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ON WARPED PRODUCT MANIFOLDS ADMITTING τ-QUASI RICCI-HARMONIC METRICS

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Abstract. In this paper, we study warped product manifolds admitting τ-quasi Ricci-harmonic(RH) metrics. We prove that the metric of the fibre is harmonic Einstein when warped product metric is τ-quasi RH metric. We also provide some conditions for M to be a harmonic Einstein manifold. Finally, we provide necessary and sufficient conditions for a metric g to be τ-quasi RH metric by using a differential equation system.
Keywords: warped products, gradient Ricci-Harmonic soliton, τ-quasi Ricci- Harmonic metric, Harmonic Einstein

1. Introduction

Various geometric flows have been studied recently and one of them is Ricci flow coupled with harmonic map flow(shortly RH for Ricci-harmonic), defined by Müller [14, 15]. Let $(M^n, g(t))$ and (N^m, h) be smooth Riemannian manifolds and $\phi(t) : M \longrightarrow N$ is a family of smooth maps between $(M^n, g(t))$ with the metric $g(t)$ evolving along the RH flow and a fixed Riemannian manifold (N, h). The Ricci-harmonic flow is the coupled system

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Ric} + 2c d\phi \otimes d\phi \\ \frac{\partial}{\partial t}\phi = \tau_g \phi \end{cases}$$

where $c(t) > 0$ is a time dependent constant, $d\phi \otimes d\phi = \phi^* h$ is the pullback of h via ϕ and $\tau_g\phi = \mathrm{tr}\nabla d\phi$ is the tension field of ϕ. The RH flow behaves less singular than

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Ricci flow and many fundamental results in Ricci flow have been extended to the RH flow. A self-similar solution to RH flow is defined by Müller[14, 15] as follows.

Definition 1.1. Let (M,g) and (N,h) be two smooth Riemannian manifolds and $\phi : M \longrightarrow N$ be a smooth map. If there is a smooth function $f : M \longrightarrow \mathbb{R}$ and constants $c \geq 0$, λ such that the coupled system

$$\begin{cases} \mathrm{Ric} + \nabla^2 f - cd\phi \otimes d\phi = \lambda g, \\ \tau(\phi) - d\phi(\nabla f) = 0, \end{cases}$$

is satisfied, then (M, g, f, ϕ, λ) is called as a gradient Ricci-harmonic soliton and f is called the potential function. There have been many studies involving gradient Ricci-harmonic solitons such as [9, 18, 20, 21, 23]. When f is a constant, gradient RH soliton is called harmonic Einstein, i.e.,

$$\begin{cases} \mathrm{Ric} - cd\phi \otimes d\phi = \lambda g, \\ \tau_g \phi = 0. \end{cases}$$

It is well-known that for $\tau > 0$, the Bakry-Émery curvature is defined by

$$\mathrm{Ric}_{u,\tau} = \mathrm{Ric} + \nabla^2 u - \frac{1}{\tau} du \otimes du,$$

and g is called a τ-quasi Einstein metric for some constant τ if there is a constant λ and a potential function u such that

$$\mathrm{Ric}_{u,\tau} = \lambda g \tag{1.1}$$

is satisfied. From this point of view, τ-quasi Ricci-harmonic metric is defined in [20].

Definition 1.2. Let (N,h) be a fixed Riemannian manifold. A metric g of M is called $\tau(> 0)$-quasi RH (with respect to h), if for a map $\phi : M \longrightarrow N$, potential function $u : M \longrightarrow \mathbb{R}$ and constants $\alpha \geq 0$, λ, g satisfies the coupled system

$$\begin{cases} \mathrm{Ric} + \nabla^2 u - \frac{1}{\tau} du \otimes du - cd\phi \otimes d\phi = \lambda g, & (1.2) \\ \tau(\phi) - d\phi(\nabla u) = 0. & (1.3) \end{cases}$$

In [17], the authors studied a structure such that the warping function and the potential function are not the same. This idea provided interesting results and led a growing interest in warped products on Ricci solitons [1, 5, 6, 8, 11, 13, 17], almost Ricci solitons [7], Yamabe solitons [10, 19] and RH solitons [2].

