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NEW TRAPEZOID TYPE INEQUALITIES FOR DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper, we first establish that an identity involving generalized fractional integrals for twice differentiable functions. By using this equality, we obtain some trapezoid type inequalities for the functions whose second derivatives in absolute value are convex.
Keywords: differentiable functions, inequalities, generalized fractional integrals.

1. Introduction

In the literature, the theory of inequalities has an important place in mathematics. There are many studies on the well-known Hermite-Hadamard inequality. Many researchers have studied the Hermite-Hadamard inequality and related inequalities such as trapezoid, midpoint, Simpson's inequality, and Bullen's inequality and have contributed to science.

Over the years, numerous articles have focused on obtaining trapezoid and midpoint type inequalities that give bounds for the right-hand side and left-hand side of the Hermite-Hadamard inequality, respectively. For example, Dragomir and Agarwal first established trapezoid inequalities for convex functions in [9], whereas Kirmacı first, obtained midpoint inequalities for convex functions in [13]. Moreover in [17], Qaisar and Hussain presented several generalized midpoint type inequalities. Sarikaya et al. and Iqbal et al. proved some fractional trapezoid and midpoint

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type inequalities for convex functions in [20] and [11], respectively. In [6] and [7], researchers established some generalized midpoint type inequalities for Riemann-Liouville fractional integrals.

Researches on the differentiable functions of these inequalities also have an important place in the literature. Many researchers have focused on twice differentiable functions to obtain many important inequalities. For example, Barani et al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality in [3, 4]. In [14], some new generalized fractional integral inequalities of midpoint and trapezoid type for twice differentiable convex functions are obtained. In [18], authors obtained some new inequalities of the Simpson and the Hermite–Hadamard type for functions whose absolute values of derivatives are convex. In [5] and [10], several fractional Simpson's inequality for twice differentiable functions were obtained. In [8], some generalizations of integral inequalities of Bullen-type for twice differentiable functions involving Riemann-Liouville fractional integrals were obtained. Fore more results please refer to [2, 15, 16].

Here, we give some definitions and notations which are used frequently in main section.

The well-known gamma and beta functions are defined as follows: For $0 < x, y < \infty$,

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt \tag{1.1}$$

and

$$\begin{aligned} \beta(x,y) \quad : \quad &= \int_0^1 t^{x-1} (1-t)^{y-1} dt \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \end{aligned}$$

The generalized fractional integrals were introduced by Sarikaya and Ertuğral as follows:

Definition 1.1. [19] Let us note that a function $\varphi : [0,\infty) \to [0,\infty)$ satisfy the following condition:

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty. \tag{1.2}$$

We consider the following left-sided and right-sided generalized fractional integral operators

$$_{a+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t) dt, \quad x > a \tag{1.3}$$

and

$$_{b-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t) dt, \quad x < b, \tag{1.4}$$

respectively.

The most significant feature of generalized fractional integrals is that they generalize some important types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, conformable fractional integral, etc. These important special cases of the integral operators (1.3) and (1.4) are mentioned as follows:

1. Let us consider $\varphi(t) = t$. Then, the operators (1.3) and (1.4) reduce to the Riemann integral.

2. If we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\alpha > 0$, then the operators (1.3) and (1.4) reduce to the Riemann-Liouville fractional integrals $J_{a+}^{\alpha}f(x)$ and $J_{b-}^{\alpha}f(x)$, respectively. Here, Γ is Gamma function.

3. For $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the operators (1.3) and (1.4) reduce to the k-Riemann-Liouville fractional integrals $J_{a+,k}^{\alpha}f(x)$ and $J_{b-,k}^{\alpha}f(x)$, respectively. Here, Γ_k is k-Gamma function defined by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt, \quad \mathcal{R}(\alpha) > 0 \tag{1.5}$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0. \tag{1.6}$$

2. A new identity for twice differentiable functions

In this section we prove an equality for twice differentiable functions by view of generalized fractional integrals.

