FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 1 (2016), 187–200

CERTAIN CURVES IN TRANS-SASAKIAN MANIFOLDS *

Avijit Sarkar and Ashis Mondal

Abstract. In the present paper, biharmonic almost contact curves with respect to generalized Tanaka Webster Okumura connections have been studied on three-dimensional trans-Sasakian manifolds. Locally ϕ-symmetric almost contact curves on three-dimensional trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connections have been introduced. Hyperbolic space as a particular case of trans-Sasakian manifolds has been studied. Examples of almost contact curves are given.

Keywords: Almost contact curve, trans-Sasakian manifold, hyperbolic space, generalized Tanaka Webster Okumura Connections.

1. Introduction

In the study of contact manifolds, Legendre curves play an important role, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair in the paper [3]. Originally the notion of Legendre curve was defined for curves in contact three manifolds with the help of contact form. This notion of Legendre curves can be also extended to almost contact manifolds [31]. Curves satisfying the properties of Legendre curves in almost contact metric manifolds are known as almost contact curves [15].

During the last few years biharmonic maps on Riemannian manifolds have been given much attention by differential geometers. A smooth map $\phi : (N, h) \to (M, g)$ between Riemannian manifolds is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2}|\tau(\phi)|^2 dv_h.$$

A harmonic map is biharmonic. A biharmonic map is said to be proper if it is not harmonic. A smooth map $\phi : (N, h) \to (M, g)$ is biharmonic if and only if its tension field $\tau(\phi)$ is the kernel of the Jacobi operator of ϕ. There exists no non-geodesic

Received May 06, 2015; Accepted October 27, 2015
2010 *Mathematics Subject Classification.* Primary 53 C 25; Secondary 53 D 15
*

biharmonic Legendre curves in S^3 with respect to Levi-Civita connection [7], [14]. The study of Legendre curves on S^3 with Tanaka Webster connection has been initiated by Cho and Lee[8].

On the other hand J. Welyzcko [31], studied Legendre curves on three-dimensional trans-Sasakian manifolds with respect to Levi-Civita connections. In [15], the authors have introduced a 1-parameter family of linear connections on three-dimensional almost contact metric manifolds to study biharmonic curves on almost contact manifolds. Linear connections in the family are regarded as a generalization of Tanaka Webster connections [28], [32]. These connections are called generalized Tanaka Webster Okumura connections or in brief gTWO-connections. Inoguchi and Lee [15], studied almost contact curves in the model spaces of three-dimensional Thurston geometry with gTWO-connectons. The first author of the present paper have also studied Legendre curves on trans-Sasakian manifolds [26]. In the present paper, we are interested to study biharmonic almost contact curves with respect to gTWO-connections on a three-dimensional trans-Sasakian manifold. We also introduce locally ϕ-symmetric almost contact curves with respect to gTWO-connections on a three-dimensional trans-Sasakian manifold. Next we show the relation between the biharmonicity and local ϕ-symmetry of an almost contact curves in the hyperbolic space $H^3(-1)$. Existence of biharmonic almost contact curves on almost contact metric manifolds with respect to gTWO-connections has been established in the paper [15]. Legendre curves with respect to pseudo-Hermitian connections have been studied in the papers [8], [17]. We may find works on biharmonic curves and maps in the papers [12], [20]. Legendre curves and slant curves have also been studied in the papers [23], [24].

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by D. Chinea and C. Gonzalez [5], and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermite manifolds [13], there appears a class W_4 of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [21] if the product manifold $M \times R$ belongs to the class W_4. The class $C_6 \oplus C_5$ [18], [19] coincides with the class of trans-Sasakian structures of type (α, β). In [19], the local nature of the two subclasses C_5 and C_6 of trans-Sasakian structures is characterized completely. In [4], some curvature identities and sectional curvatures for C_5, C_6 and trans-Sasakian manifolds are obtained. It is known that [16] trans-Sasakian structures of type (0,0), $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic, β-Kenmotsu and α-Sasakian respectively.

The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by J. C. Marrero [18]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu manifold. So proper trans-Sasakian manifolds exist only for dimension three. The first author of the present paper have studied trans-Sassakian manifold of dimension three [10]. In this paper we are involved with three-dimensional trans-Sasakian manifolds.

The present paper is organized as follows: After the introduction, we give some required preliminaries in Section 2. Section 3 deals with examples of almost contact curves in three dimensional trans-Sasakian manifolds. In Section 4, we study biharmonic almost contact curves with respect to gTWO-connections. In Section 5, we introduce the notion of locally ϕ-symmetric almost contact curves on three-dimensional trans-Sasakian manifolds and study them with respect to gTWO-connections. The last section contains study of almost contact curves in the hyperbolic space $H^3(-1)$.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in T(M)$ [1]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.4}$$

for $X, Y \in T(M)$.