In this paper, we will investigate a generalized version on the warped product manifolds which admits τ-quasi RH metric. We prove that the metric of the fibre is harmonic Einstein when warped product metric is τ-quasi RH metric. We also provide some conditions for M to be a harmonic Einstein manifold. Finally, we provide necessary and sufficient conditions for a metric g to be τ-quasi RH metric by using a differential equation system.

2. Preliminaries

Our aim is to remind the warped product $M = B \times_f F$, and the notion of lift by following the notation and terminology of O'Neill [16].

Definition 2.1. Let (B^n, g_B) and (F^m, g_F) be two Riemannian manifolds, and f be a positive smooth function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$ with the metric tensor g defined by

$$g = \pi^* g_B + (f \circ \pi)^2 \sigma^* g_F.$$

Here π and σ are the projections of $B \times F$ onto B and F respectively. The function f is called the warping function, B is the base and F is the fiber. When f is a constant function, M is simply a Riemannian product.

The lift of V to M is the unique element of $\mathfrak{X}(M)$ that is $\sigma-$related to V and $\pi-$related to zero vector field on B. The set of all such vertical lifts $\tilde{V}$ is denoted by $\mathfrak{L}(F)$. The set of all horizontal lifts $\tilde{X}$ is denoted by $\mathfrak{L}(B)$. In the same way, functions defined on B and F can be lifted to M. Let u_B, h_F be a smooth functions on B and F, respectively. The lift of u_B to M is the function $u = u_B \circ \pi$, and the lift of h_F to M is the function $h = h_F \circ \sigma$. Moreover, one can extend the idea to a mapping $\phi : M = B^n \times_f F^m \longrightarrow N$ by component-wise and consider ϕ as $\phi = \phi_B \circ \pi$ or $\phi = \phi_F \circ \sigma$. Throughout this paper, we will use the same notation for a vector field(and for a function) and its lift for simplicity. We denote the Levi-Civita connections by D, ∇ and $\overset{F}{\nabla}$; Ricci tensors by Ric, ${}^B\text{Ric}$ and ${}^F\text{Ric}$ of the M, B and F, respectively.

Now, we recall the following propositions.

Proposition 2.1. *On $M = B^n \times_f F^m$, if $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, then*

1. *$D_X Y \in \mathfrak{L}(B)$ is the lift of $\nabla_X Y$ on B,*
2. *$D_X V = D_V X = \dfrac{Xf}{f} V$,*
3. *nor $D_V W = -\dfrac{g(V,W)}{f} \nabla f$,*
4. *tan $D_V W \in \mathfrak{L}(F)$ is the lift of $\overset{F}{\nabla}_V W$ on F.*

Proposition 2.2. *On a warped product $M = B^n \times_f F^m$ with $m > 1$, let $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$. Then,*

1. *$\text{Ric}(X,Y) = {}^B\text{Ric}(X,Y) - \frac{m}{f} \nabla^2 f(X,Y)$,*
2. *$\text{Ric}(X,V) = 0$,*

3. $\mathrm{Ric}(V,W) = {}^F\mathrm{Ric}(V,W) - \left(\frac{\Delta f}{f} + (m-1)\frac{|\nabla f|^2}{f^2}\right) g(V,W)$.

In [12], the authors give the following corollary from Proposition 2.2.

Corollary 2.1. *The warped product $M = B^n \times_f F^m$ is Einstein with Ric $= \lambda g$ if and only if*

1. ${}^B\mathrm{Ric} = \lambda g_B + \frac{m}{f}\nabla^2 f$,
2. (F, g_F) *is Einstein with* ${}^F\mathrm{Ric} = \mu g_F$,
3. $\lambda f^2 + f\Delta f + (m-1)|\nabla f|^2 = \mu$.

3. Main Results

Inspiring from [17], we investigate the potential function u and conclude the next proposition.

Proposition 3.1. *Let the metric g of warped product manifold $M = B^n \times_f F^m$ be a*
τ-quasi Ricci-harmonic metric. Then in a neighbourhood of a point $(p,q) \in B^n \times F^m$, the non-constant map ϕ is $\phi = \phi_B \circ \pi$ or $\phi = \phi_F \circ \sigma$ if and only if the potential u is the lift of a function defined on B.

