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SOME SPECIAL SPACELIKE CURVES IN R_2^4

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Abstract. In this study, we define spacelike curves in R_2^4 and characterize such curves in terms of Frenet frame. Also, we examine some special spacelike curves of R_2^4, taking into account their curvatures. In addition, we study spacelike slant helices, spacelike B_2 slant helices in R_2^4. And then we obtain an approximate solution for spacelike-B_2 slant helix.
Key words: spacelike curves, slant helices, approximate solution.

1. Introduction

The curves are the common denominator of many different vital necessities such as nature, art, technology and science. It is geometrically important to describe the behavior of the curve in the vicinity a point on the curve. For this, we introduce a frame of mutually orthogonal vectors and curvatures. Thanks to these curvatures and frames that are shaped differently in different spaces, the curves become special. For example, "if all the curvatures $\kappa_r(s)$, $(r = 1, ..., n-1)$ of the curve nowhere vanish in $I \subset R$, then the curve is called a non-degenerate curve in E^n" or "a helix in E^3 is a curve whose tangent vector make a constant angle with a fixed direction" [20]. On the other hand, the curves are generally, presented in parametric format, and arc-length parameter is preferred as the parameter in theoretical treatments because of its simplicity of expression. For practical uses, the parameter is changed from arc length s to a more manageable variable parameter t, which monotonically increases with arc length [10].

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Izumiya and Takeuchi obtained a characterization of slant helices, which are the basis of our study, in E^3 [12]. Kula and Yaylı worked on spherical images of the slant helices [15]. Önder et al. presented a new type of the slant helices in Euclidean 4-space and called it the B_2-slant helix [18]. In 2009, Gök et al. transported the slant helices of E^3 to E^n, $n > 3$, which they called them V_n slant helix and obtained some characterizations of V_n-slant helix in E^n [9]. The studies have been carried out on the spacelike W-curve in E_1^3, the ccr-curves in E^n and the spherical curves [11], [19]. On the other hand, different approximate solution methods based on matrices for differential equations characterizing special curves were presented by Aydın et al [7, 8]. In addition, the issue of investigating the existence of solutions of different types of equations is still up to date [3, 6, 16].

There are many studies about the special curves we have discussed in this study, but there is no study done in R_2^4 and according to the frame we use [13]. The study is important in this respect.

2. Preliminaries

This section contains the definitions and terms that will be used in the following parts of the study.

Let $\gamma : I \subset R \to E^m$ be a regular curve, $(\|\gamma'\| \neq 0)$. Then γ is called a Frenet curve of osculating order k; $(2 \leq k \leq m)$ if γ', γ'',..., $\gamma^{(k)}$ are linearly independent and γ', γ'',..., $\gamma^{(k+1)}$ linearly dependent [22]. In this case, $Im(\gamma)$ lies in an k-dimensional Euclidean subspace of E^m. If $k = m$ the Frenet curve γ is called a generic curve [22, 5].

Definition 2.1. A generic curve in E^4 for which $\kappa_1, \kappa_2, \kappa_3$ are constant is called W-curve or (generalized) helix in E^4 [14].

Definition 2.2. A slope curve in E^4 is the curve that satisfies the relations $\frac{\kappa_2}{\kappa_1} = \alpha$ and $\frac{\kappa_3}{\kappa_1} = \beta$ for the curvatures $\kappa_1 \neq 0$, κ_2 and κ_3, where α and β are nonzero constants [4].

Definition 2.3. A curve $\gamma : I \to E^m$ has constant curvature ratios (ccr-curve) if all the quotients $\frac{\kappa_{i+1}}{\kappa_i}$ are constant [17]. Frenet curve of rank 4 with constant curvature ratios is called a ccr-curve in E^4 (see, [19]). We remark that a regular curve in E^4 is a ccr-curve if $\frac{\kappa_2}{\kappa_1}$ and $\frac{\kappa_3}{\kappa_2}$ are constant functions.

A regular curve in E^n is said to have constant curvature ratios if the ratios of the consecutive curvatures are constant [17].

Definition 2.4. A regular curve $\gamma = \gamma(s) : I \subset R \to E^n$ is named spherical if it lies on a $(n-1)$-sphere $S^{(n-1)}$ of R^n [5].

Definition 2.5. If the tangent vector T of a curve makes a fixed angle with a unit vector U of E^4 then this curve is named a general helix (or inclined curve) in E^4 [19].

Definition 2.6. A unit speed curve $\gamma : I \to E^4$ is called slant helix if its unit principal normal vector N makes a constant angle with a fixed direction [1].

