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SPHERES AND CIRCLES WITH RESPECT TO AN INDEFINITE METRIC ON A RIEMANNIAN MANIFOLD WITH A SKEW-CIRCULANT STRUCTURE

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Abstract. We study hyper-spheres, spheres and circles, with respect to an indefinite metric, in a tangent space on a 4-dimensional differentiable manifold. The manifold is equipped with a positive definite metric and an additional tensor structure of type $(1, 1)$. The fourth power of the additional structure is minus the identity and its components form a skew-circulant matrix in some local coordinate system. The both structures are compatible and they determine an associated indefinite metric on the manifold.
Keywords: indefinite metric, tangent space, tensor structure, manifold.

1. Introduction

There are various applications of the correspondences between circles and ellipses (circles and hyperbolas, circles and parabolas), as well as between spheres and other quadratic surfaces, for example in geometry, mechanics, astrophysics. Circles and spheres could be determined with respect to an indefinite metric and then their images could be obtained in Euclidean space. In this vein, we consider a circle determined with respect to an associated indefinite metric on a Riemannian manifold and the corresponding quadratic curve in Euclidean space. Also, we study a sphere (a hyper-sphere) determined with respect to an associated indefinite

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metric on a Riemannian manifold and the corresponding quadratic surface (hypersurface) in Euclidean space. We will mention some papers which concern models of hyper-spheres, spheres and circles with respect to some indefinite metrics and their relations with the corresponding quadratic geometrical objects ([1, 5, 11, 12, 13]).

The Hermitian manifolds form a class of manifolds with an integrable almost complex structure J ([9]). One subclass consists of the so-called locally conformal Kähler manifolds, determined by a special property of the covariant derivative of J. Some of the recent investigations of locally conformal Kähler manifolds are made in [2, 3, 10, 14, 15, 16].

We consider a 4-dimensional Riemannian manifold M, endowed with a positive definite metric g and an endomorphism S in a tangent space T_pM at an arbitrary point p on M. The fourth power of S is minus the identity and the components of S form a skew-circulant matrix with respect to some basis of T_pM. It is supposed that S is compatible with g. Such a manifold (M, g, S) is defined in [6]. In [7] it is proved that (M, g, J), where $J = S^2$, is a locally conformal Kähler manifold. We consider an associated metric $\tilde{g}$ on (M, g, S), defined by both structures g and S. The metric $\tilde{g}$ is necessarily indefinite and it determines space-like vectors, isotropic vectors and time-like vectors in every T_pM. We study hyper-spheres in T_pM, spheres and circles in some special subspaces of T_pM with respect to $\tilde{g}$.

The paper is organized as follows. In Sect. 2., we recall some necessary facts, definitions and statements about the manifold (M, g, S) obtained in [6] and [7]. In Sect. 3., we find the equation of a central hyper-sphere in T_pM with respect to the associated metric $\tilde{g}$. In Sect. 4., we consider spheres with respect to $\tilde{g}$ in special 3-dimensional subspaces of T_pM and obtain their equations. In Sect. 5., we consider some special 2-planes in T_pM and we get the equations of circles with respect to $\tilde{g}$ in these 2-planes. We interpret all equations of the curves and surfaces, studied in Sect. 3., Sect. 4. and Sect. 5., in terms of g.

2. Preliminaries

The skew-circulant matrices are Toeplitz matrices, which are well-studied in [4] and [8]. In our work we consider a tensor structure on a 4-dimensional differentiable manifold, whose component matrix is skew-circulant. Therefore we recall the following definition. *A real skew-circulant matrix* with the first row $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ is a square matrix of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_4 & a_1 & a_2 & a_3 \\ -a_3 & -a_4 & a_1 & a_2 \\ -a_2 & -a_3 & -a_4 & a_1 \end{pmatrix}.$$

We now introduce a manifold (M, g, S) in detail. Let M be a 4-dimensional Riemannian manifold equipped with a tensor S of type $(1, 1)$. Let the components of S form the following skew-circulant matrix in a local coordinate system:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then S has the property

$$S^4 = -\mathrm{id}. \tag{2.1}$$

We assume that g is a positive definite metric on M, which satisfies the equality

$$g(Su, Sv) = g(u, v), \quad u, v \in \mathfrak{X}M. \tag{2.2}$$

Such a manifold (M, g, S) is introduced in [6]. The manifold (M, g, J), where $J = S^2$, is a locally conformal Kähler manifold (Theorem 5.3 in [7]).