Proof. Let $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$. Assume that g be a τ-quasi RH metric on $M = B^n \times_f F^m$, then we have

$$\mathrm{Ric}(X,V) + \nabla^2 u(X,V) - \frac{1}{\tau} du \otimes du(X,V) - c d\phi \otimes d\phi(X,V) = \lambda g(X,V). \tag{3.1}$$

Since $\mathrm{Ric}(X,V) = 0$ and $g(X,V) = 0$, (3.1) becomes,

$$\nabla^2 u(X,V) - \frac{1}{\tau} du \otimes du(X,V) - c d\phi \otimes d\phi(X,V) = 0.$$

Now suppose that u is the lift of a function defined on F, and therefore $\nabla u \in \mathfrak{L}(F)$. Then the equation (3.2) is reduced to

$$\begin{aligned} 0 = \nabla^2 u(X,V) &= \langle \nabla_X \nabla u, V\rangle \\ &= \frac{Xf}{f}\langle \nabla u, V\rangle \end{aligned}$$

meaning u is a constant which contradicts the hypothesis. As a result, u is the lift of a function defined on B.

Conversely, suppose that u is a lift of a function defined on B, and therefore $\nabla u \in \mathfrak{L}(B)$. From Proposition 2.2, we have

$$\nabla^2 u(X,V) - \frac{1}{\tau} du \otimes du(X,V) = 0. \tag{3.2}$$

and

$$cd\phi(X)d\phi(V)=0. \tag{3.3}$$

Since ϕ is not constant from the hypothesis, there is a vector field $W=X+V$ in M such that $d\phi(W)d\phi(W)\neq 0$ in a neighbourhood of $(p,q)\in M$. By taking square of both sides we have that

$$(d\phi(X))^2+2\phi(X)d\phi(V)+(d\phi(V))^2\neq 0.$$

Using (3.3) in the above, we can conclude that $(d\phi(X))^2+(d\phi(V))^2\neq 0$, hence $d\phi(X)=0$ or $d\phi(V)=0$. □

Remark 3.1. Notice that the function f on the second line cannot be a constant because in that case M is simply a Riemannian product.

Now we can state our first theorem by using Proposition 3.1.

Theorem 3.1. *The metric g of warped product $M=B^n\times_f F^m$ is a τ-quasi Ricci-harmonic metric if and only if*

(i) If $\phi=\phi_B\circ\pi$, then

$$^B\mathrm{Ric}-\frac{m}{f}\nabla^2 f+\nabla^2 u-\frac{1}{\tau}du\otimes du-cd\phi\otimes d\phi=\lambda g_B, \tag{3.4}$$

and F is Einstein with $^F\mathrm{Ric}=\mu g_F$.

(ii) If $\phi=\phi_F\circ\sigma$, then

$$^B\mathrm{Ric}-\frac{m}{f}\nabla^2 f+\nabla^2 u-\frac{1}{\tau}du\otimes du=\lambda g_B, \tag{3.5}$$

and F is harmonic Einstein with

$$\begin{cases} ^F\mathrm{Ric}-cd\phi\otimes d\phi=\mu g_F,\\ \tau_g\phi=0.\end{cases}$$

In both cases μ is

$$\mu=f\Delta f+(m-1)|\nabla f|^2+\lambda f^2+f\nabla f(u). \tag{3.6}$$

Proof. Case (i): Let $\phi=\phi_B\circ\pi$. Using Proposition 2.2 for $X,Y\in\mathfrak{L}(B)$ in (1.2), we get (3.4). For $V,W\in\mathfrak{L}(F)$, the equation (1.2) is

$$\mathrm{Ric}(V,W)+\nabla^2 u(V,W)-\frac{1}{\tau}du\otimes du(V,W)-cd\phi\otimes d\phi(V,W)=\lambda g(V,W).$$

From Proposition 3.1, we know that u is lifted from B. So we can conclude that $du(V) = 0$ and similarly $d\phi(V) = 0$. Using Proposition 2.2 above we reach

$$ {}^F\mathrm{Ric}(V,W) - \left(\frac{\Delta f}{f} + (m-1)\frac{|\nabla f|^2}{f^2}\right) g(V,W) + \nabla^2 u(V,W) = \lambda g(V,W). \tag{3.7} $$

Using Proposition 2.1, we compute

$$ \begin{aligned} \nabla^2 u(V,W) &= g\left(\nabla_V \nabla u, W\right) \\ &= g\left(\frac{\nabla u(f)}{f} V, W\right) \\ &= f g_F\left(V,W\right) \nabla u(f) \end{aligned} \tag{3.8} $$

and substitute the result in (3.7) so we get

$$ {}^F\mathrm{Ric}(V,W) = \big(f\Delta f + (m-1)|\nabla f|^2 + \lambda f^2 + f\nabla f(u)\big)\, g_F(V,W) $$

which means F is Einstein.