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping (a,b) such that $f'' \in L_1([a,b])$. Then, the following equality holds:*

$$\begin{aligned}
&\left[\frac{(x-a)(A_1(1)-B_1(1))}{2A_1(1)} - \frac{(b-x)(A_2(1)-B_2(1))}{2A_2(1)}\right]f'(x) \\
&+\frac{f(a)+f(b)}{2} - \frac{1}{2}\left[\frac{{}_{x-}I_\varphi f(a)}{A_1(1)} + \frac{{}_{x+}I_\varphi f(b)}{2A_2(1)}\right] \\
=\;& \frac{(x-a)^2}{2A_1(1)}\int_0^1 (tA_1(1)-B_1(t))f''(tx+(1-t)a)dt \\
&+\frac{(b-x)^2}{2A_2(1)}\int_0^1 (tA_2(1)-B_2(t))f''(tx+(1-t)b)dt.
\end{aligned}$$

Here,

$$\begin{aligned} A_1(s) &= \int_0^s \frac{\varphi((x-a)u)}{u}du, \\ A_2(s) &= \int_0^s \frac{\varphi((b-x)u)}{u}du, \\ B_1(t) &= \int_0^t A_1(s)ds, \\ B_2(t) &= \int_0^t A_2(s)ds. \end{aligned}$$

Proof. By using integration by parts, we obtain

$$\begin{aligned} J_1 &= \int_0^1 (tA_1(1)-B_1(t))f''(tx+(1-t)a)dt \\ &= (tA_1(t)-B_1(t))\frac{f'(tx+(1-t)a)}{x-a}\Big|_0^1 \\ &\quad -\frac{1}{x-a}\int_0^1 (A_1(1)-A_1(t))f'(tx+(1-t)a)dt \\ &= (A_1(1)-B_1(1))\,\frac{f'(x)}{x-a} \\ &\quad -\frac{1}{x-a}\left[(A_1(1)-A_1(t))\frac{f(tx+(1-t)a)}{x-a}\Big|_0^1 \right. \\ &\quad \left. +\frac{1}{x-a}\int_0^1 \frac{\varphi((x-a)t)}{t}f(tx+(1-t)a)dt\right] \\ &\quad (A_1(1)-B_1(1))\,\frac{f'(x)}{x-a} \\ &\quad -\frac{1}{x-a}\left[-A_1(1)\frac{f(a)}{x-a}+\frac{1}{x-a}\int_a^x (\frac{\varphi(u-a)}{u-a}f(u)du\right] \\ &= \frac{(A_1(1)-B_1(1))f'(x)}{x-a}+\frac{A_1(1)f(a)}{(x-a)^2}-\frac{1}{(x-a)^2}\left[{}_{x-}I_\varphi f(a)\right]. \end{aligned} \tag{2.1}$$

Similar way, we get

$$\begin{aligned} J_2 &= \int_0^1 (tA_2(1)-B_2(t))f''(tx+(1-t)b)dt \\ &= \frac{(B_2(1)-A_2(1))f'(x)}{b-x}+\frac{A_2(1)f(a)}{(b-x)^2}-\frac{1}{(b-x)^2}\left[{}_{x+}I_\varphi f(b)\right]. \end{aligned} \tag{2.2}$$

From equations (2.1) and (2.2), we have

$$J_1\frac{(x-a)^2}{2A_1(1)}+J_2\frac{(b-x)^2}{2A_2(1)}$$

$$= \left[\frac{(x-a)(A_1(1)-B_1(1))}{2A_1(1)} - \frac{(b-x)(A_2(1)-B_2(1))}{2A_2(1)}\right] f'(x)$$
$$+\frac{f(a)+f(b)}{2} - \frac{1}{2}\left[\frac{{}_{x-}I_{\varphi}f(a)}{A_1(1)} + \frac{{}_{x+}I_{\varphi}f(b)}{2A_2(1)}\right]$$

This ends the proof of Lemma 2.1. □

3. Some trapezoid type inequalities for generalized fractional integrals

In this section, by utilizing generalized fractional integrals, we prove some trapezoid type inequalities for functions whose various power of absolute value of second derivatives are convex function.

Theorem 3.1. *Let us consider that the assumptions of Lemma 2.1 are valid. Let us also consider that the mapping $|f''|$ is convex on $[a,b]$. Then, we get the following inequality for generalized fractional integrals*

$$\begin{aligned}(3.1)\quad & \left|\left[\frac{(x-a)(A_1(1)-B_1(1))}{2A_1(1)} - \frac{(b-x)(A_2(1)-B_2(1))}{2A_2(1)}\right] f'(x)\right. \\ & \left. +\frac{f(a)+f(b)}{2} - \frac{1}{2}\left[\frac{{}_{x-}I_{\varphi}f(a)}{A_1(1)} + \frac{{}_{x+}I_{\varphi}f(b)}{2A_2(1)}\right]\right| \\ \leq\ & \frac{(x-a)^2}{2A_1(1)}\left[Q_1^{\varphi}\left|f''(x)\right| + Q_2^{\varphi}\left|f''(a)\right|\right] + \frac{(b-x)^2}{2A_2(1)}\left[Q_3^{\varphi}\left|f''(x)\right| + Q_4^{\varphi}\left|f''(b)\right|\right]\end{aligned}$$

