FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 4 (2024), 579–590
https://doi.org/10.22190/FUMI211101039K
Original Scientific Paper

Δ^f −LACUNARY STATISTICAL CONVERGENCE OF ORDER β

Mithat Kasap[1] **and Hifsi Altinok**[2]

[1] **Department of Accounting, Sirnak University, 73000 Sirnak, Turkey**
[2] **Department of Mathematics, Firat University, 23119 Elazig, Turkey**

ORCID IDs: Mithat Kasap https://orcid.org/0000-0002-2064-2823
Hifsi Altinok https://orcid.org/0000-0001-7836-8946

Abstract. The main object of this article is to introduce the concepts of Δ^f −lacunary statistical convergence of order β and strong Δ^f −lacunary summability of order β for sequences of fuzzy numbers and define some sequence classes related to these concepts. We give some inclusion relations between those sequence classes.
Keywords: Δ^f −lacunary statistical convergence, strong Δ^f −lacunary summability.

1. Introduction

In 1951, Steinhaus [31] and Fast [13] introduced the concept of statistical convergence and later in 1959, Schoenberg [30] reintroduced independently. Connor [7], Çakallı [8], Çınar *et al.* [9], Et *et al.* ([11]), Fridy [15], Işık [19], Salat [29] and many others investigated some arguments related to this notion.

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan [17] and after then statistical convergence of order α was studied by Çolak [10]. Later, the concept of statistical convergence of order β for fuzzy sequences defined by Altinok *et al* [2].

Aizpuru *et al.* [1] defined the f−density of the subset A of $\mathbb{N}$ by using an unbounded modulus function. After then, Bhardwaj and Dhawan [5] introduced f−statistical convergence of order α with respect to a modulus function f for real sequences and later studied lacunary statistical convergence [4]. Sengül and Et [32]

Received November 01, 2021, revised: October 23, 2022, accepted: April 06, 2024
Communicated by Jelena Ignjatović
Corresponding Author: Hifsi Altinok. E-mail addresses: fdd_mithat@hotmail.com (M. Kasap), hifsialtinok@gmail.com (H. Altinok)
2020 *Mathematics Subject Classification.* Primary 40A05; Secondary 40C05,46A45

introduced the concepts of $f-$lacunary statistical convergence of order α and strong $f-$lacunary summability of order α of sequences of real number.

The idea of fuzzy numbers was developed and studied by Zadeh [33] as an extension of the concept of classical (crisp) set. Matloka [23] applied this idea in the theory of sequence space and summability and proved some fundamental theorems related to sequences of fuzzy numbers. Nanda [26] used this theory in topology and vector spaces by helping a fuzzy metric. This idea was applied in scientific areas such as Linguistic and numerical modeling, computer programming, fuzzy optimization, summability theory, etc. ([6],[18],[24]). Later, the notion of statistical convergence for sequences of fuzzy numbers was defined and studied by Nuray and Savaş [27].

Kızmaz [21] introduced the difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, consisting of all real valued sequences $x=(x_k)$ such that $\Delta x=\Delta^1 x=(x_k-x_{k+1})$ in the sequence spaces ℓ_∞, c and c_0. Later, Altinok and Mursaleen [3] generalized this definition by using a difference operator Δ, where (X_k) is a sequence of fuzzy numbers and $\Delta X=X_k-X_{k+1}$.

The purpose of this paper is to generalize the studies of Bhardwaj and Dhawan [5] and Sengül and Et [32] applying to sequences of fuzzy numbers so as to fill up the existing gaps in the summability theory of fuzzy numbers.

Nakano [25] introduced the concept of modulus function. According to this definition, a mapping $f:[0,\infty)\to[0,\infty)$ is said to be a modulus if following conditions hold: *i)* $f(x)=0$ iff $x=0$, *ii)* $f(x+y)\le f(x)+f(y)$ for $x,y\ge 0$, *iii)* f is increasing, *iv)* f is right-continuous at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. A modulus may be unbounded or bounded.

Aizpuru et al. [1] defined $f-$density of a subset $E\subset\mathbb{N}$ for any unbounded modulus f by

$$d^f(E)=\lim_{n\to\infty}\frac{f(|\{k\le n:k\in E\}|)}{f(n)}, \text{if the limit exists}$$

and defined $f-$statistical convergence for any unbounded modulus f by

$$d^f(\{k\in\mathbb{N}:|x_k-\ell|\ge\varepsilon\})=0$$

i.e.

$$\lim_{n\to\infty}\frac{1}{f(n)}f(|\{k\le n:|x_k-\ell|\ge\varepsilon\}|)=0,$$

and we write it as $S^f-\lim x_k=\ell$ or $x_k\to\ell\left(S^f\right)$. Every $f-$statistically convergent sequence is statistically convergent, but a statistically convergent sequence need not be $f-$statistically convergent for every unbounded modulus f.