R_2^4, 4-dimensional semi-Euclidean space with index 2 is the standart vector space equipped with an indefinite flat metric $\langle , \rangle$ defined by

$$\langle , \rangle = da_1^2 + da_2^2 - da_3^2 - da_4^2,$$

where (a_1, a_2, a_3, a_4) is a rectangular coordinate system of R_2^4. A vector w in R_2^4 is called a timelike, spacelike or null (lightlike) if respectively hold $\langle w, w\rangle < 0$, $\langle w, w\rangle > 0$ or $\langle w, w\rangle = 0$ and $w \neq 0$. The norm of a vector w is defined by $\|w\| = \sqrt{|\langle w, w\rangle|}$. If $\langle w, v\rangle = 0$ then the vectors w and v are orthogonal.

An arbitrary curve $\gamma : I \to R_2^4$ can locally be timelike, spacelike or null if respectively all of its velocity vectors $\gamma'(s)$ are timelike, spacelike or null.

Let w and v be two spacelike vectors and let θ be the angel between these vectors in R_2^4.

- If $S_p\{v, w\}$ is a spacelike subspace, $\langle w, v\rangle = \|w\| \, \|v\| \cos\theta$.
- If $S_p\{v, w\}$ is a timelike subspace, $\langle w, v\rangle = \|w\| \, \|v\| \cosh\theta$.

Let w be timelike vector, v be spacelike vector. In this case, $|\langle w, v\rangle| = \|w\| \, \|v\| \sinh\theta$. Let $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame along the curve $\gamma(s)$ in R_2^4. Then T, N, B_1, B_2 are the tangent, the principal normal, the first binormal and the second binormal fields, respectively and let $\nabla_T T$ is spacelike.

Let γ be a spacelike curve in R_2^4, parametrized with arclength function s. Let the vector N be spacelike, B_1 and B_2 timelike. In this case there exists only one Frenet frame $\{T, N, B_1, B_2\}$ for which $\gamma(s)$ is a spacelike curve with Frenet equations

$$\begin{aligned} \nabla_T T &= \kappa_1 N \\ \nabla_T N &= -\kappa_1 T + \kappa_2 B_1 \\ \nabla_T B_1 &= \kappa_2 N + \kappa_3 B_2 \\ \nabla_T B_2 &= -\kappa_3 B_1, \end{aligned} \tag{2.1}$$

where the vectors T, N, B_1, B_2 satisfy the equations:

$$\langle T, T\rangle = \langle N, N\rangle = 1, \langle B_1, B_1\rangle = \langle B_2, B_2\rangle = -1. \tag{2.2}$$

The functions $\kappa_i(s)$, $(1 \leq i \leq 3)$ are called the curvatures of the spacelike curve $\gamma(s)$ [2].

Definition 2.7. Let's consider the differential equation below

$$\sum_{k=0}^{m} P_k(s) y^k(s) = g(s), (a \leq s \leq b).$$

Obviously this is m. order, linear, variable coefficient differential equation. Also, the functions are differentiable functions in the range $a \leq s \leq b$. The Morgan-Voyce polynomial method is developed to find approximate solutions of this equation under certain initial or boundary conditions. Accordingly, the approximate solution can be expressed with Morgan-Voyce polynomials as follows:

$$y(s) \cong y_N(s) = p_N(s) = \sum_{n=0}^{N} a_n B_n(s), (N \geq m).$$

Here, the coefficients a_n are defined as Morgan-Voyce polynomial coefficients that must be found. The basis of this method is based on the reduction of the unknown function $y(s)$ to an algebraic system with Morgan-Voyce coefficient a_n. For this reduction process, the matrix form of the function $y(s)$ and the collocation points

$$s_i = a + \frac{b-a}{N} i, (i = 0, 1, ..., N)$$

are used. Thus, the problem of finding the approximate solutions of a given differential equation or other functional equations becomes the problem of finding the solution of an algebraic matrix equation. Also, n. order Morgan-Voyce polynomials are expressed as

$$B_n(s) = \sum_{j=0}^{n} \begin{pmatrix} n+j+1 \\ n-j \end{pmatrix} s^j$$

or recursively as $B_n(s) = (s+2)B_{n-1}(s) - B_{n-2}(s)$, $n \geq 2$ [21].

3. The Spacelike Curves in R_2^4

In this chapter, we give definitions and characterizations of the spacelike curves by using Frenet frame in R_2^4.

Theorem 3.1. *Let $\gamma : I \to R_2^4$ be a curve parameterized by arclength. Then, the curve γ is the spacelike curve, if*

$$\nabla_T^4 T + \lambda_3 \nabla_T^3 T + \lambda_2 \nabla_T^2 T + \lambda_1 \nabla_T T + \lambda_0 T = 0. \tag{3.1}$$