where A_1, A_2, B_1 and B_2 are defined as in Lemma 2.1 and Q_i^{φ}, $i = 1,2,3,4$, are defined by

$$\begin{aligned} Q_1^{\varphi} &= \int_0^1 |tA_1(1) - B_1(t)|\, tdt \\ Q_2^{\varphi} &= \int_0^1 |tA_1(1) - B_1(t)|\,(1-t)dt \\ Q_3^{\varphi} &= \int_0^1 |tA_2(1) - B_2(t)|\, tdt \\ Q_4^{\varphi} &= \int_0^1 |tA_2(1) - B_2(t)|\,(1-t)dt. \end{aligned}$$

Proof. By taking modulus in Lemma 2.1, we have

$$\begin{aligned}(3.2)\quad & \left|\left[\frac{(x-a)(A_1(1)-B_1(1))}{2A_1(1)} - \frac{(b-x)(A_2(1)-B_2(1))}{2A_2(1)}\right] f'(x)\right. \\ & \left. +\frac{f(a)+f(b)}{2} - \frac{1}{2}\left[\frac{{}_{x-}I_{\varphi}f(a)}{A_1(1)} + \frac{{}_{x+}I_{\varphi}f(b)}{2A_2(1)}\right]\right|\end{aligned}$$

$$\leq \frac{(x-a)^2}{2A_1(1)} \int_0^1 |tA_1(1) - B_1(t)| \, |f''(tx + (1-t)a)| \, dt$$
$$+ \frac{(b-x)^2}{2A_2(1)} \int_0^1 |tA_2(1) - B_2(t)| \, |f''(tx + (1-t)b)| \, dt.$$

By using convexity of $|f''|$, we obtain

$$\left| \left[\frac{(x-a)(A_1(1) - B_1(1))}{2A_1(1)} - \frac{(b-x)(A_2(1) - B_2(1))}{2A_2(1)} \right] f'(x) \right.$$
$$\left. + \frac{f(a)+f(b)}{2} - \frac{1}{2} \left[\frac{{}_{x-}I_\varphi f(a)}{A_1(1)} + \frac{{}_{x+}I_\varphi f(b)}{2A_2(1)} \right] \right|$$
$$\leq \frac{(x-a)^2}{2A_1(1)} \int_0^1 |tA_1(1) - B_1(t)| \left[t\,|f''(x)| + (1-t)\,|f''(a)| \right] dt$$
$$+ \frac{(b-x)^2}{2A_2(1)} \int_0^1 |tA_2(1) - B_2(t)| \left[t\,|f''(x)| + (1-t)\,|f''(b)| \right] dt$$
$$= \frac{(x-a)^2}{2A_1(1)} \left[Q_1^\varphi \, |f''(x)| + Q_2^\varphi \, |f''(a)| \right] + \frac{(b-x)^2}{2A_2(1)} \left[Q_3^\varphi \, |f''(x)| + Q_4^\varphi \, |f''(b)| \right].$$

This finishes the proof of Theorem 3.1. □

Corollary 3.1. *If we choose $\varphi(t) = t$ for all $t \in [a, b]$ in Theorem 3.1, then we have the following inequality*

$$\left| (x - \frac{a+b}{2}) f'(x) + \frac{f(a)+f(b)}{2} - \frac{1}{2} \left[\frac{1}{x-a} \int_a^x f(t)dt + \frac{1}{b-x} \int_x^b f(t)dt \right] \right|$$
$$\leq \frac{(x-a)^2}{2} \left[\frac{5|f''(x)|}{24} + \frac{|f''(a)|}{8} \right] + \frac{(b-x)^2}{2} \left[\frac{5|f''(x)|}{24} + \frac{|f''(b)|}{8} \right].$$