Freedman *et al.* [14] introduced some Cesàro-type summability spaces using lacunary sequences and later Fridy and Orhan [16] defined the concepts of lacunary statistical convergence for real number sequences.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. Lacunary sequence spaces were studied in ([4],[12],[20],[32]).

2. Main Results

In this section we will introduce the concepts of Δ^f-lacunary statistically convergent sequences of order β and strongly Δ^f-lacunary summable sequences of order β of real numbers, where f is an unbounded modulus, Δ is a difference operator and give some inclusion relations between these concepts.

Definition 2.1 Let f be an unbounded modulus, $\theta = (k_r)$ be a lacunary sequence, $X = (X_k)$ be a sequence of fuzzy numbers, Δ is a difference operator and β be a real number such that $0 < \beta \leq 1$. We say that the sequence $X = (X_k)$ is Δ^f-lacunary statistically convergent of order β, if there is a fuzzy number X_0 such that

$$\lim_{r\to\infty} \frac{1}{f(h_r)^\beta}.f\left(\left|\left\{k \in I_r : d\left(\Delta X_k, X_0\right) \geq \varepsilon\right\}\right|\right) = 0,$$

where $I_r = (k_{r-1}, k_r]$ and $f(h_r)^\alpha$ denotes the βth power $f(h_r)^\beta$ of $f(h_r)$, that is $\left(f(h_r)^\beta\right) = \left(f(h_1)^\beta, f(h_2)^\beta, ..., f(h_r)^\beta, ...\right)$. This space will be denoted by $S_\theta^{f,\beta}(\Delta_F)$. In this case, we write $S_\theta^{f,\beta}(\Delta_F)-\lim X_k = X_0$ or $X_k \to X_0 \left(S_\theta^{f,\beta}(\Delta_F)\right)$.

Definition 2.2 Let f be a modulus function, $p = (p_k)$ be a sequence of strictly positive real numbers, $\theta = (k_r)$ be a lacunary sequence, $X = (X_k)$ be a sequence of fuzzy numbers, Δ is a difference operator and β be a real number such that $0 < \beta \leq 1$. We say that the sequence $X = (X_k)$ is strongly $W_\theta^{f,\beta}[\Delta_F, p]-$summable to fuzzy number X_0 if there is a fuzzy number X_0 such that

$$W_\theta^{f,\beta}[\Delta_F, p] = \left\{X = (X_k) : \lim_{r\to\infty} \frac{1}{h_r^\beta}.\sum_{k\in I_r} \left[f\left(d\left(\Delta X_k, X_0\right)\right)\right]^{p_k} = 0\right\}.$$

In the present case, we denote $W_\theta^{f,\beta}[\Delta_F, p] - \lim X_k = X_0$.

Definition 2.3 Let f be an unbounded modulus, $p = (p_k)$ be a sequence of strictly positive real numbers, $\theta = (k_r)$ be a lacunary sequence, $X = (X_k)$ be a sequence of fuzzy numbers, Δ is a difference operator and β be a real number such that $0 < \beta \leq 1$. We say that the sequence $X = (X_k)$ is strongly $W_\theta^{f,\beta}(\Delta_F, p)-$summable to fuzzy number X_0 if there is a fuzzy number X_0 such that

$$W_\theta^{f,\beta}(\Delta_F, p) = \left\{X = (X_k) : \lim_{r\to\infty} \frac{1}{f(h_r)^\beta}.\sum_{k\in I_r} \left[f\left(d\left(\Delta X_k, X_0\right)\right)\right]^{p_k} = 0\right\}.$$

In the present case, we write $W_{\theta}^{f,\beta}(\Delta_F, p) - \lim X_k = X_0$. In case of $p_k = p$ for all $k \in \mathbb{N}$ we write $W_{\theta}^{f,\beta}[\Delta_F, p]$ instead of $W_{\theta}^{f,\beta}(\Delta_F, p)$.

Definition 2.4 Let f be an unbounded modulus, $p = (p_k)$ be a sequence of strictly positive real numbers, $\theta = (k_r)$ be a lacunary sequence, $X = (X_k)$ be a sequence of fuzzy numbers, Δ is a difference operator and β be a real number such that $0 < \beta \leq 1$. We say that the sequence $X = (X_k)$ is strongly $W_{\theta,f}^{\beta}(\Delta_F, p)-$summable to fuzzy number X_0 if there is a fuzzy number X_0 such that

$$W_{\theta,f}^{\beta}(\Delta_F, p) = \left\{ X = (X_k) : \lim_{r\to\infty} \frac{1}{f(h_r)^{\beta}} . \sum_{k\in I_r} (d(\Delta X_k, X_0))^{p_k} = 0 \right\}.$$

In the present case, we write $W_{\theta,f}^{\beta}(\Delta_F, p) - \lim X_k = X_0$. In case of $p_k = p$ for all $k \in \mathbb{N}$ we write $W_{\theta,f}^{\beta}[\Delta_F, p]$ instead of $W_{\theta,f}^{\beta}(\Delta_F, p)$.

