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STEPANOV ρ-ALMOST PERIODIC FUNCTIONS IN GENERAL METRIC

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Abstract. In this paper, we analyze various classes of multi-dimensional Stepanov ρ-almost periodic functions in general metric. The main structural properties for the introduced classes of Stepanov almost periodic type functions are established. We also provide an illustrative application to the abstract degenerate semilinear fractional differential equations.
Keywords: multi-dimensional Stepanov ρ- almost periodic periodic functions, multi-dimensional Stepanov ρ- almost periodic periodic functions in general metric, abstract degenerate semilinear fractional dierential equations

1. Introduction and Preliminaries

The class of almost periodic functions was introduced by the Danish mathematician H. Bohr around 1924-1926 and later generalized by many others. Suppose that $(X, \| \cdot \|)$ is a complex Banach space and $F : \mathbb{R}^n \to X$ is a continuous function ($n \in \mathbb{N}$). Then we say that the function $F(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l) \equiv \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq l\}$ such that

$$\big\|F(\mathbf{t} + \tau) - F(\mathbf{t})\big\| \leq \epsilon, \quad \mathbf{t} \in \mathbb{R}^n;$$

here, $| \cdot - \cdot |$ denotes the Euclidean distance in $\mathbb{R}^n$. Fairly complete information about almost periodic functions and their applications can be obtained by consulting research monographs [1], [4], [7], [8], [9], [10], [16], [18] and [19].

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The multi-dimensional ρ-almost periodic type functions, the Stepanov multi-dimensional ρ-almost periodic type functions and the Weyl multi-dimensional ρ-almost periodic type functions have recently been examined in [6], [12] and [13]. In our recent research studies [14] and [15], we have introduced and analyzed the multi-dimensional ρ-almost periodic functions in general metric and the Weyl multi-dimensional ρ-almost periodic functions in general metric. The main aim of this research article is to introduce and analyze various notions of metrical Stepanov ρ-almost periodicity. Because of a certain similarity with our previous research studies (especially [15]), we omit the proofs of our structural results to a great extent. We also provide an illustrative application to the abstract degenerate semilinear fractional differential equations in the finite-dimensional spaces.

The organization and main ideas of this paper can be briefly described as follows. Various classes of metrical Stepanov ρ-almost periodic type functions are introduced and analyzed in Section 2 (see Definition 2.1-Definition 2.3). Embeddings of spaces of metrical Stepanov ρ-almost periodic functions into the corresponding spaces of metrical equi-Weyl ρ-almost periodic functions are analyzed in Proposition 2.1. The convolution invariance of metrical Stepanov almost periodicity is examined in Theorem 2.1, while a composition principle in this direction is clarified in Theorem 2.2. Subsection 2.1 investigates the invariance of metrical Stepanov c-almost periodicity under the actions of infinite convolution products; the main results of this subsection are Proposition 2.2 and Proposition 2.3. The final section of the paper is reserved for an application to the abstract degenerate semilinear fractional differential equations.

Notation, terminology and preliminaries. Suppose that X and Y are given non-empty sets. Let us recall that a binary relation between X into Y is any subset $\rho \subseteq X \times Y$. As is well known, the domain and range of ρ are defined by $D(\rho) := \{x \in X : \exists y \in Y \text{ such that } (x, y) \in X \times Y\}$ and $R(\rho) := \{y \in Y : \exists x \in X \text{ such that } (x, y) \in X \times Y\}$, respectively; $\rho(x) := \{y \in Y : (x, y) \in \rho\}$ $(x \in X)$, $x\ \rho\ y \Leftrightarrow (x, y) \in \rho$. Set $\rho(X') := \{y : y \in \rho(x) \text{ for some } x \in X'\}$ $(X' \subseteq X)$.

In the sequel, we will always assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces, $n \in \mathbb{N}$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\mathcal{B}$ is a non-empty collection of non-empty subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. By $L(X, Y)$ we denote the Banach space of all bounded linear operators from X into Y; $L(X, X) \equiv L(X)$ and I denotes the identity operator on Y. The Lebesgue measure in $\mathbb{R}^n$ is denoted by $m(\cdot)$; the restriction of function $f(\cdot)$ is denoted by $f_{|}(\cdot)$, with the meaning clear. If A and B are non-empty sets, then we define $\mathrm{B}^{\mathrm{A}} := \{f | f : \mathrm{A} \rightarrow \mathrm{B}\}$; $\chi_A(\cdot)$ stands for the characteristic function of set A. Put $\mathbb{N}_l := \{1, 2, ..., l\}$ and $\mathbb{N}_l^0 := \{0, 1, 2, ..., l\}$ $(l \in \mathbb{N})$. By $\Phi_\gamma(\cdot)$ we denote the Wright function of order $\gamma \in (0, 1)$; see [9] for the notion.

Suppose that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a non-empty Lebesgue measurable subset. By $M(\Omega : X)$ we denote the collection of all measurable functions $f : \Omega \rightarrow X$; $M(\Omega) := M(\Omega : \mathbb{R})$. Let $\mathcal{P}(\Omega)$ be the vector space of all Lebesgue measurable functions

$p : \Omega \to [1, \infty]$. For any $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, we define

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, & t \geq 0, \quad 1 \leq p(x) < \infty, \\ 0, & 0 \leq t \leq 1, \quad p(x) = \infty, \\ \infty, & t > 1, \quad p(x) = \infty \end{cases}$$

and

$$\rho(f) := \int_{\Omega} \varphi_{p(x)}(\|f(x)\|)\, dx.$$

We introduce the Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent by

$$L^{p(x)}(\Omega : X) = \Big\{ f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty \Big\}.$$

For more details about Lebesgue spaces with variable exponents, we refer to research monograph [5] by L. Diening, P. Harjulehto, P. Hästüso and M. Ruzicka.