Corollary 3.2. *If we take $x = \frac{a+b}{2}$ in Theorem 3.1, then we have the following trapezoid type inequality for generalized fractional integrals*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{\frac{a+b}{2}-}I_\varphi f(a) + {}_{\frac{a+b}{2}+}I_\varphi f(b) \right] \right|$$
$$\leq \frac{(b-a)^2}{8\Lambda(1)} \Psi^\varphi \left[|f''(a)| + |f''(b)| \right]$$

where

$$\Psi^\varphi = \int_0^1 |t\Lambda(1) - \Delta(t)| dt \tag{3.3}$$

$$\Lambda(s) = \int_0^s \frac{\varphi(\frac{b-a}{2}u)}{u}du \tag{3.4}$$

$$\Delta(t) = \int_0^t \Lambda(s). \tag{3.5}$$

Proof. For $x = \frac{a+b}{2}$ in Theorem 3.1, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{\frac{a+b}{2}-}I_{\varphi}f(a) + {}_{\frac{a+b}{2}+}I_{\varphi}f(b)\right]\right| \\ \leq\ & \frac{(b-a)^2}{8\Lambda(1)}\left[\Psi_1^{\varphi}\left|f''\left(\frac{a+b}{2}\right)\right| + \Psi_2^{\varphi}\left|f''(a)\right|\right] \\ & + \frac{(b-a)^2}{8\Lambda(1)}\left[\Psi_1^{\varphi}\left|f''\left(\frac{a+b}{2}\right)\right| + \Psi_2^{\varphi}\left|f''(b)\right|\right] \\ \leq\ & \frac{(b-a)^2}{8\Lambda(1)}\left[\Psi_1^{\varphi} + \Psi_2^{\varphi}\right]\left[\left|f''(a)\right| + \left|f''(b)\right|\right] \\ =\ & \frac{(b-a)^2}{8\Lambda(1)}\Psi^{\varphi}\left[\left|f''(a)\right| + \left|f''(b)\right|\right] \end{aligned}$$

where

$$\Psi_1^{\varphi} = \int_0^1 |t\Lambda(1) - \Delta(t)|tdt \tag{3.6}$$

and

$$\Psi_2^{\varphi} = \int_0^1 |t\Lambda(1) - \Delta(t)|\,(1-t)\,dt. \tag{3.7}$$

This finishes the proof. □

Remark 3.1. If we choose $\varphi(t) = t$ for all $t \in [a,b]$ in Corollary 3.2, then we have the following trapezoid inequality for Riemann integrals

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^2}{24}\left(\left|f''(a)\right| + \left|f''(b)\right|\right) \tag{3.8}$$

which was given by Sarikaya and Aktan in [18].

Corollary 3.3. *By choosing $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}, \alpha > 0$ for all $t \in [a,b]$ in Corollary 3.2, then we have the following trapezoid type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a) + J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \\ \leq\ & \frac{(b-a)^2}{8}\left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)}\right)\left(\left|f''(a)\right| + \left|f''(b)\right|\right). \end{aligned}$$

Corollary 3.4. *By choosing* $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}, \alpha, k>0$,*for all* $t\in[a,b]$ *in Corollary 3.2, then we have the following trapezoid type inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-,k}f(a)+J^{\alpha}_{\frac{a+b}{2}+,k}f(b)\right]\right|\\ \leq\;& \frac{(b-a)^2}{8}\left(\frac{1}{2}-\frac{k^2}{(\alpha+k)(\alpha+2k)}\right)(|f''(a)|+|f''(b)|).\end{aligned}$$

Theorem 3.2. *Let us note that the assumptions of Lemma 2.1 hold. If the mapping* $|f''|^q$, $q>1$ *is convex on* $[a,b]$, *then we have the following inequality for generalized fractional integrals*

$$\begin{aligned}&\left|\left[\frac{(x-a)(A_1(1)-B_1(1))}{2A_1(1)}-\frac{(b-x)(A_2(1)-B_2(1))}{2A_2(1)}\right]f'(x)\right.\\ &\left.+\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[\frac{{}_{x-}I_{\varphi}f(a)}{A_1(1)}+\frac{{}_{x+}I_{\varphi}f(b)}{2A_2(1)}\right]\right|\\ \leq\;& \frac{(x-a)^2}{2A_1(1)}\left(\int_0^1|tA_1(1)-B_1(t)|^p\,dt\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q+|f''(a)|^q}{2}\right)^{\frac{1}{q}}\\ &+\frac{(b-x)^2}{2A_2(1)}\left(\int_0^1|tA_2(1)-B_2(t)|^p\,dt\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\end{aligned}$$

where A_1, A_2, B_1 *and* B_2 *are defined as in Lemma 2.1.*

Proof. By using the Hölder inequality in inequality (3.2), we obtain

$$\begin{aligned}&\left|\left[\frac{(x-a)(A_1(1)-B_1(1))}{2A_1(1)}-\frac{(b-x)(A_2(1)-B_2(1))}{2A_2(1)}\right]f'(x)\right.\\ &\left.+\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[\frac{{}_{x-}I_{\varphi}f(a)}{A_1(1)}+\frac{{}_{x+}I_{\varphi}f(b)}{2A_2(1)}\right]\right|\\ \leq\;& \frac{(x-a)^2}{2A_1(1)}\left(\int_0^1|tA_1(1)-B_1(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\\ &+\frac{(b-x)^2}{2A_2(1)}\left(\int_0^1|tA_2(1)-B_2(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}.\end{aligned}$$