The coefficient functions $\lambda_i(s)$, $(0 \leq i \leq 3)$ are as follows:

$$\begin{aligned}
\lambda_0 &= \kappa_1\kappa_2\kappa_3 \left[\frac{1}{\kappa_3}(\frac{\kappa_1}{\kappa_2})'\right]' + \kappa_1^2\kappa_3^2 \\
\lambda_1 &= \kappa_1\kappa_2\kappa_3 \left\{\left[\frac{1}{\kappa_3}\left[\frac{1}{\kappa_2}(\frac{1}{\kappa_1})'\right]'\right]' - (\frac{\kappa_2}{\kappa_1\kappa_3})' + (\frac{\kappa_1}{\kappa_2\kappa_3})'\right\} + \kappa_1\kappa_2(\frac{\kappa_1}{\kappa_2})' \\
&\quad + \kappa_1\kappa_3^2(\frac{1}{\kappa_1})' \\
\lambda_2 &= \kappa_1\kappa_2\kappa_3 \left\{\left[\frac{1}{\kappa_3}(\frac{1}{\kappa_1\kappa_2})'\right]' + \left[\frac{1}{\kappa_2\kappa_3}(\frac{1}{\kappa_1})'\right]'\right\} + \kappa_1\kappa_2 \left[\frac{1}{\kappa_2}(\frac{1}{\kappa_1})'\right]'
\end{aligned}$$

$$+\kappa_1^2 - \kappa_2^2 + \kappa_3^2$$

$$(3.2)\quad \lambda_3 = \kappa_1\kappa_2\kappa_3\left(\frac{1}{\kappa_1\kappa_2\kappa_3}\right)' + \kappa_1\kappa_2\left(\frac{1}{\kappa_1\kappa_2}\right)' + \kappa_1\left(\frac{1}{\kappa_1}\right)'.$$

Proof. By using the first of the equations (2.1) we have

$$\begin{aligned} N &= \frac{1}{\kappa_1}\nabla_T T \\ B_1 &= \frac{\kappa_1}{\kappa_2}T + \frac{1}{\kappa_2}\nabla_T N \\ B_2 &= -\frac{\kappa_2}{\kappa_3}N + \frac{1}{\kappa_3}\nabla_T B_1. \end{aligned} \tag{3.3}$$

From the first of the equations (3.3) $\nabla_T N = \frac{1}{\kappa_1}\nabla_T^2 T + (\frac{1}{\kappa_1})'\nabla_T T$, and so we get

$$B_1 = \frac{1}{\kappa_1\kappa_2}\nabla_T^2 T + \frac{1}{\kappa_2}\left(\frac{1}{\kappa_1}\right)'\nabla_T T + \frac{\kappa_1}{\kappa_2}T. \tag{3.4}$$

And then we calculate the expression $\nabla_T B_1$. With similar thinking, by using the equations we found, we get B_2 and $\nabla_T B_2$. Finally, we use the equality 3.4 and the expression $\nabla_T B_2$ in the last equality of Frenet equations (2.1). Thus the proof is complete. □

Corollary 3.1. *The equation* (3.1) *is the differential equation characterizes the spacelike curves according to the tangent T field in R_2^4. Similarly, the spacelike curves can be characterized according to the N, B_1 and B_2.*

4. The Special Spacelike curves in R_2^4

Theorem 4.1. *Let $\gamma = \gamma(s) : I \subset R \to R_2^4$ be a regular spacelike curve parametrized by arc length s. Then, the curve γ is a spacelike W- curve or spacelike helix with $\nabla_T T$ spacelike if the equality*

$$\nabla_T^4 T + (\kappa_1^2 - \kappa_2^2 + \kappa_3^2)\nabla_T^2 T + (\kappa_1^2\kappa_3^2)T = 0$$

holds.

Proof. A spacelike curve $\gamma : I \to R_2^4$ parameterized by arc length provides the differential equation (3.1) in R_2^4. Since the curve γ is (generalized) helix or W-curve for which $\kappa_1, \kappa_2, \kappa_3$ are constant, with the help of the equations (3.2) the equalities

$$\begin{aligned} \lambda_0 &= \kappa_1^2\kappa_3^2 \\ \lambda_2 &= \kappa_1^2 - \kappa_2^2 + \kappa_3^2 \end{aligned}$$

and $\lambda_1 = \lambda_3 = 0$ are obtained. □

Theorem 4.2. *Let γ be a regular spacelike curve parameterized by arclength in R_2^4. Then γ is a spacelike slope curve if*

$$\nabla_T^4 T - \frac{6\kappa_1'}{\kappa_1}\nabla_T^3 T + \lambda_2\nabla_T^2 T + \lambda_1\nabla_T T + \kappa_1^2\kappa_3^2 T = 0$$

holds, where α and β are nonzero constant and the coefficient functions $\lambda_i(s), (i = 1, 2)$ are as follows:

$$\begin{aligned}\lambda_1 &= -\frac{\kappa_1'''}{\kappa_1} + \frac{10\kappa_1''\kappa_2'}{\kappa_1\kappa_2} - \frac{15\kappa_1'\kappa_2'\kappa_3'}{\kappa_1\kappa_2\kappa_3} + (\alpha^2 - \beta^2 - 1)\kappa_1\kappa_1' \\ \lambda_2 &= -\frac{4\kappa_1''}{\kappa_1} + \frac{15\kappa_1'\kappa_2'}{\kappa_1\kappa_2} - (\alpha^2 - \beta^2 - 1)\kappa_1^2.\end{aligned}$$