The associated metric $\tilde{g}$ on (M, g, S), defined by

$$\tilde{g}(u, v) = g(u, Sv) + g(Su, v), \tag{2.3}$$

is necessarily indefinite. Consequently, having in mind (2.3), for an arbitrary vector v it is valid:

$$\tilde{g}(v, v) = 2g(v, Sv) = a, \qquad a \in \mathbb{R}. \tag{2.4}$$

According to the physical terminology we give the following

Definition 2.1. Let $\tilde{g}$ be the associated metric on (M, g, S). If a vector u satisfies $\tilde{g}(u, u) > 0$ (resp. $\tilde{g}(u, u) < 0$), then u is a space-like (resp. a time-like) vector. If u is nonzero and satisfies $\tilde{g}(u, u) = 0$, then u is an isotropic vector.

It is well-known that the norm of every vector u of the tangent space T_pM and the cosine of the angle between two nonzero vectors u and v of T_pM are given by

$$\|u\| = \sqrt{g(u, u)}, \tag{2.5}$$

$$\cos\angle(u, v) = \frac{g(u, v)}{\|u\|\|v\|}. \tag{2.6}$$

A basis of type $\{u, Su, S^2u, S^3u\}$ of T_pM is called an *S-basis*. In this case we say that *the vector u induces an S-basis of T_pM*. In [6] the following assertions are proved. If a vector u induces an S-basis, then

(i) the angles between the basis vectors are

$$\begin{gathered} \angle(u, Su) = \angle(Su, S^2u) = \angle(S^2u, S^3u) = \pi - \angle(S^3u, u), \\ \angle(u, S^2u) = \angle(Su, S^3u) = \tfrac{\pi}{2}. \end{gathered} \tag{2.7}$$

(ii) the angle φ, determined by

$$\varphi = \angle(u, Su), \tag{2.8}$$

satisfies inequalities

$$\frac{\pi}{4} < \varphi < \frac{3\pi}{4}. \tag{2.9}$$

Next we have

Theorem 2.1. *Let $\tilde{g}$ be the associated metric on (M, g, S) and let the vector u induce an S-basis. The following propositions hold true.*

(i) Vector u is space-like if and only if $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

(ii) Vector u is isotropic if and only if $\varphi = \frac{\pi}{2}$.

(iii) Vector u is time-like if and only if $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{4})$.

Proof. Using (2.4), (2.5), (2.6) and (2.8) we get $\tilde{g}(u, u) = 2\|u\|^2 \cos\varphi$. Having in mind Definition 2.1 and inequalities (2.9) the proof follows. □

Evidently, due to (2.1), (2.2) and (2.3), we state

Corollary 2.1. *If u is a space-like (isotropic or time-like) vector, then Su, S^2u and S^3u are space-like (isotropic or time-like) vectors, respectively.*

In the next sections we get equations of hyper-spheres, spheres and circles with respect to $\tilde{g}$ in some subspaces of T_pM on (M, g, S). Obviously, the obtained curves and surfaces do not depend on the choice of the basis. We use orthonormal bases with respect to the metric g on (M, g, S) to find their equations easier. In Section 3., we use an orthonormal S-*basis* of T_pM. The existence of such bases is proved in [6]. In Section 4. and Section 5., we construct orthonormal bases of 3-dimensional subspaces of T_pM and of 2-dimensional subspaces of T_pM with the help of an arbitrary S-*basis*.