Case (ii): Assume that $\phi = \phi_F \circ \sigma$. Using Proposition 2.2 for $X, Y \in \mathfrak{L}(B)$ in (1.2), we get (3.5) since $d\phi(X) = 0$. For $V, W \in \mathfrak{L}(F)$, the equation (1.2) is

$$ \mathrm{Ric}(V,W) + \nabla^2 u(V,W) - \frac{1}{\tau} du \otimes du(V,W) - c\, d\phi \otimes d\phi(V,W) = \lambda g(V,W). $$

Using Proposition 2.2, the fact that $du(V) = 0$ and (3.8) we get

$$ {}^F\mathrm{Ric}(V,W) - c\, d\phi \otimes d\phi(V,W) = \mu g(V,W). $$

Since $d\phi(\nabla u) = 0$, we can conclude that F is harmonic Einstein. □

Remark 3.2. Theorem 3.1 is a generalization of Corollary 2.1 and Theorem 1.3 in [2].

In [4], if the equation (1.1) is satisfied for a smooth function λ, then the metric is called generalized τ-quasi Einstein metric. Similarly, when λ in the equation (1.2) is a function, the metric is called generalized τ-quasi RH metric [22]. Under the assumption of the gradient of the warping function f being a conformal vector field, we can conclude the following.

Corollary 3.1. *Let the metric g of warped product $M = B^n \times_f F^m$ be a τ-quasi Ricci-harmonic metric and assume that ∇f is conformal vector field on B.*

- *(i) If $\phi = \phi_B \circ \pi$, then the metric g_B of B is a generalized τ-quasi RH metric.*
- *(ii) If $\phi = \phi_F \circ \sigma$, then B is generalized τ-quasi Einsten manifold.*

Theorem 3.2. *Let the metric g of warped product $M = B^n \times_f F^m$ be a τ-quasi Ricci-harmonic metric with non-constant ϕ. If $\lambda \geq 0$ and $\frac{m}{f}\Delta f \geq {}^B R$, then u is a constant. Therefore, M is harmonic Einstein.*

Proof. Taking the trace of (3.4), we have

$$\Delta_B u = \lambda n + \frac{m}{f}\Delta_B f - {}^B R + \frac{1}{\tau}|\nabla u|^2 + \alpha|\nabla\phi|^2.$$

Using the hypothesis, we reach that $\Delta_B u \geq 0$, so we can use maximum principle to conclude that u is a constant on B and so is it's lift. Hence M is harmonic Einstein. □

The following results of this paper will be given under the assumption of the harmonic map ϕ as a real valued function, i.e., $\phi : M \longrightarrow \mathbb{R}$. Our construction in Theorem 3.1 helps us to drop the restrictions the fiber manifold F which differs from [17].

Theorem 3.3. *The metric g of warped product $M = \mathbb{R}^n \times_f F^m$ is a τ-quasi Ricci-harmonic metric with non-constant ϕ and $f = f \circ \xi$, $u = u \circ \xi$, $\varphi \circ \xi$, $\phi = \phi \circ \xi$ defined in $(\mathbb{R}^n, \varphi^{-2}g_0)$ furnished with the metric tensor $g = \varphi^{-2}g_0 + f^2 g_F$ if and only if the functions verify the system below:*

$$(n-2)\frac{\varphi''}{\varphi} - m\frac{f''}{f} - 2m\frac{\varphi'}{\varphi}\frac{f'}{f} + u'' + 2\frac{\varphi'}{\varphi}u' - \frac{1}{\tau}(u')^2 - c(\phi')^2 = 0, \tag{3.9}$$