If the set Λ is Lebesgue measurable and $\nu : \Lambda \to (0, \infty)$ is a Lebesgue measurable function, then we work with the following Banach space

$$L^{p(\mathbf{t})}_{\nu}(\Lambda : Y) := \big\{ u : \Lambda \to Y \ ; \ u(\cdot) \text{ is measurable and } ||u||_{p(\mathbf{t})} < \infty \big\},$$

where $p \in \mathcal{P}(\Lambda)$ and

$$\big\|u\big\|_{p(\mathbf{t})} := \big\|u(\mathbf{t})\nu(\mathbf{t})\big\|_{L^{p(\mathbf{t})}(\Lambda:Y)}.$$

If $\nu : \Lambda \to (0, \infty)$ is any function such that the function $1/\nu(\cdot)$ is locally bounded, then we also work with the Banach space $C_{0,\nu}(\Lambda : Y)$ consisting of all continuous functions $u : \Lambda \to Y$ satisfying that $\lim_{|\mathbf{t}|\to\infty, \mathbf{t}\in\Lambda} \|u(\mathbf{t})\|_Y \nu(\mathbf{t}) = 0$. Equipped with the norm $\| \cdot \| := \sup_{\mathbf{t}\in\Lambda} \| \cdot (\mathbf{t})\nu(\mathbf{t})\|_Y$, $C_{0,\nu}(\Lambda : Y)$ is a Banach space.

Assume now that Ω is a fixed compact subset of $\mathbb{R}^n$ with positive Lebesgue measure, $1 \leq p < \infty$, and Λ is a non-empty subset of $\mathbb{R}^n$ satisfying $\Lambda + \Omega \subseteq \Lambda$. We need the following notion (see, e.g., [10] and [13]):

Definition 1.1. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$ and $F : \Lambda \to Y$ satisfies that for each $\mathbf{t} \in \Lambda$ and $x \in X$, the function $F(\mathbf{t} + \mathbf{u}; x)$ belongs to the space $L^p(\Omega : Y)$. Then we say that $F(\cdot; \cdot)$ is Stepanov (Ω, p)-bounded if and only if

$$\sup_{\mathbf{t}\in\Lambda} \big\|F(\mathbf{t} + \mathbf{u})\big\|_{L^p(\Omega:Y)} < \infty.$$

Define $\|F\|_{S^{\Omega,p}} := \sup_{\mathbf{t}\in\Lambda} \|F(\mathbf{t} + \mathbf{u})\|_{L^p(\Omega:Y)}$; in the usually considered case $\Omega = [0, 1]^n$, we shorten $\|F\|_{S^{\Omega,p}}$ to $\|F\|_{S^p}$ and say that the function $F(\cdot)$ is Stepanov p-bounded. By Stepanov ∞-boundedness of function $F(\cdot)$ we mean its essential boundedness.

As already mentioned, various classes of multi-dimensional Weyl ρ-almost periodic functions in general metric have been analyzed in [15]. For our later purposes, we need to recall the following notion; we assume the validity of the following conditions here:

(WM1-1): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega) > 0$, $p \in \mathcal{P}(\Lambda)$, $\Lambda' + \Lambda + l\Omega \subseteq \Lambda$, $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi : [0, \infty) \to [0, \infty)$ and $\mathbb{F} : (0, \infty) \times \Lambda \to (0, \infty)$.

(WM1-2): For every $\mathbf{t} \in \Lambda$ and $l > 0$, $\mathcal{P}_{\mathbf{t},l} = (P_{\mathbf{t},l}, d_{\mathbf{t},l})$ is a metric space of functions from $\mathbb{C}^{\mathbf{t}+l\Omega}$ containing the zero function. We set $\|f\|_{P_{\mathbf{t},l}} := d_{\mathbf{t},l}(f, 0)$ for all $f \in P_{\mathbf{t},l}$. We also assume that $\mathcal{P} = (P, d)$ is a metric space of functions from $\mathbb{C}^{\Lambda}$ containing the zero function and set $\|f\|_P := d(f, 0)$ for all $f \in P$. The argument from Λ will be denoted by $\cdot\cdot$ and the argument from $\mathbf{t} + l\Omega$ will be denoted by $\cdot$.

Definition 1.2. By $e - W^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t},l},\mathcal{P})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto G_x(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, and

$$\sup_{x \in B} \left\| \mathbb{F}(l, \cdot\cdot) \left\| \phi\Big(\big\| F(\tau + \cdot; x) - G_x(\cdot) \big\|_Y \Big) \right\|_{P_{\cdot\cdot,l}} \right\|_P < \epsilon.$$

2. Metrical Stepanov ρ-Almost Periodic Type Functions

The main aim of this section is to introduce and analyze the multi-dimensional Stepanov ρ-almost periodic functions in general metric.

We will always assume the validity of the following conditions:

(SM1-1): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega) > 0$, $\Lambda' + \Lambda \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $\phi : [0, \infty) \to [0, \infty)$ and $\mathbb{F} : \Lambda \to (0, \infty)$.