3. Hyper-spheres with respect to the associated metric

Let $\{u, Su, S^2u, S^3u\}$ be an orthonormal S-*basis* of T_pM with respect to the metric g on (M, g, S). If p_{xyzt} is a coordinate system such that the vectors u, Su, S^2u and S^3u are on the axes p_x, p_y, p_z and p_t, respectively, then p_{xyzt} is orthonormal. The radius vector v of an arbitrary point (x, y, z, t) of T_pM is expressed by the equality

$$v = xu + ySu + zS^2u + tS^3u. \tag{3.1}$$

A hyper-sphere s centered at the origin p, with respect to $\tilde{g}$ on (M, g, S), is defined by (2.4). We apply (3.1) into (2.4), and bearing in mind that p_{xyzt} is an orthonormal coordinate system and also equalities (2.1) and (2.2), we obtain the equation of s as follows:

$$2(xy - xt + yz + zt) = a. \tag{3.2}$$

Now, we transform the coordinate system p_{xyzt} into $p_{x'y'z't'}$ by

$$x = \frac{1}{2}(x' - y' + z' - t'), \quad y = \frac{\sqrt{2}}{2}(-y' + t')$$
$$z = -\frac{1}{2}(x' + y' + z' + t'), \ t = \frac{\sqrt{2}}{2}(-x' + z'). \tag{3.3}$$

We substitute (3.3) into (3.2) and it takes the form

$$x'^2 + y'^2 - z'^2 - t'^2 = \frac{a}{\sqrt{2}}. \tag{3.4}$$

Evidently, in terms of g, we have that (3.4) is an equation of a 3-dimensional hyperboloid.

Therefore, we state the following

Theorem 3.1. *Let $\tilde{g}$ be the associated metric on (M, g, S) and let the vector u induce an orthonormal S-basis of T_pM. If p_{xyzt} is a coordinate system such that $u \in p_x$, $Su \in p_y$, $S^2u \in p_z$, $S^3u \in p_t$, then the hyper-sphere* (2.4) *has the equation* (3.4) *with respect to the coordinate system $p_{x'y'z't'}$, obtained by the transformation* (3.3) *of p_{xyzt}.*

Corollary 3.1. *Let s be the 3-dimensional hyperboloid* (3.4). *The following propositions are valid.*

i) Every point on s, where $a < 0$, has a time-like radius vector;

ii) Every point on s, where $a = 0$, has an isotropic radius vector;

iii) Every point on s, where $a > 0$, has a space-like radius vector.

Proof. According to Definition 2.1 and due to (2.4) the statement holds. □

Corollary 3.2. *Let s be the 3-dimensional hyperboloid* (3.4). *Then the intersections σ_1, σ_2, σ_3 and σ_4 between s and the coordinate planes of $p_{x'y'z't'}$, respectively, are the following surfaces:*

i) σ_1, σ_2 are hyperboloids of two sheets and σ_3, σ_4 are hyperboloids of one sheet, in case $a > 0$;

ii) σ_1, σ_2 are hyperboloids of one sheet and σ_3, σ_4 are hyperboloids of two sheets, in case $a < 0$;

iii) σ_1, σ_2, σ_3 and σ_4 are circular cones, in case $a = 0$.

Proof. Using (3.4) and the equation of the coordinate plane $x' = 0$ we get the surface $\sigma_1 : \ \sqrt{2}(y'^2 - z'^2 - t'^2) = a, \ x' = 0$. Consequently, if $a > 0$, then σ_1 is a hyperboloid of two sheet, if $a < 0$, then σ_1 is a hyperboloid of one sheet and if $a = 0$, then σ_1 is a circular cone. Similarly, we consider the other cases of intersections σ_2, σ_3 and σ_4. □

Theorem 3.2. *Let $\tilde{g}$ be the associated metric on (M, g, S) and let the vector u induce an orthonormal S-basis of T_pM. If p_{xyzt} is a coordinate system such that $u \in p_x$, $Su \in p_y$, $S^2u \in p_z$ and $S^3u \in p_t$, then u, Su, S^2u and S^3u are isotropic vectors and their heads lie at the surface with equations*

$$x'^2 + y'^2 = \frac{1}{2}, \quad z'^2 + t'^2 = \frac{1}{2}, \tag{3.5}$$

where $p_{x'y'z't'}$ is the coordinate system obtained by the transformation (3.3) *of p_{xyzt}.*