The proof of each of the following results is fairly straightforward, so we choose to state these results without proof, where we shall assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$, and Δ is a difference operator.

Proposition 2.5 ([28]) Let f be a modulus and $0 < \delta < 1$. Then for each $\|u\| \geq \delta$, we have $f(\|u\|) \leq 2f(1)\delta^{-1}\|u\|$.

Theorem 2.6 Let f be an unbounded modulus, $X = (X_k)$ be a sequence of fuzzy numbers, β be a real number such that $0 < \beta \leq 1$ and $p > 1$. If $\lim_{u\to\infty} \inf \frac{f(u)}{u} > 0$, then $W_{\theta}^{f,\beta}(\Delta_F, p) = W_{\theta,f}^{\beta}(\Delta_F, p)$.

Proof. Let $p > 1$ be a positive real number and $X \in W_{\theta}^{f,\beta}(\Delta_F, p)$. If

$$\lim_{u\to\infty} \inf \frac{f(u)}{u} > 0,$$

then there exists a number $c > 0$ such that $f(u) > cu$ for $u > 0$. Clearly

$$\begin{aligned} \frac{1}{f(h_r)^{\beta}} . \sum_{k\in I_r} [f(d(\Delta X_k, X_0))]^p &\geq \tfrac{1}{f(h_r)^{\beta}} . \textstyle\sum_{k\in I_r} [c.d(\Delta X_k, X_0)]^p \\ &= \tfrac{c^p}{f(h_r)^{\beta}} . \textstyle\sum_{k\in I_r} [d(\Delta X_k, X_0)]^p \end{aligned}$$

and therefore $W_{\theta}^{f,\beta}(\Delta_F, p) \subset W_{\theta,f}^{\beta}(\Delta_F, p)$.

Now let $X \in W_{\theta,f}^{\beta}(\Delta_F, p)$. Then we have

$$\frac{1}{f(h_r)^{\beta}} . \sum_{k\in I_r} [d(\Delta X_k, X_0)]^p \to 0 \text{as} r \to \infty.$$

Let $0 < \delta < 1$. We can write

$$\begin{aligned}\frac{1}{f(h_r)^{\alpha}}\sum_{k\in I_r}^{p} d(\Delta X_k, X_0) &\geq \frac{1}{f(h_r)^{\alpha}}\sum_{\substack{k\in I_r\\ d(\Delta X_k,X_0)\geq\delta}}^{p} d(\Delta X_k, X_0)\\ &\geq \frac{1}{f(h_r)^{\alpha}}\sum_{\substack{k\in I_r\\ d(\Delta X_k,X_0)\geq\delta}}\left[\frac{f(d(\Delta X_k, X_0))}{2f(1)\delta^{-1}}\right]^p\\ &\geq \frac{1}{f(h_r)^{\alpha}}\frac{\delta^p}{2^p f(1)^p}\sum_{k\in I_r}[f(d(\Delta X_k, X_0))]^p\end{aligned}$$

by Proposition 2.5. Therefore $X \in W_{\theta}^{f,\beta}(\Delta_F, p)$. So, the equality $W_{\theta}^{f,\beta}(\Delta_F, p) = W_{\theta,f}^{\beta}(\Delta^f, p)$ holds and proof completes.

If $\lim_{u\to\infty}\inf\frac{f(u)}{u} = 0$, the equality $W_{\theta}^{f,\beta}(\Delta_F, p) = W_{\theta,f}^{\beta}(\Delta^f, p)$ can not be hold as shown the following example:

Let $f(x) = x$ be a modulus function and define a fuzzy sequence $X = (X_k)$ by

$$X_k(x) = \begin{cases} \left.\begin{array}{cc} x-\sqrt{h_r}+1, & x\in\left[\sqrt{h_r}-1,\sqrt{h_r}\right] \\ -x+\sqrt{h_r}+1, & x\in\left[\sqrt{h_r},\sqrt{h_r}+1\right] \\ 0, & \text{otherwise}\end{array}\right\} & \text{if } k=k_r \\ 0, & \text{otherwise}\end{cases}$$

If we calculate the $\alpha-$level sets of sequences (X_k) and (ΔX_k), then we find the sets

$$[X_k]^{\alpha} = \begin{cases}\left[\alpha+\sqrt{h_r}-1, -\alpha+\sqrt{h_r}+1\right], & \text{if } k=k_r\\ [0,0], & \text{otherwise}\end{cases}$$

and

$$[\Delta X_k]^{\alpha} = \begin{cases}\left[\alpha+\sqrt{h_r}-1, -\alpha+\sqrt{h_r}+1\right], & \text{if } k=k_r\\ \left[\alpha-\sqrt{h_r}-1, -\alpha-\sqrt{h_r}+1\right], & \text{if } k+1=k_r\\ [0,0], & \text{otherwise}\end{cases}$$