(SM1-2): For every $\mathbf{t} \in \Lambda$, $\mathcal{P}_{\mathbf{t}} = (P_{\mathbf{t}}, d_{\mathbf{t}})$ is a metric space of functions from $\mathbb{C}^{\mathbf{t}+\Omega}$, resp. $Y^{\mathbf{t}+\Omega}$, in the case of consideration of Definition 2.1, resp. Definition 2.2-Definition 2.3, containing the zero function. We set $\|f\|_{P_{\mathbf{t}}} := d_{\mathbf{t}}(f, 0)$ for all $f \in P_{\mathbf{t},l}$. We also assume that $\mathcal{P} = (P, d)$ is a metric space of functions from $\mathbb{C}^{\Lambda}$ containing the zero function and set $\|f\|_P := d(f, 0)$ for all $f \in P$. The argument from Λ will be denoted by $\cdot\cdot$ and the argument from $\mathbf{t} + \Omega$ will be denoted by $\cdot$.

In case $0 \in \Omega$, the class of multi-dimensional Stepanov ρ-almost periodic functions introduced in [13, Definition 6] is a very special case of the following class of functions with the metric space P chosen to be $l_\infty(\Lambda : Y)$ and the metric space $P_{\mathbf{t}}$ chosen to be $L^{p(\cdot)}(\mathbf{t} + \Omega : Y)$, for any $\mathbf{t} \in \Lambda$, provided that the exponent $p(\cdot)$ has a constant value (if this is not the case, the extension can be obtained by considering the translation with tuple $\mathbf{t} \in \Lambda$ and the metric space of functions $F : \Lambda \to L^{p(\cdot)}(\Omega : \mathbb{C})$; see [11] for more details):

Definition 2.1. By $S^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto G_x(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, and

$$\sup_{x\in B}\Bigl\|\mathbb{F}(\cdot)\Bigl\|\phi\Bigl(\bigl\|F(\tau+\cdot;x)-G_x(\cdot)\bigr\|_Y\Bigr)\Bigr\|_{P_{\cdot\cdot}}\Bigr\|_P<\epsilon. \tag{2.1}$$

Definition 2.2. By $S^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto G_x(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, and

$$\sup_{x\in B}\Bigl\|\mathbb{F}(\cdot\cdot)\phi\Bigl(\bigl\|F(\tau+\cdot;x)-G_x(\cdot)\bigr\|_{P_{\cdot\cdot}}\Bigr)\Bigr\|_P<\epsilon.$$

Definition 2.3. By $S^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t},l},\mathcal{P})_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto G_x(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, and

$$\sup_{x\in B}\Bigl\|\phi\Bigl(\mathbb{F}(\cdot\cdot)\bigl\|F(\tau+\cdot;x)-G_x(\cdot)\bigr\|_{P_{\cdot\cdot}}\Bigr)\Bigr\|_P<\epsilon.$$

For simplicity, we will not consider here the notion of metrical Bochner transform. The notion of Bohr $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$-almost periodicity (see [14, Definition 3.1(i)]) can be viewed as a special case of the notion introduced in the previous three definitions only if we assume some extra conditions on the metric space $\mathcal{P}$ and the function $F(\cdot)$ under our consideration (see also [15]). Any of the above introduced classes of function spaces is denoted by $A_{X,Y}$. If $F(\cdot;\cdot)$ belongs to $A_{X,Y}$, $c_1 \in \mathbb{R}\setminus\{0\}$, $\tau \in \mathbb{R}^n$, c, $c_2 \in \mathbb{C} \setminus \{0\}$ and $x_0 \in X$, then it is not difficult to find some sufficient conditions ensuring that the function $cF(\cdot;\cdot)$, $F(c_1\cdot; c_2\cdot)$, $\|F(\cdot;\cdot)\|_Y$ or $F(\cdot+\tau; \cdot+x_0)$ also belongs to $A_{X,Y}$. As in the case of consideration of metrical Weyl ρ-almost periodicity, it is not simple to say when $A_{X,Y}$ will be a vector space, even in the case that $\rho = \mathrm{I}$; we can use the Jensen integral inequality in general measure spaces to clarify certain inclusions about the introduced classes of functions. We can also analyze the uniformly convergent sequences of functions belonging to the space $S^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, $S^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ or $S^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t},l},\mathcal{P})_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$. For example, if $(F_k(\cdot;\cdot))$ is a sequence of functions from the space $S^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ and there exists a function $F : \Lambda \times X \to Y$ such that $\lim_{k\to+\infty} F_k(\mathbf{t}; x) = F(\mathbf{t}; x)$, uniformly on $\Lambda \times B$ for each set B of collection $\mathcal{B}$, then $F \in S^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ provided that the following conditions hold true:

(i) $P_{\mathbf{t}}$ and P are Banach spaces for all $\mathbf{t} \in \Lambda$;

(ii) There exists a finite real constant $c > 0$ such that $\phi(x+y) \leq c[\phi(x)+\phi(y)]$ for all $x,\ y \geq 0$;

(iii) $\phi(\cdot)$ is continuous at zero;

(iv) $D(\rho)$ is closed, ρ is single-valued on $R(F)$ and continuous on $D(\rho)$ in the usual sense ([6]);

(v) We have $\mathbb{F}(\cdot\cdot) \in P$.