An almost contact metric manifold is normal if $[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$.

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called trans-Sasakian structure [21] if $(M \times R, J, G)$ belongs to the class W_4 [13], where J is the almost complex structure on $M \times R$ defined by

$$J(X, f d/dt) = (\phi X - f\xi, \eta(X)d/dt),$$

for all vector fields X on M, a smooth function f on $M \times R$ and the product metric G on $M \times R$. This may be expressed by the condition [2]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.5}$$

for smooth functions α and β on M. Here ∇ is Levi-Civita connection on M. We say M as the trans-Sasakian manifold of type (α, β). From (2.5) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{2.6}$$

A trans-Sasakian manifold is said to be

- cosymplectic or co Kaehler manifold if $\alpha = \beta = 0$,
- quasi-Sasakian manifold if $\beta = 0$ and $\xi(\alpha) = 0$,

- α-Sasakian manifold if α is a non-zero constant and $\beta = 0$,
- β-Kenmotsu manifold if $\alpha = 0$ and β is a non-zero constant.

Therefore, trans-Sasakian manifold generalizes a large class of almost contact manifolds.

From [11], we get the Riemannian curvature tensor R with respect to Levi-Civita connection of a three-dimensional trans-Sasakian manifold as the following:

$$\begin{aligned} R(X,Y)Z = {} & (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))(g(Y,Z)X - g(X,Z)Y) \\ & -g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi \\ & -\eta(X)(\phi\mathrm{grad}\alpha - \mathrm{grad}\beta) + (X\beta + (\phi X)\alpha)\xi] \\ & +g(X,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\ & -\eta(Y)(\phi\mathrm{grad}\alpha - \mathrm{grad}\beta) + (Y\beta + (\phi Y)\alpha)\xi] \\ & -[(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) \\ & +(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z)]X \\ & +[(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) \\ & +(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)]Y, \end{aligned} \tag{2.7}$$

where S is the Ricci tensor of type $(0,2)$, and r is the scalar curvature of the manifold M with respect to Levi-Civita connection.

The generalized Tanaka Webster Okumura connections [15] $\tilde{\nabla}$ and the Levi-Civita connection ∇ are related by

$$\tilde{\nabla}_X Y = \nabla_X Y + A(X,Y), \tag{2.8}$$

for all vector fields X, Y on M. Here

$$A(X,Y) = \alpha(g(X,\phi Y)\xi + \eta(Y)\phi X) + \beta(g(X,Y)\xi - \eta(Y)X) - l\eta(X)\phi Y, \tag{2.9}$$

where l is a real constant.

The torsion $\tilde{T}$ of the gTWO-connections $\tilde{\nabla}$ is given by

$$\tilde{T}(X,Y) = \alpha(2g(X,\phi Y)\xi - \eta(X)\phi Y + \eta(Y)\phi X) + \eta(X)(\beta Y - l\phi Y) - \eta(Y)(\beta X - l\phi X). \tag{2.10}$$

For $l = 0$, the connections become the connection introduced by Sasaki and Hatakeyama. When $l = 1$, it is the connection introduced by Cho [6]. For Sasakian case and $l = 1$, the connection is Okumura connection [22]. The connections become generalized Tanaka Webster connection introduced by Tanno [30], when $l = -1$.

Let M be a 3−dimensional Riemannian manifold. Let $\gamma : I \to M$, I being an interval, be a curve in M which is parameterized by arc length, and let $\nabla_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to the Levi-Civita connection on M. It is said that γ is a Frenet curve if one of the following three cases holds:

(a) γ is of osculating order 1, i.e., $\nabla_t t = 0$ (geodesic), $t = \dot{\gamma}$. Here, . denotes differentiation with respect to arc parameter.

(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $t(= \dot{\gamma})$, n and a non-negative function k (curvature) along γ such that $\nabla_t t = kn$, $\nabla_t n = -kt$.

(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $t(= \dot{\gamma})$, n, b and two non-negative functions k(curvature) and τ(torsion) along γ such that

$$\nabla_t t = kn, \tag{2.11}$$

$$\nabla_t n = -kt + \tau b, \tag{2.12}$$

$$\nabla_t b = -\tau n. \tag{2.13}$$

With respect to Levi-Civita connection, a Frenet curve of osculating order 3 for which k is a positive constant and $\tau = 0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if k and τ both are positive constants and the curve is called a generalized helix if $\frac{k}{\tau}$ is a constant.

Let $\tilde{\nabla}_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to gTWO-connections on M. We shall say that γ is a Frenet curve with respect to gTWO-connections if one of the following three cases holds:

(a) γ is of osculating order 1, i.e., $\tilde{\nabla}_t t = 0$ (geodesic).