$$\left[\frac{\varphi''}{\varphi} - (n-1)\left(\frac{\varphi'}{\varphi}\right)^2 + m\frac{\varphi'}{\varphi}\frac{f'}{f} - \frac{\varphi'}{\varphi}u'\right]||\alpha||^2 = \frac{\lambda}{\varphi^2}, \tag{3.10}$$

$$\left[\frac{f''}{f} - (n-2)\frac{\varphi'}{\varphi}\frac{f'}{f} + (m-1)\left(\frac{f'}{f}\right)^2 - \frac{f'}{f}u'\right]||\alpha||^2 = \frac{\mu}{f^2\varphi^2} - \frac{\lambda}{\varphi^2}, \tag{3.11}$$

$$\left[\phi'' - (n-2)\frac{\varphi'}{\varphi}\phi' + m\phi'\frac{f'}{f} - \phi'u'\right]||\alpha||^2 = 0. \tag{3.12}$$

Proof. The Theorem 3.1 gives us necessary and sufficient condition to the metric g of $B^n \times_f F^m$ be a τ-quasi Ricci-harmonic metric. By using invariant solution technique, we reach equations (3.9), (3.10), (3.11) and (3.12).

For an arbitrary choice of a nonzero vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, consider $\xi : \mathbb{R}^n \to \mathbb{R}$ given by $\xi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i x_i$. Assume that $\varphi(\xi)$, $f(\xi)$, $u(\xi)$ and $\phi(\xi)$ are functions of ξ, so we have

$$\varphi_{,x_i} = \varphi'\alpha_i, \qquad f_{,x_i} = f'\alpha_i, \qquad u_{,x_i} = u'\alpha_i, \qquad \phi_{,x_i} = \phi'\alpha_i$$

$$\varphi_{,x_ix_j} = \varphi''\alpha_i\alpha_j, \qquad f_{,x_ix_j} = f''\alpha_i\alpha_j, \qquad u_{,x_ix_j} = u''\alpha_i\alpha_j \qquad \phi_{,x_ix_j} = \phi''\alpha_i\alpha_j.$$

Notice that the functions f, φ, u and ϕ are lifted from $B = (\mathbb{R}^n, \varphi^{-2}g_0)$.

For the conformal metric $g_B = \varphi^{-2} g_0$, the Ricci curvature is given by [3]:

$$^B\mathrm{Ric} = \frac{1}{\varphi^2}\Big\{(n-2)\varphi Hess_{g_0}(\varphi) + [\varphi\Delta_{g_0}\varphi - (n-1)|\nabla_{g_0}\varphi|^2]g_0\Big\}.$$

Since $(Hess_{g_0}(\varphi))_{i,j} = \varphi''\alpha_i\alpha_j,\ \Delta_{g_0}\varphi = \varphi''||\alpha||^2$, and $|\nabla_{g_0}\varphi|^2 = \varphi'||\alpha||^2$ we have

$$(^B\mathrm{Ric})_{i,j} = \frac{1}{\varphi}\{(n-2)\varphi''\alpha_i\alpha_j\} \quad \forall i \neq j = 1, ..., n \tag{3.13}$$

$$(^B\mathrm{Ric})_{i,i} = \frac{1}{\varphi^2}\left\{(n-2)\varphi\varphi''(\alpha_i)^2 + [\varphi\varphi''||\alpha||^2 - (n-1)(\varphi')^2||\alpha||^2]\epsilon_i\right\} \quad \forall i = 1, ..., n. \tag{3.14}$$

For the metric g_B, $Hess(u)$ is

$$(Hess_{g_B}(u))_{ij} = u_{,x_i x_j} - \sum_{k=1}^{n}\Gamma_{ij}^k u_{,x_k},$$

where the Christoffel symbol Γ_{ij}^k for distinct i, j, k are given by

$$\Gamma_{ij}^k = 0,\ \Gamma_{ij}^i = -\frac{\varphi_{,x_j}}{\varphi},\ \Gamma_{ii}^k = \varepsilon_i\varepsilon_k\frac{\varphi_{,x_k}}{\varphi} \text{ and } \Gamma_{ii}^i = -\frac{\varphi_{,x_i}}{\varphi}.$$