For $X_0 = \bar{0}$, $\beta = \frac{3}{4}$, $p = \frac{5}{2}$ we have

$$\frac{1}{f(h_r)^{\beta}}\cdot\sum_{k\in I_r}[f\left(d\left(\Delta X_k, \bar{0}\right)\right)]^p = \frac{\left(\left(\sqrt{h_r}+1\right)^{\frac{1}{4}}\right)^{\frac{5}{2}}}{\left(\sqrt{h_r}\right)^{\frac{3}{4}}} = \frac{\left(\sqrt{h_r}+1\right)^{\frac{5}{8}}}{\left(\sqrt{h_r}\right)^{\frac{3}{4}}} \to 0 \ (r\to\infty)$$

hence $X_k \in W_{\theta}^{f,\beta}(\Delta_F, p)$, but

$$\frac{1}{f(h_r)^{\beta}}\cdot\sum_{k\in I_r}[d(\Delta X_k, \bar{o})]^p = \frac{\left(\left(\sqrt{h_r}+1\right)^{\frac{1}{2}}\right)^{\frac{5}{2}}}{\left(\sqrt{h_r}\right)^{\frac{3}{4}}} = \frac{\left(\sqrt{h_r}+1\right)^{\frac{5}{4}}}{\left(\sqrt{h_r}\right)^{\frac{3}{4}}} \to \infty \ (r\to\infty)$$

and so $X_k \notin W^{\beta}_{\theta,f}(\Delta_F, p)$.

Maddox [22] showed that the existence of an unbounded modulus f for which there is a positive constant c such that $f(xy) \geq cf(x)f(y)$, for all $x \geq 0$, $y \geq 0$.

Theorem 2.7 Let f be an unbounded modulus, $\theta = (k_r)$ be a lacunary sequence, $X = (X_k)$ be a sequence of fuzzy numbers, β be a real number such that $0 < \beta \leq 1$ and $p_k = 1$ for all $k \in \mathbb{N}$. If $\lim_{u\to\infty} \frac{f(u)^\beta}{u^\beta} > 0$, then $W_\theta^{f,\beta}[\Delta_F, p] \subset S_\theta^{f,\beta}(\Delta_F, p)$.

Proof. Let $x \in W_\theta^{f,\beta}[\Delta_F, p]$ and $\lim_{u\to\infty} \frac{f(u)^\beta}{u^\beta} > 0$. For $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{h_r^\beta}.\sum_{k\in I_r} f\left(d\left(\Delta X_k, X_0\right)\right) &\geq \tfrac{1}{h_r^\beta} f\left(\textstyle\sum_{k\in I_r} d\left(\Delta X_k, X_0\right)\right) \\
&\geq \tfrac{1}{h_r^\beta} f\left(\textstyle\sum_{\substack{k\in I_r \\ d(\Delta X_k, X_0)\geq\varepsilon}} d\left(\Delta X_k, X_0\right)\right) \\
&\geq \tfrac{1}{h_r^\beta}.f\left(\left|\left\{k \in I_r : d\left(\Delta X_k, X_0\right) \geq \varepsilon\right\}\right|.\varepsilon\right) \\
&\geq \tfrac{c}{h_r^\beta}.f(\left|\left\{k \in I_r : d\left(\Delta X_k, X_0\right) \geq \varepsilon\right\}\right|).f\left(\varepsilon\right) \\
&= \tfrac{c}{h_r^\beta}.\tfrac{f(\left|\left\{k\in I_r : d(\Delta X_k, X_0)\geq\varepsilon\right\}\right|)}{f(h_r)^\beta}.f\left(h_r\right)^\beta.f\left(\varepsilon\right).
\end{aligned}$$

Therefore, $X \in W_\theta^{f,\beta}[\Delta_F, p]$ implies that $X \in S_\theta^{f,\beta}(\Delta_F, p)$, that is $W_\theta^{f,\beta}[\Delta_F, p] - \lim X_k = X_0$ implies that $S_\theta^{f,\beta}(\Delta_F, p) - \lim X_k = X_0$.

Theorem 2.8 Let f be an unbounded modulus function, $X = (X_k)$ be a sequence of fuzzy numbers, β_1, β_2 be two real numbers such that $0 < \beta_1 \leq \beta_2 \leq 1$, and $\theta = (k_r)$ be a lacunary sequence, then we have $W_\theta^{f,\beta_1}(\Delta_F, p) \subset S_\theta^{f,\beta_2}(\Delta_F)$.