Proof. Bearing in mind (2.3) and Definition 2.1 we get that u, Su, S^2u and S^3u are isotropic vectors with respect to $\tilde{g}$. Therefore, their heads are on the hyper-cone (3.4) in case $a = 0$. On the other hand, these heads lie at the unit hyper-sphere with respect to g. This hyper-sphere with respect to $p_{x'y'z't'}$ has the equation

$$x'^2 + y'^2 + z'^2 + t'^2 = 1. \tag{3.6}$$

The system of (3.4), where $a = 0$, and (3.6) gives the intersection of a hyper-cone with a hyper-sphere. This intersection, with respect to the coordinate system $p_{x'y'z't'}$, is represented by the equivalent system (3.5). □

4. Spheres in a 3-dimensional subspace of T_pM

Let the unit vector u induce an S-*basis* of T_pM. Hence u induces four different pyramids spanned by the following triples $\{u, Su, S^2u\}$, $\{Su, S^2u, S^3u\}$, $\{u, Su, S^3u\}$ and $\{u, S^2u, S^3u\}$. According to (2.2) and (2.7), the first and the second pyramid constructed on these basis vectors are equal, as well as the third and the fourth pyramid are also equal. Thus we will investigate only the subspaces with bases defined by the first and the third pyramid.

4.1. A sphere in the 3-dimensional subspace of T_pM, spanned by vectors u, Su and S^2u

Lemma 4.1. *Let α_1 be a subspace of T_pM with a basis $\{u, Su, S^2u\}$. The system of vectors $\{e_1, e_2, e_3\}$, determined by the equalities*

$$e_1 = u, \quad e_2 = \frac{(-\cos\varphi)u + Su - (\cos\varphi)S^2u}{\sqrt{1 - 2\cos^2\varphi}}, \quad e_3 = S^2u, \tag{4.1}$$

where $\varphi = \angle(u, Su)$, form an orthonormal basis of α_1.

Proof. Using (2.2), (2.7) and (4.1) we obtain $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$ and $g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0$. □

The coordinate system p_{xyz} such that $e_1 \in p_x$, $e_2 \in p_y$ and $e_3 \in p_z$ is orthonormal.

A sphere s_1 in α_1 centered at the origin p, with respect to $\tilde{g}$ on (M, g, S), is defined by (2.4). In the next statement we get the equation of s_1 with respect to the orthonormal coordinate system p_{xyz}.

Theorem 4.1. *Let $\tilde{g}$ be the associated metric on (M, g, S) and let α_1 be a 3-dimensional subspace of T_pM with a basis $\{u, Su, S^2u\}$. If e_1, e_2 and e_3 are determined by* (4.1) *and p_{xyz} is a coordinate system such that $e_1 \in p_x$, $e_2 \in p_y$, $e_3 \in p_z$, then the equation of the sphere s_1 in α_1 is given by*

$$2(\cos\varphi)(x^2 - y^2 + z^2) + 2\sqrt{1 - 2\cos^2\varphi}(xy + yz) = a. \tag{4.2}$$

Proof. The radius vector v of an arbitrary point (x, y, z) on α_1 is expressed by $v = xe_1 + ye_2 + ze_3$. We apply the latter equality into (2.4) and we find

$$\begin{aligned}\tilde{g}(e_1, e_1)x^2 + \tilde{g}(e_2, e_2)y^2 + \tilde{g}(e_3, e_3)z^2 + 2\tilde{g}(e_1, e_2)xy \\ +2\tilde{g}(e_1, e_3)xz + 2\tilde{g}(e_2, e_3)yz = a.\end{aligned} \tag{4.3}$$