With the help of the convexity of $|f''|^q$, we get

$$\left|\left[\frac{(x-a)(A_1(1)-B_1(1))}{2A_1(1)}-\frac{(b-x)(A_2(1)-B_2(1))}{2A_2(1)}\right]f'(x)\right.$$

$$
\begin{aligned}
&+\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[\frac{{}_{x-}I_{\varphi}f(a)}{A_1(1)}+\frac{{}_{x+}I_{\varphi}f(b)}{2A_2(1)}\right]\Bigg| \\
\le\;& \frac{(x-a)^2}{2A_1(1)}\left(\int_0^1 |tA_1(1)-B_1(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[t\,|f''(x)|^q+(1-t)\,|f''(a)|^q\right]dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2}{2A_2(1)}\left(\int_0^1 |tA_2(1)-B_2(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[t\,|f''(x)|^q+(1-t)\,|f''(b)|^q\right]dt\right)^{\frac{1}{q}} \\
=\;& \frac{(x-a)^2}{2A_1(1)}\left(\int_0^1 |tA_1(1)-B_1(t)|^p\,dt\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q+|f''(a)|^q}{2}\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2}{2A_2(1)}\left(\int_0^1 |tA_2(1)-B_2(t)|^p\,dt\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof of Theorem 3.2. □

Corollary 3.5. *If we choose $\varphi(t)=t$ for all $t\in[a,b]$ in Theorem 3.2, then we have the following inequality*

$$
\begin{aligned}
&\left|(x-\frac{a+b}{2})f'(x)+\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\le\;& \left(\frac{(\Gamma(p+1))^2}{2\Gamma(2p+2)}\right)^{\frac{1}{p}}\left[(x-a)^2\left(\frac{|f''(x)|^q+|f''(a)|^q}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+(b-x)^2\left(\frac{|f''(x)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Proof. For $\varphi(t)=t$, we have

$$
\begin{aligned}
\int_0^1 |tA_1(1)-B_1(t)|^p\,dt &= \int_0^1 \left|t(x-a)-(x-a)\frac{t^2}{2}\right|^p dt \\
&= (x-a)^p\int_0^1 t^p\left(1-\frac{t}{2}\right)^p dt \\
&= 2^{p-1}(x-a)^p\int_0^1 t^p\,(1-t)^p\,dt \\
&= 2^{p-1}(x-a)^pB(p+1,p+1) \\
&= 2^{p-1}(x-a)^p\frac{(\Gamma(p+1))^2}{\Gamma(2p+2)}
\end{aligned}
$$

and similarly

$$\int_0^1 |tA_2(1) - B_2(t)|^p \, dt = 2^{p-1}(b-x)^p \frac{(\Gamma(p+1))^2}{\Gamma(2p+2)}. \tag{3.9}$$

This completes the proof. □

Corollary 3.6. *If we take $x = \frac{a+b}{2}$ in Theorem 3.2, then we have the ollowing trapezoid type inequality for generalized fractional integrals*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{\frac{a+b}{2}-}I_\varphi f(a) + {}_{\frac{a+b}{2}+}I_\varphi f(b) \right] \right| \\ \leq & \frac{(b-a)^2}{8\Lambda(1)} \left(\int_0^1 |t\Lambda(1) - \Delta(t)|^p \, dt \right)^{\frac{1}{p}} \\ & \times \left[\left(\frac{3|f''(a)|^q + |f''(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|f''(a)|^q + 3|f''(b)|^q}{4} \right)^{\frac{1}{q}} \right] \\ \leq & \frac{(b-a)^2}{8\Lambda(1)} \left(4 \int_0^1 |t\Lambda(1) - \Delta(t)|^p \, dt \right)^{\frac{1}{p}} [|f''(a)| + |f''(b)|] \end{aligned}$$

where Λ and Δ are defined as in Corollary 3.2.