In the remainder of this paper, we will mainly deal with the notion introduced in Definition 2.1. Concerning embeddings of spaces of metrical Stepanov ρ-almost periodic functions into spaces of metrical equi-Weyl ρ-almost periodic functions, we will clarify only one result here (this result generalizes some already known results in the classical approach; see [10]):

Proposition 2.1. *Suppose that* (SM1-1)-(SM1-2) *hold,* $F \in S^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, $\mathbb{F}_1 : (0,\infty) \times \Lambda \to (0,\infty)$, *and* (WM1-1)-(WM1-2) *hold with the metric space* $\mathcal{P}$ *and the metric spaces* $\mathcal{P}_{\mathbf{t},l}$ *replaced therein by the metric space* $\mathcal{P}_1$ *and the metric spaces* $\mathcal{P}^1_{\mathbf{t},l}$ *(*$\mathbf{t} \in \Lambda$; $l > 0$*). Suppose, further, that the following conditions hold:*

(i) *The space* $\mathcal{P}_1$ *is a Banach space;*

(ii) *If* $f,\ g \in P_1$ *and* $0 \leq f \leq g$, *then* $\|f\|_{P_1} \leq \|g\|_{P_1}$;

(iii) $P = L^\infty_\nu(\Lambda : \mathbb{C})$ *with some Lebesgue measurable function* $\nu(\cdot)$;

(iv) *There exist two finite real numbers* $c > 0$, $M > 0$ *and a positive integer* $l \in \mathbb{N}$ *such that*

$$\left\| \mathbb{F}_1(l,\cdot\cdot) \sum_{k \in (\mathbb{N}_0^{l-1})^n} \frac{1}{\nu(\cdot\cdot+k)\mathbb{F}(\cdot\cdot+k)} \right\|_{P_1} \leq M, \tag{2.2}$$

and, for every $\mathbf{t} \in \Lambda$ *and* $\mathbf{t}' \in \Lambda$, *the assumption* $f \in P_{\mathbf{t},l}$ *implies* $f_{|(\mathbf{t}'+\Omega)} \in P_{\mathbf{t}'}$ *whenever* $\mathbf{t}' + \Omega \subseteq \mathbf{t} + l\Omega$, *and*

$$\left\| \phi\Bigl(\bigl\| F(\tau+\cdot;x) - G_x(\cdot) \bigr\|_Y \Bigr) \right\|_{P^1_{\mathbf{t},l}} \leq c \sum_{k \in (\mathbb{N}_0^{l-1})^n} \left\| \phi\Bigl(\bigl\| F(\tau+\cdot;x) - G_x(\cdot) \bigr\|_Y \Bigr) \right\|_{P_{\mathbf{t}+k}}. \tag{2.3}$$

Then $F \in e - W^{(\phi,\mathbb{F}_1,\rho,\mathcal{P}^1_{\mathbf{t},l},\mathcal{P}_1)}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$.

Proof. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be fixed. Then there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every

$x \in B$, the mapping $\mathbf{u} \mapsto G_x(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, and (2.1) holds. Using (2.3), we get that

$$\Bigl\| \mathbb{F}_1(l, \cdot\cdot) \Bigl\| \phi\Bigl(\bigl\| F(\tau + \cdot; x) - G_x(\cdot) \bigr\|_Y \Bigr) \Bigr\|_{P^1_{\cdot\cdot, l}} \Bigr\|_{P_1}$$

$$\leq \Bigl\| c\mathbb{F}_1(l, \cdot\cdot) \sum_{k \in (\mathbb{N}_0^{l-1})^n} \Bigl\| \phi\Bigl(\bigl\| F(\tau + \cdot; x) - G_x(\cdot) \bigr\|_Y \Bigr) \Bigr\|_{P_{\cdot\cdot+k}} \Bigr\|_{P_1}.$$

Due to our choice of space P, we have ($\mathbf{t} \in \Lambda$; $k \in (\mathbb{N}_0^{l-1})^n$):

$$\Bigl\| \phi\Bigl(\bigl\| F(\tau + \cdot; x) - G_x(\cdot) \bigr\|_Y \Bigr) \Bigr\|_{P_{\mathbf{t}+k}} \leq \frac{\epsilon}{\nu(\mathbf{t} + k)\mathbb{F}(\mathbf{t} + k)}.$$

Then the final conclusion simply follows from our assumption that $\mathcal{P}_1$ is a Banach space, (ii) and the estimate (2.2). $\square$

Before proceeding further, let us emphasize that it is too much to expect that the functions of the form $\chi_K(\cdot)$, where K is a non-empty compact set in $\mathbb{R}^n$ with a positive Lebesgue measure, belong to the spaces of metrical Stepanov ρ-almost periodic functions introduced in this paper. As is well known, these functions are always equi-Weyl c-almost periodic for all values of complex parameter c; see [10] for more details.

The proof of following theorem is almost the same as the proof of [15, Theorem 3.4] and therefore omitted (we can just follow the argumentation given in the proof of this theorem, with $l = 1$):

Theorem 2.1. *Suppose that $c \in \mathbb{C} \setminus \{0\}$, $\varphi : [0, \infty) \to [0, \infty)$, $\phi : [0, \infty) \to [0, \infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leq \varphi(x)\phi(y)$ for all $x,\ y \geq 0$, $h \in L^1(\mathbb{R}^n)$, $\Omega = [0, 1]^n$, $F \in S^{(\phi, \mathbb{F}, c\mathrm{I}, \mathcal{P}_{\mathbf{t}}, \mathcal{P})}_{\Omega, \Lambda', \mathcal{B}}(\mathbb{R}^n \times X : Y)$, where $p,\ q \in \mathcal{P}(\mathbb{R}^n)$, $1/p(\cdot) + 1/q(\cdot) = 1$, $\nu : \mathbb{R}^n \to (0, \infty)$ is a Lebesgue measurable function, $\nu(\mathbf{t}) \geq d > 0$ for some positive real number $d > 0$, $P_{\mathbf{t}} = L^{p(\cdot)}_{\nu}(\mathbf{t} + \Omega : \mathbb{C})$, the metric $d_{\mathbf{t}}$ is induced by the norm of this Banach space ($\mathbf{t} \in \Lambda$), $P = L^\infty(\mathbb{R}^n : \mathbb{C})$ and for each $x \in X$ we have $\sup_{\mathbf{t} \in \mathbb{R}^n} \|F(\mathbf{t}; x)\|_Y < \infty$. If* (SM1-1)-(SM1-2) *hold, $\mathbb{F}_1 : (0, \infty) \times \mathbb{R}^n \to (0, \infty)$, the following two conditions*