(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $t(= \dot{\gamma})$, N and a non-negative function $\tilde{k}$ (curvature) along γ such that $\tilde{\nabla}_t t = \tilde{k}n$, $\tilde{\nabla}_t n = -\tilde{k}t$.

(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $t(= \dot{\gamma})$, n, b and two non-negative functions $\tilde{k}$(curvature) and $\tilde{\tau}$(torsion) along γ such that

$$\tilde{\nabla}_t t = \tilde{k}n, \tag{2.14}$$

$$\tilde{\nabla}_t n = -\tilde{k}t + \tilde{\tau}b, \tag{2.15}$$

$$\tilde{\nabla}_t b = -\tilde{\tau}n. \tag{2.16}$$

The vector fields t and n along γ in the above equations are related by $n = \phi t$ and hence $b = \xi$ along γ. With respect to gTWO-connection, a Frenet curve of osculating order 3 for which $\tilde{k}$ is a positive constant and $\tilde{\tau} = 0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if $\tilde{k}$ and $\tilde{\tau}$ both are positive constants and the curve is called a generalized helix with respect to gTWO-connections if $\frac{\tilde{k}}{\tilde{\tau}}$ is a constant.

A Frenet curve γ in an almost contact metric manifold is said to be almost contact curve if it is an integral curve of the distribution $\mathcal{D} = \ker\eta$. Formally, it is also said that a Frenet curve γ in an almost contact metric manifold is an almost contact curve if and only if $\eta(\dot{\gamma}) = 0$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$. For more details we refer [3], [8], [17], [31].

It is to be mentioned that in the paper [15], curves satisfying the above properties on almost contact manifolds have been termed as almost contact curve while Welyczko [31] has termed such curves on almost contact manifolds as Legendre curves. Henceforth by Legendre curves on almost contact manifolds we shall mean almost contact curves.

If we consider the binormal vector field b along ξ, then by (2.6) and (2.8) $\tilde{\nabla}_t b = 0$ holds for an almost contact curve with respect to gTWO-connections. Hence in that case we can say that the torsion with respect to gTWO-connections

$$\tilde{\tau} = 0 \tag{2.17}$$

for an almost contact curve on a three-dimensional trans-Sasakian manifold.

Theorem 5.5 of [15]: *If γ is an almost contact curve in $H^3(-1)$, then γ is biharmonic with respect to gTWO connections if and only if γ is a geodesic with respect to gTWO-connections.*

3. Examples of almost contact curves on a proper trans-Sasakian manifold of dimension three

In this section, we give an example of proper trans-Sasakian manifold and then several examples of almost contact curves on it. We consider the three-dimensional manifold $M = \{(x, y, z) \in R^3, z > 0\}$, where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric whose matrix representation with respect to (e_1, e_2, e_3) is

$$\begin{pmatrix} z + y^2 & 0 & -y \\ 0 & z & 0 \\ -y & 0 & 1 \end{pmatrix}$$

Let η be the 1-form defined by $\eta = dz - ydx$ and ϕ be the (1,1) tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1 - ye_3$, $\phi(e_3) = 0$ and $\xi = e_3$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z)e_3, \qquad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.

By Koszul formula

$$\begin{aligned} \nabla_{e_1} e_3 &= \tfrac{1}{2z} e_1 + \tfrac{1}{2z} e_2, & \nabla_{e_1} e_2 &= \tfrac{y}{2z} e_1. \\ \nabla_{e_2} e_3 &= -\tfrac{1}{2z} e_1 + \tfrac{1}{2z} e_2 - \tfrac{y}{2z} e_3 & \nabla_{e_2} e_2 &= -\tfrac{y}{2z} e_1 - \tfrac{y^2+z}{2z} e_3. \\ \nabla_{e_3} e_3 &= 0 & \nabla_{e_3} e_2 &= -\tfrac{1}{2z} e_1 + \tfrac{1}{2z} e_2 - \tfrac{y}{2z} e_3. \end{aligned}$$

$$\nabla_{e_1} e_1 = -\frac{y}{z} e_2 - \frac{y^2+z}{2z} e_3,$$
$$\nabla_{e_2} e_1 = \frac{y}{2z} e_1 + \frac{y^2-z}{2z} e_3,$$
$$\nabla_{e_3} e_1 = \frac{1}{2z} e_1 + \frac{1}{2z} e_2.$$

From the above we see that the manifold satisfies (2.5) for $\alpha = -\frac{1}{2z}$, $\beta = \frac{1}{2z}$, and $e_3 = \xi$. Hence the manifold is a proper trans-Sasakian manifold.