Hence,

$$\begin{aligned}(Hess_{g_B}(u))_{ij} &= u_{,x_i x_j} + \varphi^{-1}(\varphi_{,x_i}u_{,x_j} + \varphi_{,x_j}u_{,x_i}) - \delta_{ij}\varepsilon_i\sum_k \varepsilon_k\varphi^{-1}\varphi_{,x_k}u_{,x_k} \\ &= \alpha_i\alpha_j u'' + (2\alpha_i\alpha_j - \delta_{ij}\varepsilon_i||\alpha||^2)\varphi^{-1}\varphi' u'. \end{aligned}\tag{3.15}$$

Clearly, the Laplacian $\Delta_{g_B} f = \sum_k \varphi^2\varepsilon_k(Hess_{g_B}(f))_{kk}$ of f is

$$\Delta_{g_B} f = ||\alpha||^2\varphi^2(f'' - (n-2)\varphi^{-1}\varphi' f'). \tag{3.16}$$

Since g_B is a conformal metric, the terms $\nabla f(u)$, $|\nabla f|^2$ and $(\nabla\phi\otimes\nabla\phi)_{ij}$ can be given by

$$\begin{aligned}\nabla_{g_B} f(u) = \langle\nabla_{g_B} f, \nabla_{g_B} u\rangle = \varphi^2\sum_k \varepsilon_k f_{,x_k}u_{,x_k} &= ||\alpha||^2\varphi^2 f' u', \\ |\nabla_{g_B} f|^2 = \varphi^2\sum_k \varepsilon_k f_{,x_k}^2 &= ||\alpha||^2\varphi^2(f')^2, \\ (\nabla_{g_B}\phi\otimes\nabla_{g_B}\phi)_{ij} = \phi_{,x_i}\phi_{,x_j} &= \alpha_i\alpha_j(\phi')^2.\end{aligned}\tag{3.17}$$

Plugging in (3.14), (3.15) and (3.17) for $i = j$ into (3.4) we get (3.10).

When $i \neq j$, substituting (3.13) and (3.15) into (3.4) we obtain

$$\alpha_i\alpha_j\left((n-2)\frac{\varphi''}{\varphi} - m\frac{f''}{f} - 2m\frac{\varphi'}{\varphi}\frac{f'}{f} + u'' + 2\frac{\varphi'}{\varphi}u' - \frac{1}{\tau}(u')^2 - \theta(\phi')^2\right) = 0.$$

If there exist i, j, $i \neq j$ such that $\alpha_i \alpha_j \neq 0$, we have the equation (3.9). If $\alpha_i \alpha_j = 0$, $\forall i \neq j$, then consider for a fixed $k_0 \neq k$ such that $\alpha_{k_0} = 1$, $\alpha_k = 0$. For $i \neq k_0$, substituting (3.14), (3.15) and (3.17) into (3.4) we get the equation (3.10), i.e, $\alpha_i = 0$. For $i = k_0$, we have the equation (3.9), i.e., $\alpha_{k_0} = 1$.

Similarly, we obtain (3.11) by substituting (3.16), (3.17) in (3.6). Considering (1.3) and the laplace of ϕ, which is lifted from base, we have

$$\Delta\phi = \left[\Delta_{g_B}\phi + \frac{m}{f} g_B(\nabla\phi, \nabla f)\right] = g_B(\nabla\phi, \nabla u). \tag{3.18}$$

Using (3.16) and (3.17) in (3.18) we have (3.12) which completes the proof. □

Corollary 3.2. *Let $f = f \circ \xi$, $u = u \circ \xi$, $\varphi \circ \xi$, $\phi = \phi \circ \xi$ defined in $(\mathbb{R}^n, \varphi^{-2} g_0)$ and the metric g of warped product $\left(M = \mathbb{R}^n \times_f F^m, g = \varphi^{-2} g_0 + f^2 g_F\right)$ be a τ-quasi Ricci-harmonic metric with non-constant ϕ. If $||\alpha||^2 = 0$, then $\lambda = 0$ and $\mu = 0$, i.e., F^m is Ricci flat.*

Example 3.1. Let $||\alpha||^2 = 0$ in Theorem 3.3. For simplicity, assume that $c = 1$, $m = 4$, $n = 3$, $\tau = 1$ and $\varphi(\xi) = e^\xi$, $f(\xi) = e^\xi$ and $\phi(\xi) = \xi$. Solving (3.9), we obtain

$$u(\xi) = -\log\left(\cos(\sqrt{10}(c_1 + \xi))\right) + \xi + c_2, \; c_1, c_2 \in \mathbb{R}$$

which defines a τ-quasi RH metric on M.