Proof. Let $X \in W_\theta^{f,\beta_1}(\Delta_F, p)$ and $\varepsilon > 0$ be given and $\sum_1$, $\sum_2$ denote the sums over $k \in I_r$, $d(\Delta X_k, X_0) \geq \varepsilon$ and $k \in I_r$, $d(\Delta X_k, X_0) < \varepsilon$ respectively. Since

$f(h_r)^{\beta_1} \leq f(h_r)^{\beta_2}$ for each r, we may write

$$\begin{aligned}
&\frac{1}{f(h_r)^{\beta_1}} \cdot \sum_{k\in I_r} [f(d(\Delta X_k, X_0))]^{p_k} \\
&\quad = \frac{1}{f(h_r)^{\beta_1}} .[\sum_1 [f(d(\Delta X_k, X_0))]^{p_k} + \sum_2 [f(d(\Delta X_k, X_0))]^{p_k}] \\
&\quad \geq \frac{1}{f(h_r)^{\beta_2}} .[\sum_1 [f(d(\Delta X_k, X_0))]^{p_k} + \sum_2 [f(d(\Delta X_k, X_0))]^{p_k}] \\
&\quad \geq \frac{1}{f(h_r)^{\beta_2}} .[\sum_1 [f(\varepsilon)]^{p_k}] \geq \frac{1}{f(h_r)^{\beta_2}} .f\left[\sum_1 [\varepsilon]^{p_k}\right] \\
&\quad \geq \frac{1}{f(h_r)^{\beta_2}} . \left[f\left(\sum_1 \min\left([\varepsilon]^h, [\varepsilon]^H\right)\right)\right] \\
&\quad \geq \frac{1}{f(h_r)^{\beta_2}} .f(|\{k \in I_r : d(\Delta X_k, X_0) \geq \varepsilon\}|). \min\left([\varepsilon]^h, [\varepsilon]^H\right) \\
&\quad \geq \frac{c}{f(h_r)^{\beta_2}} .f(|\{k \in I_r : d(\Delta X_k, X_0) \geq \varepsilon\}|).f\left(\min\left([\varepsilon]^h, [\varepsilon]^H\right)\right).
\end{aligned}$$

Hence $X \in S_\theta^{f,\beta_2}(\Delta_F)$.

Theorem 2.9 Let f be an unbounded modulus function, $X = (X_k)$ be a sequence of fuzzy numbers, $\theta = (k_r)$ be a lacunary sequence and β be a fixed real number such that $0 < \beta \leq 1$. If $\liminf_r q_r > 1$ and $\lim_{u\to\infty} \frac{f(u)^\beta}{u^\beta} > 0$, then $S^{f,\beta}(\Delta_F) \subset S_\theta^{f,\beta}(\Delta_F)$.

Proof. Suppose first that $\liminf_r q_r > 1$; then there exists a $\lambda > 0$ such that $q_r \geq 1 + \lambda$ for sufficiently large r, which implies that

$$\left(\frac{h_r}{k_r}\right)^\beta \geq \left(\frac{\lambda}{1+\lambda}\right)^\beta.$$

If $S^{f,\beta}(\Delta_F) - \lim X_k = X_0$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}
&\frac{1}{k_r^\beta} \cdot f(|\{k \leq k_r : d(\Delta X_k, X_0) \geq \varepsilon\}|) \\
&\quad \geq \frac{1}{f(k_r)^\beta} .f(|\{k \in I_r : d(\Delta X_k, X_0) \geq \varepsilon\}|) \\
&\quad = \frac{f(h_r)^\beta}{f(k_r)^\beta} . \frac{1}{f(h_r)^\beta} .f(|\{k \in I_r : d(\Delta X_k, X_0) \geq \varepsilon\}|) \\
&\quad = \frac{f(h_r)^\beta}{h_r^\beta} . \frac{k_r^\beta}{f(k_r)^\beta} . \frac{h_r^\beta}{k_r^\beta} . \frac{f(|\{k \in I_r : d(\Delta X_k, X_0) \geq \varepsilon\}|)}{f(h_r)^\beta} \\
&\quad \geq \frac{f(h_r)^\beta}{h_r^\beta} . \frac{k_r^\beta}{f(k_r)^\beta} . \left(\frac{\lambda}{1+\lambda}\right)^\beta . \frac{f(|\{k \in I_r : d(\Delta X_k, X_0) \geq \varepsilon\}|)}{f(h_r)^\beta}.
\end{aligned}$$

This proves the sufficiency.

Theorem 2.10 Let f be an unbounded modulus function, $X = (X_k)$ be a sequence of fuzzy numbers, $\theta = (k_r)$ be a lacunary sequence and β be a fixed real number such that $0 < \beta \leq 1$. If $X_k \in S^f(\Delta_F) \cap S_\theta^{f,\beta}(\Delta_F)$, then $S^f(\Delta_F) - \lim X_k = S_\theta^{f,\beta}(\Delta_F) - \lim X_k$ such that $|f(x) - f(y)| = f(|x - y|)$, for $x \geq 0$, $y \geq 0$.