Now, we would like to characterize an almost contact curve for the manifold considered above. Let $\gamma : I \to M$ be defined by $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ fot $t \in I \subset R$. If the curve is almost contact curve,then $g(\dot{\gamma}, \xi) = 0$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$. The first condition implies

$$(\dot{\gamma_1}, \dot{\gamma_2}, \dot{\gamma_3}) \begin{pmatrix} z + y^2 & 0 & -y \\ 0 & z & 0 \\ -y & 0 & 1 \end{pmatrix} (0,\ 0,\ 1)^T = 0.$$

So $\dot{\gamma_3} = y\dot{\gamma_1}$, i.e.,

$$\dot{\gamma_3} = \gamma_2 \dot{\gamma_1} \tag{3.1}$$

Again the condition $g(\dot{\gamma}, \dot{\gamma}) = 1$ gives

$$(\dot{\gamma_1}, \dot{\gamma_2}, \dot{\gamma_3}) \begin{pmatrix} z + y^2 & 0 & -y \\ 0 & z & 0 \\ -y & 0 & 1 \end{pmatrix} (\dot{\gamma_1}, \dot{\gamma_2}, \dot{\gamma_3})^T = 0.$$

So $(\dot{\gamma_1})^2 + (\dot{\gamma_2})^2 = \frac{1}{z}$, i.e.,

$$(\dot{\gamma_1})^2 + (\dot{\gamma_2})^2 = \frac{1}{\gamma_3}. \tag{3.2}$$

Using (3.1) and (3.2) we construct following almost contact curves on this manifold.

(i) $\gamma : I \to M$ given by $\gamma(t) = (1, \frac{t}{2}, 4)$.

(ii) $\gamma : I \to M$ given by $\gamma(t) = (t, 0, 1)$.

(iii) $\gamma : I \to M$ given by $\gamma(t) = (\frac{9}{4}t^{\frac{2}{3}} + \frac{2}{3}, 1, \frac{9}{4}t^{\frac{2}{3}})$, $t > 0$.

(iv) $\gamma : I \to M$ given by $\gamma(t) = (1, \frac{t}{3}, 9)$.

(v) $\gamma : I \to M$ given by $\gamma(t) = (\sqrt{2t}, \sqrt{2t}, t)$, $t > 0$.

4. Biharmonic almost contact curves with respect to generalized Tanaka Webster Okumura connections

In this section we study biharmonic almost contact curves on a three-dimensional trans-Sasakian manifold with respect to gTWO-connections.

Definition 4.1. *An almost contact curve γ on a three-dimensional trans-Sasakian manifold is called biharmonic with respect to gTWO-connections if it satisfies the equation [15]*

$$\tilde{\nabla}^3_t t + \tilde{\nabla}_t \tilde{T}(\tilde{\nabla}_t t, t) + \tilde{R}(\tilde{\nabla}_t t, t)t = 0, \tag{4.1}$$

where $\dot{\gamma} = t$.

If instead of gTWO-connections $\tilde{\nabla}$, we take Levi-Civita connection ∇, then the above equation becomes [8]

$$\nabla^3_t t + R(\nabla_t t, t)t = 0.$$

Existence of biharmonic almost contact curves with respect to gTWO-connections has been established in the paper [15].

Let us consider a biharmonic almost contact curve with respect to gTWO-connections on a three dimensional trans-Sasakian manifold. The vector fields t and n in (2.14) to (2.16) along γ are related by $n = \phi t$ and hence $b = \xi$ along γ. Let T be a vector field on a tubular neighborhood of the image of γ with $T \circ \gamma = t$ and set $N := -\phi T$ and $B := \xi$. We take $\{T, \phi T, \xi\}$ as right handed system. Now

$$\begin{aligned}\tilde{\nabla}_t t &= \tilde{\nabla}_T T\\ &= \tilde{k}(\phi T)\\ &= -\tilde{k}N.\end{aligned}$$

By (2.10), $\tilde{T}(\tilde{\nabla}_T T, T) = 2\alpha\tilde{k}\xi$. Again, since $\tilde{\nabla}_t \xi = 0$, we get

$$\tilde{\nabla}_T \tilde{T}(\tilde{\nabla}_T T, T) = 2(\alpha\tilde{k})'\xi. \tag{4.2}$$

Then the equation (4.1) reduces to the following

$$\tilde{\nabla}^3_T T + 2(\alpha\tilde{k})'\xi - \tilde{k}\tilde{R}(N, T)T = 0. \tag{4.3}$$

In view of (2.8), we can obtain

$$\begin{aligned}\tilde{R}(N, T)T &= R(N, T)T + A(N, \nabla_T T) - A(T, \nabla_N T)\\ &+ \tilde{\nabla}_N A(T, T) - \tilde{\nabla}_T A(N, T) - A([N, T], T).\end{aligned} \tag{4.4}$$

