots$ in Q_p. Then

$$0 < |\lambda_n|_p = \frac{1}{|p|_p^n} \nearrow \infty, n \to \infty,$$

where $|\cdot|_p$ is the p-adic valuation. Now,

$$\left| \frac{\lambda_n}{\lambda_{n-1}} \right|_p = \left| \frac{1/p^n}{1/p^{n-1}} \right|_p = \frac{1}{|p|_p}, n = 0, 1, 2, \ldots,$$

so that

$$\left\{\frac{\lambda_n}{\lambda_{n-1}}\right\} \in m.$$

Consequently, the Y-method preserves λ-boundedness for the above choice of $\lambda = \{\lambda_n\}$.

For the sake of completeness, we recall the following definition from [7]. Let, as usual, $\lambda = \{\lambda_n\}$ be a sequence in K such that

$$0 < |\lambda_n| \nearrow \infty, n \to \infty.$$

Definition 2.6. A sequence $\{x_n\}$ in K is said to be convergent with speed λ or λ-convergent if $\{x_n\} \in c$ with $\lim_{n\to\infty} x_n = s$ and

$$\lim_{n\to\infty} \lambda_n(x_n - s) \text{ exists.}$$

Let c^λ denote the set of all λ-convergent sequences in K. By definition,

$$c^\lambda \subset m^\lambda \subset c.$$

We now have the following result, the proof of which is very similar to the proof of Theorem 2.1.

Theorem 2.3. *$A = a_{nk} \in (c^\lambda, m^\mu)$ if and only if $A \in (m^\lambda, m^\mu)$. In other words, $A \in (c^\lambda, m^\mu)$ if and only if (2.1) – (2.5) are satisfied.*

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