(i) *If $0 \leq |f(\cdot)| \leq |g(\cdot)|$ and $f,\ g \in P_1$ ($f,\ g \in P^1_{\mathbf{t}}$ for some $\mathbf{t} \in \mathbb{R}^n$), then $\|f\|_{P_1} \leq \|g\|_{P_1}$ ($\|f\|_{P^1_{\mathbf{t}}} \leq \|g\|_{P^1_{\mathbf{t}}}$);*

(ii) *If $\epsilon > 0$ and $f \in P_1$ ($f \in P^1_{\mathbf{t}}$ for some $\mathbf{t} \in \mathbb{R}^n$), then $\epsilon f \in P_1$ ($\epsilon f \in P^1_{\mathbf{t}}$) and $\|\epsilon f\|_{P_1} \leq \epsilon \|f\|_{P_1}$ ($\|\epsilon f\|_{P^1_{\mathbf{t}}} \leq \epsilon \|f\|_{P^1_{\mathbf{t}}}$),*

hold, and there exists a sequence $(a_k)_{k \in \mathbb{Z}^n}$ of positive real numbers such that $\sum_{k \in \mathbb{Z}^n} a_k = 1$ and

$$\Bigl\| \mathbb{F}_1(\cdot\cdot) \Bigl\| 2 \sum_{k \in \mathbb{Z}^n} a_k \Bigl[\varphi\bigl(a_k^{-1} h(\cdot - \mathbf{v}) \bigr) \Bigr]_{L^{q(\mathbf{v})}(\cdot - k + \Omega)} \bigl[\mathbb{F}(\cdot - k) \bigr]^{-1} \frac{\nu(\cdot)}{d} \Bigr\|_{P^1_{\cdot\cdot}} \Bigr\|_{P^1} \leq 1,$$

then $h * F \in S^{(\phi,\mathbb{F}_1,c\mathrm{I},\mathcal{P}^1_{\mathbf{t}},\mathcal{P}^1)}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$; *here,*

$$(h * F)(\mathbf{t}; x) := \int_{\mathbb{R}^n} h(\sigma)F(\mathbf{t}-\sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X.$$

The statements of [15, Proposition 3.5, Corollary 3.6, Theorem 3.10] admit straightforward reformulations in our context. Furthermore, the conclusions established in [3, Example 2.13, Example 2.15], showing the importance of case $\Lambda' \neq \Lambda$, and the conclusions established in [10, Example 6.1.15], showing the importance of case $\Lambda' \nsubseteq \Lambda$, can be formulated in our new framework. Details can be left to the interested readers.

Concerning composition principles for the introduced classes of functions, we will only state the following slight extension of [9, Theorem 2.7.1]; the proof is omitted since it follows from the insignificant modification of the argumentation contained in the proofs of [17, Lemma 2.1, Theorem 2.2] (let us also note that an analogue of [9, Theorem 6.2.30] can be formulated for the metrical Stepanov almost periodicity as well as that we can consider an analogue of [9, Theorem 2.7.2] with the usually considered Lipschitz assumption, when the situation in which $p = q$ can appear):

Theorem 2.2. *Suppose that* $c \in \mathbb{C} \setminus \{0\}$, $\Lambda = [0,\infty)$ *or* $\Lambda = \mathbb{R}$, $\emptyset \neq \Lambda' \subseteq \Lambda$, $\nu : \Lambda \to (0,\infty)$ *is a Lebesgue measurable function,* $F : \Lambda \times X \to Y$ *and* $f : \Lambda \to X$. *Let there exist a real number* $r \geq \max(p, p/(p-1))$ *and a Stepanov* r*-bounded function* $L_F : \Lambda \to [0,\infty)$ *such that the function* $L_F(\cdot + a)\nu(\cdot)$ *is Stepanov* r*-bounded for any* $a \in \Lambda'$,

$$\|F(t;x) - F(t;y)\|_Y \leq L_F(t)\|x-y\|, \quad t \in \Lambda,\ x,\ y \in X, \tag{2.4}$$

and let there exist a set $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{x(t) : t \in \Lambda \setminus \mathrm{E}\}$ *is relatively compact in* X. *Suppose further that, for every* $\epsilon > 0$ *and for every compact set* $K \subseteq X$, *there exist two real numbers* $l > 0$ *and* $L > 0$ *such that any interval* $\Lambda_0 \subseteq \Lambda$ *of length* L *contains a number* $\tau \in \Lambda_0 \cap \Lambda'$ *such that*

$$\sup_{t\in\Lambda, u\in K}\left[\int_t^{t+1} \left\|F(s+\tau; cu) - cF(s;u)\right\|^p \nu^p(s)\, ds\right]^{1/p} \leq \epsilon$$

and

$$\sup_{t\in\Lambda}\left[\int_t^{t+1} \left\|f(s+\tau) - cf(s)\right\|^p \nu^p(s)\, ds\right]^{1/p} \leq \epsilon.$$

Then $q := pr/(p+r) \in [1,p)$ *and* $F(\cdot; f(\cdot)) \in S^{(x,c\mathrm{I},\mathcal{P}_{t,q},\mathcal{P}_q)}_{[0,1],\Lambda'}(\Lambda : Y)$, *where* $P_{t,q} := L^q_\nu([t,t+1] : \mathbb{C})$, *the metric* $d_{t,\nu}$ *is induced by the norm of this Banach space* ($t \in \Lambda$), $P_q := L^\infty(\Lambda : \mathbb{C})$, *and the metric* d *is induced by the norm of this Banach space.*