Again in view of (2.8)

$$\nabla_T T = \tilde{\nabla}_T T - A(T, T) = \tilde{k}(\phi T) - \beta\xi.$$

Therefore, we have

$$A(N, \nabla_T T) = \tilde{k}A(\phi T, \phi T) - \beta A(\phi T, \xi) = \tilde{k}(\beta - \alpha)\xi + \alpha\beta T + \beta^2\phi T.$$

Using (2.6), from $\eta(T) = 0$ we get $\eta(\nabla_N T) = \alpha g(T, \phi N)$, by covariant differentiation with respect to Levi-Civita connection ∇. Similarly for $\eta(N) = 0$, $\eta(\nabla_T N) = -\alpha g(T, \phi N)$ Considering T as unit vector field i. e., $g(T, T) = 1$, we get

$g(\nabla_N T, T) = 0$. Since the Levi-Civita connection is torsion free $[N, T] = \nabla_N T - \nabla_T N$. Also by (2.6) and (2.8), $\tilde{\nabla}_T \xi = 0$. In the subsequent calculations we shall frequently use these results.

Using (2.8) and (2.9) we get from (4.4)

$$\begin{aligned} \tilde{R}(N,T)T &= R(N,T)T + k\beta\xi - \tilde{k}\alpha\xi + \alpha\beta T + \beta^2\phi T \\ &+ \alpha g(\nabla_N T, \phi T)\xi - \beta g(\nabla_N T, T)\xi + \tilde{\nabla}_N(\beta\xi) + \tilde{\nabla}_T(\alpha\xi) \\ &- \alpha g(\phi T, [N,T])\xi - \beta g(T,[N,T])\xi + 2l\alpha g(T,\phi N)\phi T. \end{aligned} \tag{4.5}$$

Using $g(\nabla_N T, T) = 0$ and $g(\nabla_T T, T) = 0$, the above equation can be further simplified as

$$\begin{aligned} \tilde{R}(N,T)T &= R(N,T)T + \tilde{k}\beta\xi - \tilde{k}\alpha\xi + \tilde{d}\beta(N)\xi + 3\alpha\beta T - \alpha^2\phi T \\ &+ \tilde{d}\alpha(T)\xi - \alpha g(\phi T, \nabla_T N)\xi + 2l\alpha g(T,\phi N)\phi T. \end{aligned} \tag{4.6}$$

In view of (2.7), the above equation yields

$$\begin{aligned} \tilde{R}(N,T)T &= (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))N + 3\alpha\beta T - \alpha^2\phi T \\ &+ (\tilde{d}\beta(N) + \tilde{d}\alpha(T) + \alpha g(\phi T, \nabla_N T) + \tilde{k}\beta - \tilde{k}\alpha)\xi \\ &+ 2l\alpha g(T,\phi N)\phi T. \end{aligned} \tag{4.7}$$

For $\phi T = -N$, $\phi N = T$ and $B = \xi$. the above equation reduces to

$$\begin{aligned} \tilde{R}(N,T)T &= -\alpha\beta T + (\frac{r}{2} + 2\xi\beta - 2\alpha^2 + \beta^2 - 2l\alpha)N + (\tilde{d}\beta(N) \\ &+ \tilde{d}\alpha(T) + \alpha g(N, \nabla_N T) - d\beta(N) - d\alpha(\phi T) - \tilde{k}\beta + \tilde{k}\alpha)B. \end{aligned} \tag{4.8}$$

Using (4.8), and Frenet-Serret formula in (4.3) we have

$$\begin{aligned} & (3\tilde{k}\alpha\beta + 3\tilde{k}\tilde{k}')T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2 - \tilde{k}(\frac{r}{2} + 2\xi\beta - 2\alpha^2 + \beta^2 - 2l\alpha))N \\ + \; & (2(\alpha\tilde{k})' + 2\tilde{\tau}\tilde{k}' - \tilde{k}(\tilde{d}\beta(N) + \tilde{d}\alpha(T) + \alpha g(N, \nabla_N T) + \tilde{k}\beta - \tilde{k}\alpha))B \\ = \; & 0. \end{aligned} \tag{4.9}$$

Taking inner product with T in both sides of (4.9), we get $\tilde{k}\tilde{k}' + \tilde{k}\alpha\beta = 0$. So, either $\tilde{k} = 0$ or $\tilde{k}' + \alpha\beta = 0$. Hence we obtain the following:

Theorem 4.1. *In a three-dimensional trans-Sasakian manifold of type (α, β) with gTWO connections a biharmonic almost contact curve is either a geodesic or a curve whose curvature $\tilde{k}$ satisfies $\tilde{k} + \alpha\beta = 0$.*