Proof. By choosing $x = \frac{a+b}{2}$ in Theorem 3.2, we get

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{\frac{a+b}{2}-}I_\varphi f(a) + {}_{\frac{a+b}{2}+}I_\varphi f(b) \right] \right| \\ \leq & \frac{(b-a)^2}{8\Lambda(1)} \left(\int_0^1 |t\Lambda(1) - \Delta(t)|^p \, dt \right)^{\frac{1}{p}} \\ & \times \left[\left(\frac{\left|f''(\frac{a+b}{2})\right|^q + |f''(a)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{\left|f''(\frac{a+b}{2})\right|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}} \right] \\ \leq & \frac{(b-a)^2}{8\Lambda(1)} \left(\int_0^1 |t\Lambda(1) - \Delta(t)|^p \, dt \right)^{\frac{1}{p}} \\ & \times \left[\left(\frac{3|f''(a)|^q + |f''(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|f''(a)|^q + 3|f''(b)|^q}{4} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

For the proof of second inequality, let $a_1 = |f''(a)|^q$, $b_1 = 3|f''(b)|^q$, $a_2 = 3|f''(a)|^q$ and $b_2 = |f''(b)|^q$. Using the facts that,

$$\sum_{k=1}^{n} (a_k + b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s, \tag{3.10}$$

for $0 \leq s < 1$and $1 + 3^{\frac{1}{q}} \leq 4$, then the desired result can be obtained straightforwardly. This completes the proof of Theorem 3.6. □

Remark 3.2. If we choose $\varphi(t) = t$ for all $t \in [a, b]$ in Corollary 3.6, then we have the following trapezoid inequality for Riemann integrals

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq\ & \frac{(b-a)^2}{8}\left(\frac{(\Gamma(p+1))^2}{2\Gamma(2p+2)}\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f''(a)|^q+3|f''(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)^2}{8}\left(\frac{2(\Gamma(p+1))^2}{\Gamma(2p+2)}\right)^{\frac{1}{p}}\left[|f''(a)|+|f''(b)|\right].
\end{aligned}$$

Corollary 3.7. *By choosing $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}, \alpha > 0$ for all $t \in [a, b]$ in Corollary 3.2, then we have the following trapezoid type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-}f(a) + J^\alpha_{\frac{a+b}{2}+}f(b)\right]\right| \\
\leq\ & \frac{(b-a)^2}{8}\left(\int_0^1 t^p\left(1-\frac{t^\alpha}{\alpha+1}\right)^p dt\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f''(a)|^q+3|f''(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)^2}{8}\left(4\int_0^1 t^p\left(1-\frac{t^\alpha}{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left[|f''(a)|+|f''(b)|\right].
\end{aligned}$$

Corollary 3.8. *By choosing $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}, \alpha, k > 0$,for all $t \in [a, b]$ in Corollary 3.2, then we have the following trapezoid type inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-,k}f(a) + J^\alpha_{\frac{a+b}{2}+,k}f(b)\right]\right| \\
\leq\ & \frac{(b-a)^2}{8}\left(\int_0^1 t^p\left(1-\frac{kt^{\frac{\alpha}{k}}}{\alpha+k}\right)^p dt\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f''(a)|^q+3|f''(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)^2}{8}\left(4\int_0^1 t^p\left(1-\frac{kt^{\frac{\alpha}{k}}}{\alpha+k}\right)^p dt\right)^{\frac{1}{p}}\left[|f''(a)|+|f''(b)|\right].
\end{aligned}$$

Theorem 3.3. *Let us note that the assumptions of Lemma 2.1 hold. If the mapping $|f''|^q$, $q \geq 1$ is convex on $[a, b]$, then we have the following inequality*

$$\begin{aligned}
&\left| \left[\frac{(x-a)(A_1(1)-B_1(1))}{2A_1(1)} - \frac{(b-x)(A_2(1)-B_2(1))}{2A_2(1)} \right] f'(x) \right. \\
&\left. + \frac{f(a)+f(b)}{2} - \frac{1}{2}\left[\frac{{}_{x-}I_\varphi f(a)}{A_1(1)} + \frac{{}_{x+}I_\varphi f(b)}{2A_2(1)} \right] \right| \\
\leq\ & \frac{(x-a)^2}{2A_1(1)} \left(Q_5^\varphi\right)^{1-\frac{1}{q}} \left(Q_1^\varphi |f''(x)|^q + Q_2^\varphi |f''(a)|^q \right)^{\frac{1}{q}} \\
&+ \frac{(b-x)^2}{2A_2(1)} \left(Q_6^\varphi\right)^{1-\frac{1}{q}} \left(Q_3^\varphi |f''(x)|^q + Q_4^\varphi |f''(b)|^q \right)^{\frac{1}{q}}.
\end{aligned}$$