Proof. Suppose $S^f(\Delta_F) - \lim X_k = X_0$, $S_\theta^{f,\beta}(\Delta_F) - \lim X_k = X_0'$ and $X_0 \neq X_0'$. Let $0 < \varepsilon < \frac{|\ell_1 - \ell_2|}{2}$. Then for $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{f(|\{k \leq n : d(\Delta X_k, X_0) \geq \varepsilon\}|)}{f(n)} = 0$$

and

$$\lim_{r\to\infty} \frac{f(|\{k \leq I_r : d(\Delta X_k, X_0') \geq \varepsilon\}|)}{f(h_r)^\beta} = 0.$$

On the other hand we can write

$$\frac{f(|\{k \leq n : d(X_0, X_0') \geq 2\varepsilon\}|)}{f(n)} \leq \frac{f(|\{k \leq n : d(\Delta X_k, X_0) \geq \varepsilon\}|)}{f(n)} + \frac{f(|\{k \leq n : d(\Delta X_k, X_0') \geq \varepsilon\}|)}{f(n)}.$$

Taking limit as $n \to \infty$, we get

$$1 \leq 0 + \lim_{n\to\infty} \frac{f(|\{k \leq n : d(\Delta X_k, X_0') \geq \varepsilon\}|)}{f(n)} \leq 1,$$

and so

$$\lim_{n\to\infty} \frac{f(|\{k \leq n : d(\Delta X_k, X_0') \geq \varepsilon\}|)}{f(n)} = 1.$$

We consider the subsequence

$$\frac{1}{f(k_m)} f(|\{k \leq k_m : d(\Delta X_k, X_0') \geq \varepsilon\}|)$$

of sequence

$$\frac{1}{f(n)} f(|\{k \leq n : d(\Delta X_k, X_0') \geq \varepsilon\}|).$$

Then

$$\begin{aligned}
&\frac{1}{f(k_m)} . f(|\{k \leq k_m : d(\Delta X_k, X_0') \geq \varepsilon\}|) \\
&= \frac{1}{f(k_m)} . f(|\{k \in \cup_{r=1}^m I_r : d(\Delta X_k, X_0') \geq \varepsilon\}|) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (1) \\
&= \frac{1}{f(k_m)} . f\left(\sum_{r=1}^m |\{k \in I_r : d(\Delta X_k, X_0') \geq \varepsilon\}|\right) \\
&\leq \frac{1}{f(k_m)} . \sum_{r=1}^m f(|\{k \in I_r : d(\Delta X_k, X_0') \geq \varepsilon\}|) \\
&= \frac{1}{f(k_m)} . \sum_{r=1}^m f(h_r)^\beta . \frac{1}{f(h_r)^\beta} . f(|\{k \in I_r : d(\Delta X_k, X_0') \geq \varepsilon\}|)
\end{aligned}$$

and

$$\sum_{r=1}^{m} f(h_r)^{\beta} = f(h_1)^{\beta} + f(h_2)^{\beta} + ... + f(h_m)^{\beta} \text{ .... (2)}$$

$$= f(k_1 - k_0)^{\beta} + f(k_2 - k_1)^{\beta} + ... + f(k_m - k_{m-1})^{\beta}$$
$$= f(|k_1 - k_0|)^{\beta} + f(|k_2 - k_1|)^{\beta} + ... + f(|k_m - k_{m-1}|)^{\beta}$$
$$= |f(k_1) - f(k_0)|^{\beta} + |f(k_2) - f(k_1)|^{\beta} + ... + |f(k_m) - f(k_{m-1})|^{\beta}$$
$$\leq |f(k_1) - f(k_0)| + |f(k_2) - f(k_1)| + ... + |f(k_m) - f(k_{m-1})|$$
$$= f(k_1) - f(k_0) + f(k_2) - f(k_1) + ... + f(k_m) - f(k_{m-1})$$
$$= f(k_m).$$

Using (2) in (1), we have

$$\tfrac{1}{f(k_m)}.f(|\{k \leq k_m : d(\Delta X_k, X_0') \geq \varepsilon\}|)$$
$$\leq \tfrac{\sum_{r=1}^{m} f(h_r)^{\beta}}{\sum_{r=1}^{m} f(h_r)^{\beta}}.\tfrac{1}{f(h_r)^{\beta}}.f(|\{k \in I_r : d(\Delta X_k, X_0') \geq \varepsilon\}|)$$

so

$$\frac{1}{f(k_m)} f(|\{k \leq k_m : d(\Delta X_k, X_0') \geq \varepsilon\}|) \to 0,$$

but this is a contradiction to

$$\lim_{n\to\infty} \frac{f(|\{k \leq n : d(\Delta X_k, X_0') \geq \varepsilon\}|)}{f(n)} = 1.$$

As a result, $X_0 = X_0'$.

From Theorem 2.10 we have the following result:

Corollary 2.11 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences and $0 < \beta \leq 1$. If $X_k \in S_{\theta}^{f,\beta}(\Delta_F) \cap (S_{\theta}^{f,\beta}(\Delta_F) \cap S_{\theta'}^{f,\beta}(\Delta_F))$, then $S_{\theta}^{f,\beta}(\Delta_F) - \lim X_k = S_{\theta'}^{f,\beta}(\Delta_F) - \lim X_k$.