5. Locally ϕ-symmetric almost contact curves with respect to generalized Tanaka Webster Okumura connections.

The notion of locally ϕ-symmetric manifolds was introduced by T. Takahashi [29] in the context of Sasakian geometry. Since every smooth curve is one-dimensional differentiable manifold we may apply the concept of local ϕ-symmetry on a smooth curve. In [27], locally ϕ-symmetric Legendre curves have been studied. In the next section (Section 6), we have given a justification of the study of such curves. In the following we first give the definition of locally ϕ-symmetric trans-Sasakian manifolds and then locally ϕ-symmetric Legendre curves on it with respect to gTWO-connections.

Definition 5.1. *With respect to gTWO-connections a three-dimensional trans-Sasakian manifold will be called locally ϕ-symmetric if it satisfies*

$$\phi^2(\tilde{\nabla}_W\tilde{R})(X,Y)Z = 0 \tag{5.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 5.2. *With respect to gTWO-connections an almost contact curve γ on a three-dimensional trans-Sasakian manifold will be called locally ϕ-symmetric if it satisfies*

$$\phi^2(\tilde{\nabla}_t\tilde{R})(\tilde{\nabla}_t t, t)t = 0, \tag{5.2}$$

where $t = \dot{\gamma}$.

By definition of covariant differentiation of the Riemannian curvature tensor R of type $(1,3)$ we obtain

$$(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T = \tilde{\nabla}_T\tilde{R}(\tilde{\nabla}_T T, T)T - \tilde{R}(\tilde{\nabla}_T^2 T, T)T - \tilde{R}(\tilde{\nabla}_T T, \tilde{\nabla}_T T)T - \tilde{R}(\tilde{\nabla}_T T, T)\tilde{\nabla}_T T.$$

Using Serret-Frenet formula we get, after simplification, from above equation

$$(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T = -\tilde{k}\tilde{\nabla}_T\tilde{R}(N,T)T + \tilde{k}^2\tilde{R}(N,T)N. \tag{5.3}$$

Now

$$\tilde{R}(N,T)N = (\frac{r}{2} + 2\xi\beta - 2\alpha^2 + \beta^2 - 2l\alpha)T + (\beta - \alpha\beta)N - (\tilde{k}\alpha + \tilde{k}\beta + \beta g(\nabla_N T, N))\xi.$$

With the help of equation (4.8), we get from (5.3) and the above equation

$$\begin{aligned}
(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T &= -\tilde{k}(\tilde{d}(\frac{r}{2} \\
&+ 2\xi\beta - 2\alpha^2 + 2\beta^2 - 2l\alpha)N - d(\alpha\beta)(T)T - \alpha\beta(\tilde{k}(\phi T) - \beta\xi) \\
&- (\frac{r}{2} + 2\xi\beta - 2\alpha^2 + 2\beta^2 - 2l\alpha)(-\tilde{k}T + \alpha\xi) \\
&+ \tilde{d}(\tilde{d}\beta(N) + \tilde{d}\alpha(T) + \alpha g(N, \nabla_N T) - d\beta(N) - d\alpha(\phi T) - \tilde{k}\beta)B) \\
&- \tilde{k}^2((\frac{r}{2} + 2\xi\beta - 2\alpha^2 + \beta^2 - 2l\alpha)T + (\beta - \alpha\beta)N \\
&+ (\tilde{k}\alpha + \tilde{k}\beta + \beta g(\nabla_N T, N))\xi.
\end{aligned} \tag{5.4}$$

Applying ϕ^2 in both sides of the above equation, we have

$$\begin{aligned}
\phi^2(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T &= \tilde{k}(\tilde{d}(\frac{r}{2} + 2\xi\beta - 2\alpha^2 + 2\beta^2 - 2l\alpha)N + d(\alpha\beta)(T)T - \alpha\beta\tilde{k}N \\
&- (\frac{r}{2} + 2\xi\beta - 2\alpha^2 + 2\beta^2 - 2l\alpha)(\tilde{k}T)) \\
&- \tilde{k}^2(\frac{r}{2} + 2\xi\beta - 2\alpha^2 + 2\beta^2 - 2l\alpha)T - (\beta - \alpha\beta)N.
\end{aligned} \tag{5.5}$$

By virtue of the above equation, we can state the following:

Theorem 5.1. *With respect to gTWO-connection a necessary and sufficient condition for an almost contact curve on a three-dimensional trans-Sasakian manifold to be locally ϕ-symmetric is $\tilde{k} = 0$, that is the curve is a geodesic.*