Using (2.2), (2.3), (2.7) and (4.1), we obtain

$$\begin{aligned}\tilde{g}(e_1, e_1) = 2\cos\varphi, \quad \tilde{g}(e_3, e_3) = 2\cos\varphi, \quad \tilde{g}(e_2, e_2) = -2\cos\varphi, \\ \tilde{g}(e_1, e_2) = \tilde{g}(e_2, e_3) = \sqrt{1 - 2\cos^2\varphi}, \quad \tilde{g}(e_1, e_3) = 0.\end{aligned}$$

Substituting the latter equalities into (4.3) we get (4.2). □

Now, we transform the coordinate system p_{xyz} into $p_{x'y'z'}$ by

$$x = \frac{1}{\sqrt{2}}x' + \lambda_1 y' + \mu_1 z', \quad y = \lambda_2 y' + \mu_2 z', \quad z = -\frac{1}{\sqrt{2}}x' + \lambda_1 y' + \mu_1 z',$$

where

$$\lambda_1 = \frac{1}{2}\sqrt{1 + \sqrt{2}\cos\varphi}, \quad \lambda_2 = \frac{\sqrt{2}}{2}\sqrt{1 - \sqrt{2}\cos\varphi}, \tag{4.4}$$

$$\mu_1 = \frac{1}{2}\sqrt{1 - \sqrt{2}\cos\varphi}, \quad \mu_2 = -\frac{\sqrt{2}}{2}\sqrt{1 + \sqrt{2}\cos\varphi}. \tag{4.5}$$

Therefore the equation (4.2) takes the form

$$2\cos\varphi x'^2 + \sqrt{2}y'^2 - \sqrt{2}z'^2 = a. \tag{4.6}$$

Corollary 4.1. *Let s_1 be the surface, determined by* (4.6) *in case $a = 0$. The following statements hold true.*

i) If $\varphi \neq \frac{\pi}{2}$, then s_1 is a cone;

ii) If $\varphi = \frac{\pi}{2}$, then s_1 separates into two planes $z' = \pm y'$.

Corollary 4.2. *Let s_1 be the surface, determined by* (4.6) *in case $a > 0$. The following statements hold true.*

i) If $\varphi \in (\frac{\pi}{4}, \frac{\pi}{2})$, then s_1 is a hyperboloid of one sheets;

ii) If $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{4})$*, then* s_1 *is a hyperboloid of two sheet;*

iii) If $\varphi = \frac{\pi}{2}$*, then* s_1 *is a hyperbolic cylinder.*

Corollary 4.3. *Let* s_1 *be the surface, determined by* (4.6) *in case* $a < 0$*. The following statements hold true.*

i) If $\varphi \in (\frac{\pi}{4}, \frac{\pi}{2})$*, then* s_1 *is a hyperboloid of two sheets;*

ii) If $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{4})$*, then* s_1 *is a hyperboloid of one sheet;*

iii) If $\varphi = \frac{\pi}{2}$*, then* s_1 *is a hyperbolic cylinder.*

4.2. A sphere in the 3-dimensional subspace of T_pM, spanned by vectors u, Su and S^3u

Lemma 4.2. *Let* α_2 *be a subspace of* T_pM *with a basis* $\{u, Su, S^3u\}$*. The system of vectors* $\{e_1, e_2, e_3\}$*, determined by the equalities*

$$e_1 = Su, \quad e_2 = \frac{u - (\cos\varphi)Su + (\cos\varphi)S^3u}{\sqrt{1 - 2\cos^2\varphi}}, \quad e_3 = S^3u, \tag{4.7}$$

where $\varphi = \angle(u, Su)$*, is an orthonormal basis of* α_2*.*

Proof. Using (2.1), (2.2), (2.7) and (4.7) we obtain $g(e_1, e_2) = g(e_2, e_3) = 0$, $g(e_1, e_3) = 0$ and $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$. □

The coordinate system p_{xyz} such that the vectors e_1, e_2 and e_3 are on the axes p_x, p_y and p_z, respectively, is orthonormal.

A sphere s_2 in α_2 centered at the origin p, with respect to $\tilde{g}$ on (M, g, S), is defined by (2.4). In the next statement we get the equation of s_2 with respect to the orthonormal coordinate system p_{xyz}.