Theorem 2.12 Let f be an unbounded modulus function, $\theta = (k_r)$ be a lacunary sequence and β be a fixed real number such that $0 < \beta \leq 1$. If $\lim p_k > 0$, then $W_{\theta}^{f,\beta}(\Delta_F, p) - \lim X_k = X_0$ uniquely.

Proof. Let $\lim p_k = s > 0$. Assume that $W_{\theta}^{f,\beta}(\Delta_F, p) - \lim X_k = X_0'$ and $W_{\theta}^{f,\beta}(\Delta_F, p) - \lim X_k = X_0''$. Then

$$\lim_r \frac{1}{f(h_r)^{\beta}}.\sum_{k\in I_r} [f(d(\Delta X_k, X_0'))]^{p_k} = 0$$

and

$$\lim_r \frac{1}{f(h_r)^{\beta}}.\sum_{k\in I_r} [f(d(\Delta X_k, X_0''))]^{p_k} = 0.$$

By definition to f, we have

$$\frac{1}{f(h_r)^{\beta}}\cdot\sum_{k\in I_r}[f(d(X_0',X_0''))]^{p_k}$$
$$\le \frac{D}{f(h_r)^{\beta}}\left(\sum_{k\in I_r}[f(d(\Delta X_k,X_0'))]^{p_k}+\sum_{k\in I_r}[f(d(\Delta X_k,X_0''))]^{p_k}\right)$$
$$\le \frac{D}{f(h_r)^{\alpha}}\sum_{k\in I_r}f(d(\Delta X_k,X_0'))]^{p_k}+\frac{D}{f(h_r)^{\alpha}}\sum_{k\in I_r}[f(d(\Delta X_k,X_0''))]^{p_k},$$

where $\sup_k p_k = H$ and $D=\max\left(1,2^{H-1}\right)$. Hence

$$\lim_r \frac{1}{f(h_r)^{\beta}}.\sum_{k\in I_r}[f(d(X_0',X_0''))]^{p_k}=0.$$

Since $\lim_{k\to\infty} p_k = s$ we have $d(X_0',X_0'')=0$. Thus the limit is unique.

Theorem 2.13 Let f be an unbounded modulus function, $X=(X_k)$ be a sequence of fuzzy numbers, $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subset J_r$ for all $r\in\mathbb{N}$ and let β_1,β_2 two real numbers such that $0<\beta_1\le\beta_2\le 1$. If

$$\lim_{r\to\infty}\inf\frac{f(h_r)^{\beta_1}}{f(l_r)^{\beta_2}}>0, \tag{3}$$

where $I_r=(k_{r-1},k_r]$, $h_r=k_r-k_{r-1}$ and $J_r=(s_{r-1},s_r]$, $\ell_r=s_r-s_{r-1}$, then $W_{\theta'}^{f,\beta_2}(\Delta_F,p)\subset W_{\theta}^{f,\beta_1}(\Delta_F,p)$.

Proof. Let $X\in W_{\theta'}^{f,\beta_2}(\Delta_F,p)$. We can write

$$\frac{1}{f(l_r)^{\beta_2}}.\sum_{k\in J_r}[f(d(\Delta X_k,X_0))]^{p_k}$$
$$=\frac{1}{f(l_r)^{\beta_2}}.\sum_{k\in J_r-I_r}[f(d(\Delta X_k,X_0))]^{p_k}+\frac{1}{f(l_r)^{\beta_2}}.\sum_{k\in I_r}[f(d(\Delta X_k,X_0))]^{p_k}$$
$$\ge\frac{1}{f(l_r)^{\beta_2}}.\sum_{k\in I_r}[f(d(\Delta X_k,X_0))]^{p_k}$$
$$\ge\frac{f(h_r)^{\beta_1}}{f(l_r)^{\beta_2}}.\frac{1}{f(h_r)^{\beta_1}}.\sum_{k\in I_r}[f(d(\Delta X_k,X_0))]^{p_k}.$$

Thus if $X\in W_{\theta'}^{f,\beta_2}(\Delta_F,p)$, then $X\in W_{\theta}^{f,\beta_1}(\Delta_F,p)$.

From Theorem 2.13 we have the following result:

Corollary 2.14 Let f be an unbounded modulus function, $X=(X_k)$ be a sequence of fuzzy numbers, $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subset J_r$ for all $r\in\mathbb{N}$ and let β_1,β_2 two real numbers such that $0<\beta_1\le\beta_2\le 1$. If (3) holds then

(*i*) $W_{\theta'}^{f,\beta}(\Delta_F,p)\subset W_{\theta}^{f,\beta}(\Delta_F,p)$, if $\beta_1=\beta_2=\beta$,

(*ii*) $W_{\theta'}^{f}(\Delta_F,p)\subset W_{\theta}^{f,\beta_1}(\Delta_F,p)$, if $\beta_2=1$,

(*iii*) $W_{\theta'}^{f}(\Delta_F,p)\subset W_{\theta}^{f}(\Delta_F,p)$, if $\beta_1=\beta_2=1$.