where A_1, A_2, B_1 and B_2 are defined as in Lemma 2.1, Q_i^φ, $i = 1, 2, 3, 4$, are defined by as in Theorem 3.1 and Q_5^φ and Q_6^φ are defined by

$$\left\{ \begin{array}{l} Q_5^\varphi = \int_0^1 |tA_1(1) - B_1(t)|\, dt \\ \\ Q_6^\varphi = \int_0^1 |tA_2(1) - B_2(t)|\, dt. \end{array} \right. . \tag{3.11}$$

Proof. By applying power-mean inequality in (3.2), we obtain

$$\begin{aligned}
&\left| \left[\frac{(x-a)(A_1(1)-B_1(1))}{2A_1(1)} - \frac{(b-x)(A_2(1)-B_2(1))}{2A_2(1)} \right] f'(x) \right. \\
&\left. + \frac{f(a)+f(b)}{2} - \frac{1}{2}\left[\frac{{}_{x-}I_\varphi f(a)}{A_1(1)} + \frac{{}_{x+}I_\varphi f(b)}{2A_2(1)} \right] \right| \\
\leq\ & \frac{(x-a)^2}{2A_1(1)} \left(\int_0^1 |tA_1(1) - B_1(t)|\, dt \right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 |tA_1(1) - B_1(t)| \, |f''(tx + (1-t)a)|^q \, dt \right)^{\frac{1}{q}} \\
&+ \frac{(b-x)^2}{2A_2(1)} \left(\int_0^1 |tA_2(1) - B_2(t)|\, dt \right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 |tA_2(1) - B_2(t)| \, |f''(tx + (1-t)b)|^q \, dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f''|^q$ is convex, we have

$$\left| \left[\frac{(x-a)(A_1(1)-B_1(1))}{2A_1(1)} - \frac{(b-x)(A_2(1)-B_2(1))}{2A_2(1)} \right] f'(x) \right.$$

$$
\begin{aligned}
& +\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[\frac{{}_{x-}I_{\varphi}f(a)}{A_1(1)}+\frac{{}_{x+}I_{\varphi}f(b)}{2A_2(1)}\right]\Bigg| \\
\leq\ & \frac{(x-a)^2}{2A_1(1)}\left(\int_0^1 |tA_1(1)-B_1(t)|\,dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 |tA_1(1)-B_1(t)|\left[t\,|f''(x)|^q+(1-t)\,|f''(a)|^q\right]dt\right)^{\frac{1}{q}} \\
& +\frac{(b-x)^2}{2A_2(1)}\left(\int_0^1 |tA_2(1)-B_2(t)|\,dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 |tA_2(1)-B_2(t)|\left[t\,|f''(x)|^q+(1-t)\,|f''(b)|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{(x-a)^2}{2A_1(1)}\left(Q_5^{\varphi}\right)^{1-\frac{1}{q}}\left(Q_1^{\varphi}\,|f''(x)|^q+Q_2^{\varphi}\,|f''(a)|^q\right)^{\frac{1}{q}} \\
& +\frac{(b-x)^2}{2A_2(1)}\left(Q_6^{\varphi}\right)^{1-\frac{1}{q}}\left(Q_3^{\varphi}\,|f''(x)|^q+Q_4^{\varphi}\,|f''(b)|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Then, we obtain the desired result of Theorem 3.3. □

Corollary 3.9. *If we choose $\varphi(t)=t$ for all $t\in[a,b]$ in Theorem 3.3, then we have the following inequality*

$$
\begin{aligned}
& \left|(x-\frac{a+b}{2})f'(x)+\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & \frac{(x-a)^2}{6}\left(\frac{5|f''(x)|^q+3\,|f''(a)|^q}{8}\right)^{\frac{1}{q}}+\frac{(b-x)^2}{6}\left(\frac{5|f''(x)|^q+3\,|f''(b)|^q}{8}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 3.10. *If we take $x=\frac{a+b}{2}$ in Theorem 3.3, then we have the ollowing trapezoid type inequality for generalized fractional integrals*