Proof. A spacelike curve $\gamma : I \to R_2^4$ parameterized by arc length provides the differential equation (3.1) in R_2^4. Since the curve γ is slope curve for which the curvatures $\kappa_1 \neq 0$, κ_2 and κ_3 satisfy the relations $\frac{\kappa_2}{\kappa_1} = \alpha$ and $\frac{\kappa_3}{\kappa_1} = \beta$. Thus, with the help of the equations (3.2), λ_1 and λ_2 are obtained. □

Theorem 4.3. *Let $\gamma = \gamma(s) : I \subset R \to R_2^4$ be a unit speed spacelike curve. Then, the curve γ is the constant curvature ratios (ccr-curve) spacelike curve if*

$$\nabla_T^4 T - \frac{6\kappa_1'}{\kappa_1}\nabla_T^3 T + \lambda_2\nabla_T^2 T + \lambda_1\nabla_T T + \kappa_1^2\kappa_3^2 T = 0$$

holds, where the coefficient functions $\lambda_i(s), (i = 1, 2)$ are as follows:

$$\begin{aligned}\lambda_1 &= -\frac{\kappa_1'''}{\kappa_1} + \frac{10\kappa_1''\kappa_2'}{\kappa_1\kappa_2} - \frac{15\kappa_1'\kappa_2'\kappa_3'}{\kappa_1\kappa_2\kappa_3} - (\kappa_1\kappa_1' - \kappa_2\kappa_2' + \kappa_3\kappa_3') \\ \lambda_2 &= -\frac{4\kappa_1''}{\kappa_1} + \frac{15\kappa_1'\kappa_2'}{\kappa_1\kappa_2} + \kappa_1^2 - \kappa_2^2 + \kappa_3^2.\end{aligned}$$

Proof. A spacelike curve $\gamma : I \to R_2^4$ parameterized with arc length provides the differential equation (3.1) in R_2^4. Since the curve γ is the constant curvature ratios (ccr-curve) space curve, the $\frac{\kappa_2}{\kappa_1}$ and $\frac{\kappa_3}{\kappa_2}$ are constant functions. Thus, with the help of the equations (3.2), λ_1 and λ_2 are obtained. □

Theorem 4.4. *A curve $\gamma = \gamma(s) : I \subset R \to R_2^4$ is spacelike spherical, i.e., it is contained in a sphere of radius r, if*

$$\begin{aligned}&\rho''' + [\kappa_3(\frac{1}{\kappa_2\kappa_3})' + (\frac{1}{\kappa_2})']\kappa_2\rho'' \\ &+[\kappa_2\kappa_3(\frac{1}{\kappa_3}(\frac{1}{\kappa_2})')' + \kappa_3^2 - \kappa_2^2]\rho' + \kappa_2\kappa_3(\frac{\kappa_2}{\kappa_3})'\rho = 0, \qquad (4.1)\end{aligned}$$

where $\rho = \frac{1}{\kappa_1}$.

Proof. The proof here is similar to that for spherical curves in R^4. It consists of obtaining information thanks to successive derivatives of the expression $<\gamma(s)-m, \gamma(s)-m>=r^2$, where m is the center of the sphere. □

Lemma 4.1. *A curve $\gamma=\gamma(s): I\subset R\rightarrow R^4_2$ is spacelike spherical constant curvature ratios (ccr-curve), i.e., it is contained in a sphere of radius r, if*

$$\rho'''+[\kappa_3(\frac{1}{\kappa_2\kappa_3})'+(\frac{1}{\kappa_2})']\kappa_2\rho''+[\kappa_2\kappa_3(\frac{1}{\kappa_3}(\frac{1}{\kappa_2})')'+\kappa_3^2-\kappa_2^2]\rho'=0. \tag{4.2}$$

Proof. In this case, the equation (4.1) is rearranged by taking $\frac{\kappa_2}{\kappa_1}$ and $\frac{\kappa_3}{\kappa_2}$ as constant functions. Thus, the equality (4.2) is obtained. □

Lemma 4.2. *A curve $\gamma=\gamma(s): I\subset R\rightarrow R^4_2$ is spacelike spherical slope curve, if*

$$\rho^2\rho'''+4\rho\rho'\rho''+(\rho')^3+(\beta^2-\alpha^2)\rho'=0. \tag{4.3}$$

Proof. In this case, we can rewrite (4.1) in terms of curvature, κ_1, $\kappa_2=\alpha\kappa_1$ and $\kappa_3=\beta\kappa_1$, where α, β are constants. Thus, the equality (4.3) is obtained. □