REFERENCES

1. A. Aizpuru, M. C. Listán-García and F. Rambla-Barreno: *Density by moduli and statistical convergence*. Quaest. Math. **37**(4) (2014), 525–530.
2. H. Altinok, Y. Altin and M. Işik: *Statistical Convergence and Strong $p-$Cesàro Summability of Order β in Sequences of Fuzzy Numbers*. Iranian J. of Fuzzy Systems **9**(2) (2012), 65–75.
3. H. Altinok and M. Mursaleen: *$\Delta-$Statistical boundedness for sequences of fuzzy numbers*. Taiwanese Journal of Mathematics, **15**(5), (2011), 2081–2093.
4. V. K. Bhardwaj and S. Dhawan: *Density by moduli and lacunary statistical convergence*. Abstr. Appl. Anal. (2016), Art. ID 9365037, 11 pp.
5. V. K. Bhardwaj and S. Dhawan: *$f-$statistical convergence of order α and strong Cesaro summability of order α with respect to a modulus*. J. Inequal. Appl. **2015**(332) (2015).
6. R. Colak, Y. Altin, and M. Mursaleen: *On some sets of difference sequences of fuzzy numbers*. Soft Computing **15** (2011), 787–793.
7. J. S. Connor: *The statistical and strong $p-$Cesaro convergence of sequences*. Analysis **8** (1988), 47–63.
8. H. Çakalli: *A study on statistical convergence*. Funct. Anal. Approx. Comput. **1**(2) (2009), 19–24.
9. M. Çinar, M. Karakaş and M. Et: *On pointwise and uniform statistical convergence of order α for sequences of functions*. Fixed Point Theory And Applications, Article Number: **33** (2013).
10. R. Çolak: *Statistical convergence of order α*. Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub, (2010), 121–129.
11. M. Et, Y. Altin and H. Altinok: *On some generalized difference sequence spaces defined by a modulus function*. Filomat **17** (2003), 23–33.
12. M. Et and H. Şengül: *Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α*. Filomat **28(8)** (2014), 1593–1602.
13. H. Fast: *Sur la convergence statistique*. Colloq. Math. **2** (1951), 241–244.
14. A. R. Freedman, J. J. Sember and M. Raphael: *Some Cesaro-type summability spaces*. Proc. Lond. Math. Soc. **37**(3) (1978), 508–520.
15. J. Fridy: *On statistical convergence*. Analysis **5** (1985), 301–313.
16. J. Fridy and C. Orhan: *Lacunary statistical convergence*. Pacific J. Math. **160** (1993), 43–51.
17. A. D. Gadjiev and C. Orhan: *Some approximation theorems via statistical convergence*. Rocky Mountain J. Math. **32**(1) (2002), 129–138.
18. B. Hazarika, A. Alotaibi and S. A. Mohiuddine: *Statistical convergence in measure for double sequences of fuzzy-valued functions*. Soft Computing **24** (2020), 6613–6622.
19. M. Işik: *Generalized vector-valued sequence spaces defined by modulus functions*. J. Inequal, Appl. 2010, Art. ID 457892, 7 pp.
20. M. Işik and K. E. Et: *On lacunary statistical convergence of order α in probability*. AIP Conference Proceedings 1676, 020045 (2015). doi: http://dx.doi.org/10.1063/1.4930471.

21. H. Kizmaz: *On certain sequence spaces*. Canad. Math. Bull. **24**(2) (1981), 169–176.

22. I. J. Maddox: *Sequence spaces defined by a modulus*. Math. Proc. Camb. Philos. Soc. **100** (1986), 161–166.

23. M. Matloka: *Sequences of fuzzy numbers*. BUSEFAL **28** (1986), 28–37.

24. S. A. Mohiuddine, A. Asiri and B. Hazarika: *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*. Int. J. Gen. Syst. **48**(5) (2019), 492–506.

25. H. Nakano: *Modulared sequence spaces*. Proc. Japan Acad. **27** (1951), 508–512.

26. S. Nanda: *On sequences of fuzzy number*. Fuzzy Sets and Systems **33** (1989), 123–126.

27. F. Nuray and E. Savaş: *Some new sequence spaces defined by a modulus function*. Indian J. Pure Appl. Math. **24**(11) (1993), 657–663.

28. S. Pehlivan and B. Fisher: *Some sequence spaces defined by a modulus*. Math. Slovaca. **45**(3) (1995), 275–280.

29. T. Salat: *On statistically convergent sequences of real numbers*. Math. Slovaca **30** (1980), 139–150.

30. I. J. Schoenberg: *The integrability of certain functions and related summability methods*. Amer. Math. Monthly **66** (1959), 361–375.

31. H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique*. Colloq. Math. **2** (1951),73–74.

32. H. Şengül and M. Et: *On lacunary statistical convergence of order α*. Acta Math. Sci. Ser. B Engl. Ed. **34**(2) (2014), 473–482.

33. L. A. Zadeh: *Fuzzy sets*. Information and Control **8**, (1965), 338–353.