Before proceeding to Subsection 2.1, we would like to propose an open problem concerning the following well-known example of H. Bohr and E. Følner [2]:

Example 2.1. Let a real number $P > 1$ be fixed. Then there exists a Stepanov P-bounded function $F : \mathbb{R} \to \mathbb{R}$ which is Stepanov 1-almost periodic but not Stepanov

P-almost periodic (see [2, Main example 2, pp. 70–73], and [9] for the notion). We would like to ask whether there exists a Lebesgue measurable function $\nu(\cdot)$ such that $F \in S_{\Omega,\mathbb{R}}^{(x,1,\mathrm{I},\mathcal{P}_{\mathbf{t}},\mathcal{P})}(\mathbb{R} : \mathbb{C})$ with $\Omega = [0,1]$, $P_t = L_\nu^P([t,t+1] : \mathbb{C})$ for all $t \in \mathbb{R}$ and $P = L^\infty(\mathbb{R} : \mathbb{C})$?

2.1. Invariance of metrical Stepanov c-almost periodicity under the actions of infinite convolution products

We start this subsection by observing that the statement of [15, Theorem 3.11] admits a straightforward reformulation in our context:

Proposition 2.2. *Assume that $\phi(x) \equiv x$, $P = L_\nu^\infty(\mathbb{R} : \mathbb{C})$, $P_1 = L_{\nu_1}^\infty(\mathbb{R} : \mathbb{C})$, $P_t = L_\sigma^\infty([t,t+1] : \mathbb{C})$ and $P_t^1 = L_{\sigma_1}^\infty([t,t+1] : \mathbb{C})$ for all $t \in \mathbb{R}$. Assume, further, that $c \in \mathbb{C} \setminus \{0\}$, $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family, $\nu : \mathbb{R} \to (0,\infty)$, $\nu_1 : \mathbb{R} \to (0,\infty)$, $\sigma : \mathbb{R} \to (0,\infty)$ and $\sigma_1 : \mathbb{R} \to (0,\infty)$ are Lebesgue measurable functions as well as that the spaces P, P_1, P_t and P_t^1 are given as above ($t \in \mathbb{R}$). Let* (SM1-1) *hold with $\Lambda = \mathbb{R}$, let $\mathbb{F}_1 : \mathbb{R} \to (0,\infty)$, and let*

$$\sup_{t\in\mathbb{R}} \mathbb{F}_1(t) \sup_{u\in[t,t+1]} \sigma_1(u)\nu_1(t) \int_0^\infty \frac{\|R(s)\|}{\nu(t-s)\mathbb{F}(t-s)\sigma(u-s)}\, ds < +\infty.$$

If $f \in S_{\Omega,\Lambda'}^{(x,\mathbb{F},c\mathrm{I},\mathcal{P}_{\mathbf{t}},\mathcal{P})}(\mathbb{R} : X)$, then $F \in S_{\Omega,\Lambda'}^{(x,\mathbb{F},c\mathrm{I},\mathcal{P}_{\mathbf{t}}^1,\mathcal{P}^1)}(\mathbb{R} : Y)$, provided that the function $F(\cdot)$, given by

$$F(t) := \int_{-\infty}^t R(t-s)f(s)\, ds, \quad t \in \mathbb{R}, \tag{2.5}$$

is well defined.

Now we will prove a result which shows that the notion introduced and analyzed in the previous part of this paper is not fully complete in the study of subject under our consideration, especially when we use the weighted function spaces. For the sake of completeness, we will provide all details of the proof of the following result:

Proposition 2.3. *Suppose that $c \in \mathbb{C} \setminus \{0\}$, $1 \le p < \infty$, $1/p + 1/q = 1$, $\nu : \mathbb{R} \to (0,\infty)$ and $\sigma : \mathbb{R} \to (0,\infty)$ are Lebesgue measurable functions, the function $1/\nu^p(\cdot)$ is locally integrable, and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^\infty \|R(\cdot)\|_{L_\nu^q[k,k+1]} < \infty$. If $f : \mathbb{R} \to X$ satisfies*

$$\sup_{t\in\mathbb{R},k\in\mathbb{Z}} \big\| f(t-\cdot)/\nu(\cdot) \big\|_{L^p[k,k+1]} < +\infty, \tag{2.6}$$

and for every $\epsilon > 0$ there exists a finite real number $l > 0$ such that, for every $t_0 \in \Lambda'$, there exists a number $\tau \in [t_0 - l, t_0 + l]$ such that

$$\sup_{t\in\mathbb{R},k\in\mathbb{Z}} \sigma(t) \big\| [f(\cdot+\tau) - cf(\cdot)]/\nu(t-\cdot) \big\|_{L^p[t-k-1,t-k]} < \epsilon, \tag{2.7}$$

then the function $F : \mathbb{R} \to Y$, given by (2.5)*, is well-defined, continuous and satisfies that, for every $\epsilon > 0$, there exists a finite real number $l > 0$ such that, for every*

$t_0 \in \Lambda'$, there exists a number $\tau \in [t_0 - l, t_0 + l]$ such that

$$\sup_{t\in\mathbb{R}} \sigma(t)\big\|F(t+\tau) - cF(t)\big\|_Y \leq \epsilon. \tag{2.8}$$