5. The Spacelike Slant Helix in R^4_2

Theorem 5.1. *Let $\gamma: I\rightarrow R^4_2$ be a regular spacelike curve given with arc-length parameter s and $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame at the point $\gamma(s)$ of the curve γ. If the curve γ is a spacelike slant helix, their position vector satisfies the equation*

$$\begin{aligned}\frac{\kappa_2^2-\kappa_1^2}{\kappa_1\kappa_2\kappa_3}\mu_1''+[(\frac{\kappa_2}{\kappa_1\kappa_3})'-\frac{1}{\kappa_3}(\frac{\kappa_1}{\kappa_2})'-(\frac{\kappa_1}{\kappa_2\kappa_3})']\mu_1'\\ -[(\frac{1}{\kappa_3}(\frac{\kappa_1}{\kappa_2})')'+\frac{\kappa_1\kappa_3}{\kappa_2}]\mu_1=0,\end{aligned} \tag{5.1}$$

where μ_1 is the coefficient function of the tangent of a spacelike constant vector taken in the fixed direction studied.

Proof. We call γ as spacelike slant helix if its principial normal vector makes a constant angle with a fixed direction. From this definition of the slant helix we write

$$<N, U>=constant. \tag{5.2}$$

where U is a spacelike constant vector and we can compose U as

$$U=\mu_1T+\mu_2N+\mu_3B_1+\mu_4B_2. \tag{5.3}$$

The coefficient functions are $\mu_1=<T, U>, \mu_2=<N, U>, \mu_3=-<B_1, U>, \mu_4=-<B_2, U>$. Because the vector U is constant, differentiation of the equation 5.3 and considering Frenet equations, we have

$$\begin{aligned}(\mu_1'-\kappa_1\mu_2)T+(\kappa_1\mu_1+\mu_2'+\kappa_2\mu_3)N+(\kappa_2\mu_2+\mu_3'-\kappa_3\mu_4)B_1\\ +(\kappa_3\mu_3+\mu_4')B_2=0.\end{aligned} \tag{5.4}$$

Also, the function μ_2 is constant from the equality 5.2, and so $\mu_2'(s) = 0$ for all s. Then we find the following system

$$
\begin{aligned}
\mu_1' - \kappa_1\mu_2 &= 0 \\
\kappa_1\mu_1 + \kappa_2\mu_3 &= 0 \\
\kappa_2\mu_2 + \mu_3' - \kappa_3\mu_4 &= 0 \\
\kappa_3\mu_3 + \mu_4' &= 0.
\end{aligned} \tag{5.5}
$$

From the third equation of the system of equation 5.5 $\mu_4 = \frac{\kappa_2}{\kappa_3}\mu_2 + \frac{1}{\kappa_3}\mu_3'$, and so we get

$$
[\frac{\kappa_2}{\kappa_3}\mu_2 + \frac{1}{\kappa_3}\mu_3']' = -\kappa_3\mu_3. \tag{5.6}
$$

By using the equalities $\mu_2 = \frac{1}{\kappa_1}\mu_1'$ and $\mu_3 = -\frac{\kappa_1}{\kappa_2}\mu_1$ in the equation 5.6, we obtain the equation 5.1 . Thus the proof is completed. □

Corollary 5.1. *The equation* (5.1) *is the differential equation characterizes the spacelike slant helix according to the coefficient function* μ_1 *in* R_2^4*. Obviously, the spacelike slant helix can be characterized similarly according to the other coefficient functions* μ_3 *and* μ_4*,but, since* μ_2 *is already fixed, a characterization based on* μ_2 *cannot be given.*

Theorem 5.2. *Let* $\gamma : I \to R_2^4$ *be a regular spacelike curve given by arc-length parameter s and* $\{T(s), N(s), B_1(s), B_2(s)\}$ *be the moving Frenet frame at the point* $\gamma(s)$ *of the curve* γ*. If the curve* γ *is a spacelike slant helix, their position vector satisfy the equations*

$$
\begin{aligned}
\frac{\kappa_2^2 - \kappa_1^2}{\kappa_1^2\kappa_3}\mu_3'' + [(\frac{\kappa_2^2}{\kappa_1^2\kappa_3})' + \frac{\kappa_2}{\kappa_3\kappa_1}(\frac{\kappa_2}{\kappa_1})' - (\frac{1}{\kappa_3})']\mu_3' - \{\left[\frac{\kappa_2}{\kappa_3\kappa_1}(\frac{\kappa_2}{\kappa_1})'\right]' - \kappa_3\}\mu_3 &= 0, \\
\frac{\kappa_1^2 - \kappa_2^2}{\kappa_1^2\kappa_3}\mu_4'' + [\frac{1}{\kappa_3}(\frac{1}{\kappa_3})' - \frac{\kappa_2}{\kappa_1\kappa_3}(\frac{\kappa_2}{\kappa_1\kappa_3})']\mu_4' + \mu_4 &= 0,
\end{aligned}
$$

where μ_3 *and* μ_4 *are the coefficient functions of the first binormal* B_1 *and the second binormal* B_2*, respectively, of a spacelike constant vector taken in the fixed direction studied.*