Theorem 3.4. *The metric g of warped product $M = B^n \times_f F^m$ is a τ-quasi Ricci-harmonic metric with non-constant ϕ, $f = f \circ \xi$, $u = u \circ \xi$, $\varphi \circ \xi$, $\phi = \phi \circ \zeta$ defined in $(\mathbb{R}^n, \varphi^{-2} g_0)$ and $(\mathbb{R}^m, \psi^{-2} g_0)$, respectively, and furnished with the metric tensor $g = \varphi^{-2} g_0 + f^2 \psi^{-2} g_F$ if and only if the functions verify the system below:*

$$(n-2)\frac{\varphi''}{\varphi} - m\frac{f''}{f} - 2m\frac{\varphi'}{\varphi}\frac{f'}{f} + u'' + 2\frac{\varphi'}{\varphi}u' - \frac{1}{\tau}(u')^2 = 0, \tag{3.19}$$

$$\left[\frac{\varphi''}{\varphi} - (n-1)\left(\frac{\varphi'}{\varphi}\right)^2 + m\frac{\varphi'}{\varphi}\frac{f'}{f} - \frac{\varphi'}{\varphi}h'\right]||\alpha||^2 = \frac{\lambda}{\varphi^2}, \tag{3.20}$$

$$\begin{aligned}\left[f''\varphi^2 f - (n-2)\varphi'\varphi f f' + (m-1)(f')^2\varphi^2 - f' f \varphi^2 h'\right]||\alpha||^2 + \lambda f^2 \\ = \left[\frac{\psi''}{\psi} - (m-1)\left(\frac{\psi'}{\psi}\right)^2\right]||\beta||^2,\end{aligned} \tag{3.21}$$

$$(m-2)\frac{\psi''}{\psi} - c(\phi')^2 = 0, \tag{3.22}$$

$$\left[\psi^2\phi'' - (m-2)\psi\psi'\phi'\right]||\beta||^2 = 0. \tag{3.23}$$

Proof. We use the same technique as in the proof of the Theorem 3.3 for both the base and the fiber. When $i \neq j$, substituting the equation (3.13) and (3.17) in (3.5) we have the equation (3.19) and when $i = j$, plugging in (3.14) and (3.17) in (3.5) we get (3.20).

From Theorem 3.1, F is harmonic Einstein,

$$ {}^{F}\mathrm{Ric} - c d\phi \otimes d\phi = \mu g_F \tag{3.24} $$

where $c > 0$ and

$$ \mu = f\Delta_{g_B} f + (m-1)|\nabla f|^2 + \lambda f^2 + f\nabla f(u). \tag{3.25} $$

For an arbitrary choice of a nonzero vector $\beta = (\beta_1, \ldots, \beta_m)$, let $\psi : \mathbb{R}^m \to \mathbb{R}^+$ be the conformal factor of the fiber and $\zeta : \mathbb{R}^m \to \mathbb{R}$ be the invariant function so that $u(\zeta)$ is a function of ζ which gives

$$ (\nabla_{g_F}\phi_F \otimes \nabla_{g_F}\phi_F)_{ij} = \phi_{,y_i}\phi_{,y_j}\beta_i\beta_j \qquad \forall i,j = 1,...,m. \tag{3.26} $$

Using (3.17) in (3.25) we obtain

$$ \left[f''\varphi^2 f - (n-2)\varphi'\varphi f f' + (m-1)(f')^2\varphi^2 - f' f \varphi^2 u'\right] ||\alpha||^2 + \lambda f^2 = \mu. \tag{3.27} $$

Replacing (3.13), (3.14), (3.26) and (3.27) in (3.24) we get the equations (3.21) and (3.22) for $i = j$ or $i \neq j$.

From (ii) of Theorem 3.1 we have $\Delta_{g_F}\phi = 0$, by using (3.16), we obtain (3.23). □

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