Proof. We will follow the proofs of [9, Proposition 2.6.11, Proposition 3.5.3] with appropriate changes; w.l.o.g., we may assume that $X = Y$. Clearly, we have $G(t) = \int_0^\infty R(s)g(t-s)\,ds$ for all $t \in \mathbb{R}$, since the last integral is absolutely convergent by the Hölder inequality and condition (2.6):

$$\int_0^\infty \|R(s)\|\|f(t-s)\|\,ds = \sum_{k=0}^\infty \int_k^{k+1} \big[\|R(s)\|\nu(s)\big] \cdot \big[\|f(t-s)\|/\nu(s)\big]\,ds$$

$$\leq \sum_{k=0}^\infty \|R(\cdot)\|_{L^q_\nu[k,k+1]} \sup_{x\in\mathbb{R},k\in\mathbb{Z}} \big\|f(x-\cdot)/\nu(\cdot)\big\|_{L^p[k,k+1]}, \quad t \in \mathbb{R}.$$

Let a number $\epsilon > 0$ be fixed. Then there exists a finite real number $l > 0$ such that, for every $t_0 \in \Lambda'$, there exists a number $\tau \in [t_0 - l, t_0 + l]$ such that (2.7) holds. Using the Hölder inequality, we get

$$\sigma(t)\|F(t+\tau) - cF(t)\| \leq \sigma(t) \int_0^\infty \|R(r)\| \cdot \|f(t+\tau-r) - cf(t-r)\|\,dr$$

$$= \sigma(t) \sum_{k=0}^\infty \int_k^{k+1} \|R(r)\| \cdot \|f(t+\tau-r) - cf(t-r)\|\,dr$$

$$\leq \sigma(t) \sum_{k=0}^\infty \|R(\cdot)\|_{L^q_\nu[k,k+1]} \left(\int_k^{k+1} \|f(t+\tau-r) - cf(t-r)\|^p \nu^{-p}(r)\,dr \right)^{1/p}$$

$$= \sigma(t) \sum_{k=0}^\infty \|R(\cdot)\|_{L^q_\nu[k,k+1]} \left(\int_{t-k-1}^{t-k} \|f(s+\tau) - cf(s)\|^p \nu^{-p}(t-r)\,ds \right)^{1/p}$$

$$\leq \sum_{k=0}^\infty \|R(\cdot)\|_{L^q_\nu[k,k+1]} \epsilon = M\epsilon, \quad t \in \mathbb{R}.$$

Hence, (2.8) holds; we only need to prove yet that the function $F(\cdot)$ is continuous. Define $F_k(t) := \int_k^{k+1} R(s)f(t-s)\,ds$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Since $M < +\infty$ and (2.6) holds, the Weierstrass criterion implies that $\sum_{k\geq 0} F_k(\cdot) = F(\cdot)$ uniformly on the real line. The continuity of function $F_k(\cdot)$ for a fixed non-negative integer k can be shown using our assumption that the function $1/\nu^p(\cdot)$ is locally integrable and repeating almost verbatim the argumentation given in the proof of [9, Proposition 3.5.3] (the only thing worth noting is that, for given numbers $t \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we can choose a sequence $(f_l(\cdot))$ of test functions converging to the function $f(\cdot)$ in the space $L^p_{\nu(t+\cdot)}[t+k, t+k+2]$). $\square$

3. An Application to the Abstract Degenerate Semilinear Fractional Differential Equations

In this paper, we have not considered the notion of metrical Stepanov uniform recurrence by now. This notion is sometimes crucial because there exist many serious problems in applying Theorem 2.2 and Proposition 2.2 (Proposition 2.3) to the abstract semilinear Cauchy problems; fortunately, these results can be reformulated for the metrically Stepanov uniformly recurrent functions. In this section, we will provide an illustrative application of such analogues of Theorem 2.2 and Proposition 2.3 in the qualitative analysis of metrically Stepanov uniformly recurrent solutions for a class of abstract degenerate semilinear fractional differential equations in the finite-dimensional spaces (the bounded uniformly recurrent solutions for this class of semilinear problems has recently been analyzed in [10]). Before doing this, we would like to note that all applications made to the various classes of the abstract (semilinear) Volterra integro-differential equations and inclusions, given in [15, Section 3, Points 1.-2.], can be simply reformulated for the metrical Stepanov almost periodicity. For example, we can analyze the existence and uniqueness of metrical Stepanov almost periodic solutions of the inhomogeneous heat equation in $\mathbb{R}^n$ and the inhomogeneous fractional Poisson heat equation with Weyl-Liouville fractional derivatives; we can also revisit our recent analyses of the famous Kirchhoff formula, the Poisson formula and d'Alembert formula, governing solutions of the inhomogeneous wave equation in $\mathbb{R}^3$, $\mathbb{R}^2$ and $\mathbb{R}$, respectively.

Consider the finite-dimensional space $X := \mathbb{C}^n$, where $n \geq 2$; let $c > 0$, and $A,\ B \in \mathbb{C}^{n,n}$ (the space of all complex matrices of format $n \times n$). Suppose that the matrix B is not invertible, the degree of complex polynomial $P(\lambda) := \det(\lambda B - A)$, $\lambda \in \mathbb{C}$ is equal to n and its roots lie in the region $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -c(|\operatorname{Im}\lambda| + 1)\}$. We know that these assumptions imply the existence of a positive real constant $M > 0$ such that condition [10, (P)] holds with $\beta = 1$, so that the multivalued linear operator $\mathcal{A} = AB^{-1}$ generates an exponentially decaying, analytic strongly continuous degenerate semigroup $(T(t))_{t\geq 0}$; cf. [9]-[10] for the notion and more details.