Proof. It is obvious from proof of Theorem 5.1. □

6. The Spacelike-B_1 Slant Helix in R_2^4

Theorem 6.1. *Let* $\gamma : I \to R_2^4$ *be a regular spacelike curve given by arc-length parameter s and* $\{T(s), N(s), B_1(s), B_2(s)\}$ *be the moving Frenet frame at the point* $\gamma(s)$ *of the curve* γ*. If the curve* γ *is a spacelike-*B_1 *slant helix, their position vector satisfy the equation*

$$
\frac{\kappa_3^2 - \kappa_2^2}{\kappa_1^2\kappa_3^3}\mu_1'' + [\frac{1}{\kappa_1}(\frac{1}{\kappa_1})' - \frac{\kappa_2}{\kappa_1\kappa_3}(\frac{\kappa_2}{\kappa_1\kappa_3})']\mu_1' + \mu_1 = 0, \tag{6.1}
$$

where μ_1 is the coefficient function of the tangent of a spacelike constant vector taken in the fixed direction studied.

Proof. We call γ as B_1 slant helix if its first binormal vector makes a constant angle with a fixed direction. From this definition of the B_1 slant helix we can write

$$<B_1, U>= constant, \tag{6.2}$$

where U is a spacelike constant vector and we can compose U as

$$U = \mu_1 T + \mu_2 N + \mu_3 B_1 + \mu_4 B_2. \tag{6.3}$$

The coefficient functions are $\mu_1 =< T, U >, \mu_2 =< N, U >, \mu_3 = - < B_1, U > , \mu_4 = - < B_2, U >$ in R_2^4. Because the vector U is constant, differentiation of the equation 6.3 and considering Frenet equations, we have

$$\begin{aligned} (\mu_1' - \kappa_1\mu_2)T + (\kappa_1\mu_1 + \mu_2' + \kappa_2\mu_3)N + (\kappa_2\mu_2 + \mu_3' - \kappa_3\mu_4)B_1 \\ +(\kappa_3\mu_3 + \mu_4')B_2 = 0. \end{aligned} \tag{6.4}$$

Also, the function μ_3 is constant from the equality 6.2, and so $\mu_3'(s) = 0$ for all s. Then we find the following system of ordinary differential equations

$$\begin{aligned} \mu_1' - \kappa_1\mu_2 &= 0 \\ \kappa_1\mu_1 + \mu_2' + \kappa_2\mu_3 &= 0 \\ \kappa_2\mu_2 - \kappa_3\mu_4 &= 0 \\ \kappa_3\mu_3 + \mu_4' &= 0. \end{aligned} \tag{6.5}$$

From the second equation of this system of equation, the equality

$$\mu_3 = -\frac{\kappa_1}{\kappa_2}\mu_1 - \frac{1}{\kappa_2}\mu_2' \tag{6.6}$$

is obtained. By using the equalities $\mu_2 = \frac{1}{\kappa_1}\mu_1'$, $\mu_3 = -\frac{1}{\kappa_3}\mu_4'$ and $\mu_4 = -\frac{\kappa_2}{\kappa_3}\mu_2$ in the equation 6.6, we obtain the equation 6.1. Thus the proof is completed. □

Corollary 6.1. *The equation* (6.1) *is the differential equation characterizes the spacelike-B_1 slant helix according to the coefficient function μ_1 in R_2^4. Obviously, the spacelike-B_1 slant helix can be characterized similarly according to the other coefficient functions μ_2 and μ_4,but, since μ_3 is already fixed, a characterization based on μ_3 cannot be given.*

Theorem 6.2. *Let $\gamma : I \to R_2^4$ be a regular spacelike curve given by arc-length parameter s and $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving Frenet frame at the point $\gamma(s)$ of the curve γ. If the curve γ is a spacelike-B_1 slant helix, their position vector satisfy the equations*

$$\begin{aligned} \frac{\kappa_2^2+\kappa_3^2}{\kappa_1\kappa_3^2}\mu_2'' + [(\frac{\kappa_2^2}{\kappa_1\kappa_3^2})' + \frac{\kappa_2}{\kappa_1\kappa_3}(\frac{\kappa_2}{\kappa_3})' + (\frac{1}{\kappa_1})']\mu_2' + \{\left[\frac{\kappa_2}{\kappa_3\kappa_1}(\frac{\kappa_2}{\kappa_3})'\right]' - \kappa_1\}\mu_2 &= 0, \\ \frac{\kappa_2^2-\kappa_3^2}{\kappa_1\kappa_2\kappa_3}\mu_4'' + [(\frac{\kappa_2}{\kappa_1\kappa_3})' - (\frac{\kappa_3}{\kappa_1\kappa_2})' - \frac{1}{\kappa_1}(\frac{\kappa_3}{\kappa_2})']\mu_4' - \{\left[\frac{1}{\kappa_1}(\frac{\kappa_3}{\kappa_2})'\right]' + \frac{\kappa_1\kappa_3}{\kappa_2}\}\mu_4 &= 0, \end{aligned}$$

where μ_2 and μ_4 are the coefficient functions of the principal normal N and the second binormal B_2, respectively, of a spacelike constant vector taken in the fixed direction studied.