6. Almost contact curves on Hyperbolic space with gTWO connections

The hyperbolic space $H^3(-1)$ is a trans-Sasakian manifold with $\alpha = 0$. So Theorem 5.1 is also true for that space. On the other hand Inoguchi and Lee [15] have proved that an almost contact curve on the hyperbolic space $H^3(-1)$ is biharmonic if and only if it is geodesic. Thus, Theorem 5.1 leads us to state the following:

Theorem 6.1. *With respect to gTWO connections an almost contact curve on the hyperbolic space $H^3(-1)$ is biharmonic if and only if it is locally ϕ-symmetric.*

Remark 6.1. Apparently it seems that the condition of ϕ-symmetry is an artificial condition but the power of this notion is that it makes equations (4.1) and (5.2) equivalent for Hyperbolic space. So for Hyperbolic space it relives us from the torsion of the gTWO connection to describe biharmonicity, i.e., we can characterize biharmonic almost contact curves by (5.2), where only the curvature R is required.

Now we give an examples of the hyperbolic space $H^3(-1)$ and then an almost contact curve on it.

We consider the three-dimensional manifold $M = \{(x, y, z) \in R^3, z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y}, \qquad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \qquad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z)e_3, \qquad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = -e_2, \qquad [e_1, e_3] = -e_1.$$

By Koszul formula

$$\begin{array}{lll}
\nabla_{e_1} e_3 = -e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = e_3, \\
\nabla_{e_2} e_3 = -e_2 & \nabla_{e_2} e_2 = e_3, & \nabla_{e_2} e_1 = 0, \\
\nabla_{e_3} e_3 = 0 & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0.
\end{array}$$

From above we see that the manifold satisfies (2.5) for $\alpha = 0, \beta = -1$, and $e_3 = \xi$. Hence the manifold is a trans-Sasakian manifold of type $(0, -1)$. With the help of the above results it can be verified that

$$
\begin{array}{lll}
R(e_1, e_2)e_3 = 0, & R(e_2, e_3)e_3 = -e_2, & R(e_1, e_3)e_3 = -e_1, \\
R(e_1, e_2)e_2 = -e_1, & R(e_2, e_3)e_2 = e_3, & R(e_1, e_3)e_2 = 0, \\
R(e_1, e_2)e_1 = e_2, & R(e_2, e_3)e_1 = 0, & R(e_1, e_3)e_1 = e_3.
\end{array}
$$

Also from the above expressions of the curvature tensor we observe that the manifold is a manifold of constant curvature -1. Hence the manifold is the hyperbolic space $H^3(-1)$.

Using (2.8) and (2.9), it can be easily calculated that

$$\tilde{\nabla}_{e_3} e_1 = -le_2, \qquad \tilde{\nabla}_{e_3} e_2 = le_1.$$

The other $\tilde{\nabla}_{e_i} e_j$ are 0.

Consider a curve $\gamma : I \to M$ defined by $\gamma(s) = (\sqrt{\frac{2}{3}}s, \sqrt{\frac{1}{3}}s, 1)$. Hence $\dot{\gamma}_1 = \sqrt{\frac{2}{3}}$, where $\dot{\gamma}_2 = \sqrt{\frac{1}{3}}$ and $\dot{\gamma}_3 = 0$, $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$. Now

$$\eta(\dot{\gamma}) = g(\dot{\gamma}, e_3) = g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, e_3) = 0.$$

$$
\begin{aligned}
g(\dot{\gamma}, \dot{\gamma}) &= g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, \dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3) \\
&= \dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dot{\gamma}_3^2 \\
&= \dot{\gamma}_1^2 + \dot{\gamma}_2^2 \\
&= \frac{2}{3} + \frac{1}{3} \\
&= 1.
\end{aligned}
$$

Hence the curve is an almost contact curve. For this curve $\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = 0$. Hence the curve is a geodesic with respect to gTWO-connections. So the curve is biharmonic and locally ϕ-symmetric with respect to gTWO connections by Theorem 6.1.

REFERENCES

1. D. E. Blair, *Riemannian geometry of contact and symplectic manifolds, Progress in Math.*, Vol. 203, Birkhäuser, Boston, 2002.
2. D. E. Blair and J. A. Oubina, *Conformal and related changes of metric on the product of two almost contact metric manifolds, Publications Mathematiques,* 34(1990), 199-207.
3. C. Baikoussis and D. E. Blair, *On Legendre curves in contact 3-manifolds, Geom. Dedicata,* 49(1994), 135-142.
4. D. Chinea and C. Gonzalez, *Curvature relations in trans-Sasakian manifolds, Proceedings of the XIIth Portuguese-Spanish conference on Mathematics, Vol. II(Portuguese) (Braga, 1987), 564-571, Univ. Minho, Braga,* 1987.