Further on, let $0 < \gamma < 1$ and $\nu > -1$. Define

$$T_{\gamma,\nu}(t)x := t^{\gamma\nu}\int_0^\infty s^\nu \Phi_\gamma(s)T\big(st^\gamma\big)x\,ds, \quad t > 0,\ x \in X, \text{ and } P_\gamma(t) := \gamma T_{\gamma,1}(t)/t^\gamma, \quad t > 0.$$

If $\beta \in (0, 1]$, then there exists a finite real constant $M_1 > 0$ such that

$$\big\|P_\gamma(t)\big\| \leq M_1 t^{\gamma(\beta-1)}, \quad t > 0 \quad \text{and} \quad \big\|P_\gamma(t)\big\| \leq M_2 t^{-2\gamma},\ t \geq 1.$$

Set $R_\gamma(t) := t^{\gamma-1}P_\gamma(t)$, $t > 0$.

Of concern is the following abstract fractional inclusion

$$D_+^\gamma \vec{u}(t) \in -\mathcal{A}\vec{u}(t) + F(t, \vec{u}(t)), \quad t \in \mathbb{R}, \tag{3.1}$$

where $D_+^\gamma u(t)$ denotes the Weyl-Liouville fractional derivative of order γ and $F : \mathbb{R} \times X \to X$ (see [9]-[10] for the notion); after the usual substitution $\vec{v}(t) \in B^{-1}\vec{u}(t)$,

$t \in \mathbb{R}$, this inclusion becomes

$$D_+^\gamma \big[B\vec{v}(t)\big] = -A\vec{v}(t) + F\big(t, B\vec{v}(t)\big), \quad t \in \mathbb{R}.$$

We say that a continuous function $u : \mathbb{R} \to X$ is a mild solution of (3.1) if and only if

$$\vec{u}(t) = \int_{-\infty}^{t} R_\gamma(t-s) F\big(s, \vec{u}(s)\big)\, ds, \quad t \in \mathbb{R}.$$

Fix now a strictly increasing sequence (α_k) of positive real numbers tending to plus infinity as well as a number $c \in \mathbb{C} \setminus \{0\}$ and an essentially bounded function $\nu : \mathbb{R} \to (0, \infty)$. By $Y_{(\alpha_k),\nu}(\mathbb{R} : X)$ we denote the set of all bounded continuous functions $\vec{u} : \mathbb{R} \to X$ such that

$$\lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \Big[\big\|\vec{u}(t+\alpha_k) - c\vec{u}(t)\big\|\nu(t)\Big] = 0.$$

It can be simply proved that the set $Y_{(\alpha_k),\nu}(\mathbb{R} : X)$ equipped with the metric $d(\cdot,\cdot) := \|\cdot - \cdot\|_\infty$ forms a complete metric space.

We have the following result:

Theorem 3.1 *Suppose that $p \in (1, \infty)$, $1/p + 1/q = 1$, and $F : \mathbb{R} \times X \to X$ is a measurable function. Let there exist a real number $r \geq \max(p, p/(p-1))$ and a Stepanov r-bounded function $L_F : \mathbb{R} \to [0, \infty)$ such that* (2.4) *holds. Suppose further that, for every compact set $K \subseteq X$, we have*

$$\lim_{k\to+\infty} \sup_{t\in\mathbb{R}, u\in K} \left[\int_t^{t+1} \big\|F(s+\alpha_k; cu) - cF(s; u)\big\|^p \nu^p(s)\, ds\right]^{1/p} = 0.$$

Suppose that $p(\gamma\beta - 1) > 1$, $\int_0^\infty \|R_\gamma(s)\| L_F(s)\, ds < 1$ and there exists an essentially bounded function $\varphi : \mathbb{R} \to (0, \infty)$ such that $\nu(t-s) \leq \varphi(t)\nu(s)$ for all $t,\ s \in \mathbb{R}$. Then there exists a unique mild solution of problem (3.1) *which belongs to the space $Y_{(\alpha_k),\nu}(\mathbb{R} : X)$.*

Proof. Define $\Lambda' := \{\alpha_k : k \in \mathbb{N}\}$ and $\Upsilon : Y_{(\alpha_k),\nu}(\mathbb{R} : X) \to Y_{(\alpha_k),\nu}(\mathbb{R} : X)$ by

$$(\Upsilon\vec{u})(t) := \int_{-\infty}^{t} R_\gamma(t-s) F(s, \vec{u}(s))\, ds, \quad t \in \mathbb{R}.$$

We claim that the mapping $\Upsilon(\cdot)$ is well defined. Suppose that $\vec{u} \in Y_{(\alpha_k),\nu}(\mathbb{R} : X)$. Then the set $R(\vec{u}) = B$ is bounded and therefore relatively compact in X. Applying an analogue of Theorem 2.2 for metrically Stepanov uniformly recurrent functions, we get that the function $H(\cdot) \equiv F(\cdot, \vec{u}(\cdot))$ satisfies

$$\lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \left[\int_t^{t+1} \big\|H(s+\alpha_k) - cH(s; u)\big\|^q \nu^q(s)\, ds\right]^{1/q} = 0.$$

Since $p(\gamma\beta - 1) > 1$ and there exists an essentially bounded function $\varphi : \mathbb{R} \to (0, \infty)$ with the prescribed assumptions, we can repeat verbatim the proof of Proposition

2.3, with the functions $\sigma(\cdot)$ and $1/\nu(\cdot)$ replaced therein with the function $\nu(\cdot)$, and the number q replaced therein with the number p, in order to see that $\Upsilon\vec{u} \in Y_{(\alpha_k),\nu}(\mathbb{R} : X)$; hence, the mapping $\Upsilon(\cdot)$ is well defined. Due to our assumption $\int_0^\infty \|R_\gamma(s)\| L_F(s)\, ds < 1$, this mapping is a contraction so that the proof of theorem completes an application of the Banach contraction principle. $\square$

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