Proof. It is obvious from proof of Theorem 6.1. □

7. Approximate solution with Morgan-Voyce polynomial approach

In this section, approximate solution of the differential equation 6.1 that characterizes the spacelike-B_1 slant helix based on the coefficient μ_1, will be obtained by the Morgan-Voyce polynomial approximation. Similar solution can be applied for characterizations linked to the coefficients μ_2 and μ_4.

Firstly, the differential equation (6.1) is generally expressed as follows:

$$\sum_{k=0}^{2} P_k(s)y^{(k)}(s) = g(s), \tag{7.1}$$

for the coefficient functions

$$P_2(s) = \frac{\kappa_3^2 - \kappa_2^2}{\kappa_1^2\kappa_3^3}, P_1(s) = \frac{1}{\kappa_1}(\frac{1}{\kappa_1})' - \frac{\kappa_2}{\kappa_1\kappa_3}(\frac{\kappa_2}{\kappa_1\kappa_3})', P_0(s) = 1, y(s) = \mu_1(s), g(s) = 0.$$

Suppose that this equation has an approximate solution in $(0 \leq s \leq 1)$, under the initial conditions $y^{(k)}(0) = \omega_k$, $(k = 0, 1)$, in the form of Morgan-Voyce polynomials as

$$y(s) = \sum_{n=0}^{N} a_n B_n(s). \tag{7.2}$$

Let $N = 3$ for convenience. Here, P_k and g functions are known functions and ω is suitable constant, a_n are unknown coefficients, B_n are Morgan-Voyce polynomials. The first four of the Morgan-Voyce polynomials are as follows:

$$B_0(s) = 1, B_1(s) = s + 2, B_2(s) = s^2 + 4s + 3, B_3(s) = s^3 + 6s^2 + 10s + 4.$$

Basic matrix relations First of all, the approximate solution can be converted into matrix form $y(s) = B(s)A$, with

$$B(s) = \begin{bmatrix} B_0(s) & B_1(s) & B_2(s) & B_3(s) \end{bmatrix}, A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}$$

for $N = 3$. On the other hand, there is a matrix relation $B(s) = S(s)R^T$ for

$$S(s) = \begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 10 & 6 & 1 \end{bmatrix}$$

from the definition of polynomial [21]. Also, it is clearly seen that the relation between the matrix $B(s)$ and its derivative $B'(s)$ is $B'(s) = S'(s)R^T$ and that repeating the process $B^{(k)}(s) = S(s)(T^T)^k R^T$, where

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

and T^0 is unite matrix. $y^{(k)}(s) = S(s)(T^T)^k R^T A$ are obtained with the help of these matrices. Also, the matrix relations of the differential part are obtained in the form $\sum_{k=0}^{2} P_k Y^{(k)} = G$ by using standard collocation points $s_i = \frac{1}{3}i$ $(i = 0, 1, 2, 3)$, in the equation 7.1, in the range of $0 \leq s \leq 1$ for $N = 3$. The matrices

$$\begin{aligned} P_k &= diag[\ P_k(0) \quad P_k(\tfrac{1}{3}) \quad P_k(\tfrac{2}{3}) \quad P_k(1)\ ], \\ Y^{(k)} &= [\ y^{(k)}(0) \quad y^{(k)}(\tfrac{1}{3}) \quad y^{(k)}(\tfrac{2}{3}) \quad y^{(k)}(1)\ ]^T \end{aligned}$$

are obvious and the matrix $W = \sum_{k=0}^{2} P_k S(s)(T^T)^k R^T$ is calculated, for $WA = G$ and the equation is written as the augmented matrix $[W; G]$.