$$
\begin{aligned}
& \left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\frac{a+b}{2}-}I_{\varphi}f(a)+{}_{\frac{a+b}{2}+}I_{\varphi}f(b)\right]\right| \\
\leq\ & \frac{(b-a)^2}{8\Lambda(1)}\left(\Psi^{\varphi}\right)^{1-\frac{1}{q}}\left(\frac{(\Psi_1^{\varphi}+2\Psi_2^{\varphi})\,|f''(a)|^q+\Psi_1^{\varphi}\,|f''(b)|^q}{2}\right)^{\frac{1}{q}} \\
& +\frac{(b-a)^2}{8\Lambda(1)}\left(\Psi^{\varphi}\right)^{1-\frac{1}{q}}\left(\frac{\Psi_1^{\varphi}\,|f''(a)|^q+(\Psi_1^{\varphi}+2\Psi_2^{\varphi})\,|f''(b)|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}
$$

where Λ, Δ *and* Ψ^{φ} *are defined as in Corollary 3.2. Here* Ψ_1^{φ} *and* Ψ_2^{φ} *are given by as in (3.6) and (3.7), respectively.*

Remark 3.3. If we choose $\varphi(t) = t$ for all $t \in [a, b]$ in Corollary 3.10, then we have the following trapezoid inequality for Riemann integrals

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{(b-a)^2}{24}\left[\left(\frac{11\left|f''(a)\right|^q + 5\left|f''(b)\right|^q}{16}\right)^{\frac{1}{q}} + \left(\frac{5\left|f''(a)\right|^q + 11\left|f''(b)\right|^q}{16}\right)^{\frac{1}{q}}\right].$$

Corollary 3.11. *By choosing* $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}, \alpha > 0$ *for all* $t \in [a, b]$ *in Corollary 3.10, then we have the following trapezoid type inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a) + J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{8}\left(\vartheta_1(\alpha)\right)^{1-\frac{1}{q}}\left(\frac{\left(\vartheta_2(\alpha)+2\vartheta_3(\alpha)\right)\left|f''(a)\right|^q + \vartheta_2(\alpha)\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}$$
$$+\frac{(b-a)^2}{8}\left(\vartheta_1(\alpha)\right)^{1-\frac{1}{q}}\left(\frac{\vartheta_2(\alpha)\left|f''(a)\right|^q + \left(\vartheta_2(\alpha)+2\vartheta_3(\alpha)\right)\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

where

$$\vartheta_1(\alpha) = \frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)} \tag{3.12}$$

$$\vartheta_2(\alpha) = \frac{1}{3} - \frac{1}{(\alpha+1)(\alpha+3)} \tag{3.13}$$

and

$$\vartheta_3(\alpha) = \frac{1}{6} - \frac{1}{(\alpha+1)(\alpha+2)\alpha+3)}. \tag{3.14}$$

Corollary 3.12. *By choosing* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}, \alpha, k > 0$,*for all* $t \in [a, b]$ *in Corollary 3.10, then we have the following trapezoid type inequality for k-Riemann-Liouville fractional integrals*

$$\left|\frac{f(a)+f(b)}{2} - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-,k}f(a) + J^{\alpha}_{\frac{a+b}{2}+,k}f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{8}\left(\vartheta_1(\alpha,k)\right)^{1-\frac{1}{q}}$$

$$\times \left[\left(\frac{(\vartheta_2(\alpha,k) + 2\vartheta_3(\alpha,k)) \left| f''(a) \right|^q + \vartheta_2(\alpha,k) \left| f''(b) \right|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{\vartheta_2(\alpha,k) \left| f''(a) \right|^q + (\vartheta_2(\alpha,k) + 2\vartheta_3(\alpha,k)) \left| f''(b) \right|^q}{2} \right)^{\frac{1}{q}} \right]$$

where

$$\vartheta_1(\alpha,k) = \frac{1}{2} - \frac{k^2}{(\alpha+k)(\alpha+2k)} \tag{3.15}$$

$$\vartheta_2(\alpha,k) = \frac{1}{3} - \frac{k^2}{(\alpha+k)(\alpha+3k)} \tag{3.16}$$

and

$$\vartheta_3(\alpha,k) = \frac{1}{6} - \frac{k^3}{(\alpha+k)(\alpha+2k)\alpha+3k)}. \tag{3.17}$$

4. Conclusion

In this study, trapezoid type inequality for twice differentiable functions using generalized fractional integrals are obtained. Also, we prove that our results generalize the inequalities obtained in earlier works. Some new inequalities for k-Riemann-Liouville fractional integrals are obtained by special choices of main findings. In the future works, authors can try to generalize our results by utilizing other kinds of convex function classes.

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