5. D. Chinea and C. Gonzalez, *A classification of almost contact metric manifolds, Ann. Mat. Pura Appl.,* 156(1990), 15-36.
6. J. T. Cho, *On some classes of almost contact metric manifolds, Tsukuba J. Math.,* 19(1995), 201-217.
7. J. T. Cho, J. Inoguchi and J. E. Lee, *Biharmonic curves in three-dimensional Sasakian space forms, Ann. Mat. Pura. Appl.* 186(2007), 685-701.
8. J. T. Cho and J. E. Lee, *Slant curves in contact pseudo-Hermitian manifolds, Bull. Austral. Math. Soc.,* 78(2008), 383-396.
9. J. T. Cho, J. Inoguchi and J. E. Lee, *Affine biharmonic submanifolds in 3-dimensional pseudo-Hermitian geometry, Abh math. Semin.Univ. Hambg.,* 79(2009), 113-133.
10. U. C. De and A. Sarkar, *On three-dimensional trans-Sasakian manifolds, Extracta Mathematicae,* 23(2008), 265-277.
11. U. C. De and M. M. Tripathi, *Ricci tensor in three-dimensional trans-Sasakian manifolds, Kyangpook Math. J.,* 43(2003), 247-255.
12. D. Fetcu, *Biharmonic Legendre curves in Sasakian space forms, J. Korean Math. Soc.,* 45(2008), 393-404.
13. A. Gray, and L. M. Hervella, *The sixteen classes of almost Harmite manifolds and their linear invariants, Ann. Mat, Pura Appl.*, 123(1980), 35-58.
14. J. Inoguchi, *Submanifolds with harmonic mean curvature in contact 3-manifolds, Colloq. Math.,* 100(2004), 163-179.
15. J. Inoguchi and J. E. Lee, *Affine biharmonic curves in 3-dimensional homogenious geometries, Mediterr. J. Math.,* 10(2013), 571-592.
16. D. Janssens and L. Vanheck, *Almost contact structures and curvature tensors, Kodai Math. J.,* 4(1981),1-27.
17. J. E. Lee, *On Legendre curves in contact pseudo-Hermitian* $3-$*manifolds, Bull. Aust. Math. Soc.,* 81(2010), 156-164.
18. J. C. Marrero, *The local structure of trans-Sasakian manifolds, Ann. Mat. Pura Appl.,* 162(1992), 77-86.
19. J. C. Marrero and D. Chinea, *On trans-Sasakian manifolds, Proceedings of the XIVth Spanish-Portuguese conference on Mathematics, Vol. I-III(Spanish)(Puerto de la Cruz, 1989), 655-659, Univ. La Laguna La Laguna,* 1990.
20. S. Montaldo and C. Oniciuc, *A short survey on biharmonic maps between Riemannian manifolds, Rev. Un. Mat. Argentina,* 47(2006), 1-22.
21. J. A. Oubina, *New classes of almost contact metric structures, Publ. Math. Debrecen,* 32(1985), 187-193.
22. M. Okumura, *Some remarks on spaces with a certain contact structures, Tohoku Math. Journal,* 14(1962), 135-145.
23. C. Özgur and S. Güvenc, *On some types of slant curves in contact pseudo-Hermitian three manifolds, Ann. Polon. Math.,* 104(2012), 217-228.
24. C. Özgur and S. Güvenc, *On some classes of biharmonic Legendre curve in generalized Sasakian space forms, Collect. Math.* 65(2014), 203-218.
25. S. Sasaki and Y. Hatakeyama, *On differentiable manifolds with certain structures which are closely related to almost contact structures II, Tohoku Math. J.* 13(1961), 281-294.

26. A. Sarkar, S. K. Hui and M. Sen, *A study on Legendre curves in 3-dimensional trans-Sasakian manifolds, Lobachevskii, J. Math.,* 35(2014), 11-18.

27. A. Sarkar and D. Biswas, *Legendre curves in three-dimensional Heisenberg group, Facta Univ.,* 28(2013), 241-248.

28. N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebra and Cartan connections, Japan J. Math.,* 2(1976), 131-190.

29. T. Takahashi, Sasakian ϕ-symmetric spaces, Tohoku Math J., 29(1977), 91-113.

30. S. Tanno, *Variational problems on contact Riemannian manifolds, Trans. Amer. Math. Soc.,* 314(1989), 349-379.

31. J. Welyczko, *On Legendre curves in 3-dimensional normal almost contact metric manifolds, Soochow J. Math.,* 33(2007), 929-937.

32. S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface, J. Diff. Geom.,* 13(1978), 25-41.

A. Sarkar and Ashis Mondal
Department of Mathematics
University of Kalyani
741235
Nadia
West Bengal
India
avjaj@yahoo.co.in