Matrix calculations for initial conditions Under the initial conditions given as $y(0) = 0$, $y'(0) = 1$ the matrix expression of the conditions is calculated as

$$U_0 = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}, U_1 = \begin{bmatrix} 0 & 1 & 4 & 10 \end{bmatrix}.$$

The Solution If the matrix form of conditions is used in the matrix form $[W; G]$ the following matrix is obtained:

$$[W^*; G^*] = \begin{bmatrix} 1 & \zeta_{01} & \zeta_{02} & \zeta_{03} & ; & 0 \\ 1 & 2 & 3 & 4 & ; & 0 \\ 0 & 1 & 4 & 10 & ; & 1 \\ 1 & \zeta_{31} & \zeta_{32} & \zeta_{33} & ; & 0 \end{bmatrix},$$

where

$$\begin{aligned} \zeta_{01} &= 2 + P_1(0), \\ \zeta_{02} &= 3 + 4P_1(0) + 2P_2(0), \\ \zeta_{03} &= 4 + 10P_1(0) + 12P_2(0), \\ \zeta_{31} &= 3 + P_1(1), \\ \zeta_{32} &= 8 + 6P_1(1) + 2P_2(1), \\ \zeta_{33} &= 21 + 25P_1(1) + 18P_2(1). \end{aligned}$$

Finally, with the help of equality $A = (W^*)^{-1}G$, the unknowns a_n are calculated as follows:

$$\begin{aligned}
a_0 &= -\frac{8P_2(0)[2P_1(1) - 3P_2(1) + 3] - 2P_1(0)[52P_2(1) + 47P_1(1) + 25]}{-P_1(0)[40P_1(1) + 52P_2(1) + 18] - P_2(0)[6P_1(1) + 12P_2(1) + 2]} \\
a_1 &= \frac{2P_2(0)[11P_1(1) - 6P_2(1) + 13] - 2P_1(0)[26P_2(1) + 9 + 20P_1(1)]}{-P_1(0)[40P_1(1) + 52P_2(1) + 18] - P_2(0)[6P_1(1) + 12P_2(1) + 2]} \\
a_2 &= -\frac{[10P_1(0) + 12P_1(1)][P_2(0) + 1]}{-P_1(0)[40P_1(1) + 52P_2(1) + 18] - P_2(0)[6P_1(1) + 12P_2(1) + 2]} \\
a_3 &= \frac{[4P_1(0) + 2P_1(1)][P_2(0) + 1]}{-P_1(0)[40P_1(1) + 52P_2(1) + 18] - P_2(0)[6P_1(1) + 12P_2(1) + 2]}.
\end{aligned}$$

If these values are substituted in the equation 7.2, the solution is obtained as follows:

$$y(s) = \mu_1(s) = a_0 + a_1(s + 2) + a_2(s^2 + 4s + 3) + a_3(s^3 + 6s^2 + 10s + 4).$$

Corollary 7.1. *The equations found for the special curves we study are generally homogeneous, linear differential equations with variable coefficients. So the solution method we present can be applied to other equations as well.*

Example 7.1. Let's find the coefficient μ_1 for the spacelike-B_1 slant helix given with its curvatures $\kappa_1 = \frac{1}{s+1}$, $\frac{\kappa_2}{\kappa_3} = \sin s$. The vector position of such a curve provides the following differential equation

$$[(1 + s)\cos s]^2 \mu_1'' + \{[\cos s - (1 + s)\sin s](1 + s)\cos s\}\mu_1' + \mu_1 = 0.$$

If the method presented is applied for

$$P_2(s) = [(1+s)\cos s]^2, P_1(s) = [\cos s-(1+s)\sin s](1+s)\cos s, P_0(s) = 1, y(s) = \mu(s), g = 0$$

the approximate solution is calculated under the initial conditions given as $y(0) = 0$, $y'(0) = 1$, in the range of $0 \leq s \leq 1$ for $N = 3$. Firstly, from the matrix

$$(W^*)^{-1} = \begin{bmatrix} -0.168\,04 & 1.037\,1 & -2.476\,1 & 0.130\,91 \\ 0.668\,04 & -0.537\,12 & 0.976\,06 & -0.130\,91 \\ -0.643\,44 & 0.587\,34 & 0.367\,4 & 0.05\,610\,6 \\ 0.190\,57 & -0.181\,22 & -0.144\,57 & -0.009\,351 \end{bmatrix}$$

and $A = (W^*)^{-1}G$, the unknowns a_n are calculated as follows:

$$\begin{aligned}
a_0 &= -2.476\,1 \\
a_1 &= 0.976\,06 \\
a_2 &= 0.367\,4 \\
a_3 &= -0.144\,57.
\end{aligned}$$

Thus, the solution is obtained as

$$\mu(s) = -0.144\,57s^3 - 0.500\,02s^2 + 0.999\,96s - 0.000\,06.$$

8. Conclusion

In this study, the characterizations are given for the spacelike curves according to the Frenet frame in R_2^4. These characterizations are interpreted for some special curves such as W-curve, Ccr curve, slope curve based on curvature properties. Also, the spacelike spherical curve is presented with a differential equation in R_2^4 and the spacelike sphericity of the special curves discussed.

In addition, the spacelike slant helix and the spacelike-B_2 slant helix concepts are defined in R_2^4 and the differential equations for vector positions are presented. These equations are homogeneous, linear, differential equations with variable coefficients. The Morgan Voyce matrix collocation method is given for the approximate solution of such differential equations. This method is applied in the differential equation that characterizes the spacelike-B_2 slant helix. An example has also been presented.

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