Theorem 4.2. *Let* $\tilde{g}$ *be the associated metric on* (M, g, S) *and let* α_2 *be a 3-dimensional subspace of* T_pM *with a basis* $\{u, Su, S^2u\}$*. If* e_1*,* e_2 *and* e_3 *are determined by* (4.7) *and* p_{xyz} *is a coordinate system such that* $e_1 \in p_x$*,* $e_2 \in p_y$*,* $e_3 \in p_z$*, then the equation of the sphere* s_2 *in* α_2 *is given by*

$$2(\cos\varphi)(x^2 - y^2 + z^2) + 2\sqrt{1 - 2\cos^2\varphi}(xy - yz) = a. \tag{4.8}$$

Proof. The radius vector v of an arbitrary point (x, y, z) on α_2 is expressed by $v = xe_1 + ye_2 + ze_3$. Then (2.4) takes the form

$$\begin{aligned} &\tilde{g}(e_1, e_1)x^2 + \tilde{g}(e_2, e_2)y^2 + \tilde{g}(e_3, e_3)z^2 + 2\tilde{g}(e_1, e_2)xy \\ &\qquad\qquad +2\tilde{g}(e_1, e_3)xz + 2\tilde{g}(e_2, e_3)yz = a. \end{aligned} \tag{4.9}$$

By (2.2), (2.3), (2.7) and (4.7) we obtain

$$\tilde{g}(e_1, e_1) = 2\cos\varphi, \quad \tilde{g}(e_3, e_3) = 2\cos\varphi, \quad \tilde{g}(e_2, e_2) = -2\cos\varphi,$$
$$\tilde{g}(e_1, e_2) = \sqrt{1 - 2\cos^2\varphi}, \quad \tilde{g}(e_1, e_3) = 0, \quad \tilde{g}(e_2, e_3) = -\sqrt{1 - 2\cos^2\varphi}.$$

Substituting the latter equalities into (4.9) we get (4.8). □

After transformation of the coordinate system p_{xyz} into $p_{x'y'z'}$ by

$$x = \frac{1}{\sqrt{2}}x' + \lambda_1 y' + \mu_1 z', \quad y = \lambda_2 y' + \mu_2 z', \quad z = \frac{1}{\sqrt{2}}x' - \lambda_1 y' - \mu_1 z'$$

with (4.4) and (4.5), the equation (4.8) takes the form

$$2\cos\varphi x'^2 + \sqrt{2}y'^2 - \sqrt{2}z'^2 = a.$$

The above equation is the same as (4.6).

5. Circles in a special 2-planes of T_pM

Let the unit vector u induce an *S-basis* of T_pM. Now we study circles in three different subspaces β_1, β_2 and β_3 spanned by 2-planes $\{u, S^2u\}$, $\{u, Su\}$ and $\{u, S^3u\}$, respectively.

5.1. Circles in the 2-plane β_1

Due to (2.7) it is true that both vectors u and S^2u form an orthonormal basis of β_1. We construct a coordinate system p_{xy} on β_1, such that u is on the axis p_x and S^2u is on the axis p_y. Therefore p_{xy} is an orthonormal coordinate system of β_1.

Lemma 5.1. *The system $\{u, S^2u\}$ satisfies the following equalities:*

$$\tilde{g}(u, u) = \tilde{g}(S^2u, S^2u) = 2\cos\varphi, \quad \tilde{g}(u, S^2u) = 0. \tag{5.1}$$

Proof. From (2.2), (2.3), (2.7) we get (5.1) by direct calculations. □

A circle k_1 in β_1 centered at the origin p, with respect to $\tilde{g}$ on (M, g, S), is defined by (2.4). Now we obtain the equation of k_1 with respect to p_{xy}.

Theorem 5.1. *Let $\tilde{g}$ be the associated metric on (M, g, S) and let β_1 be a 2-plane in T_pM with a basis $\{u, S^2u\}$. If p_{xy} is a coordinate system such that $u \in p_x$, $S^2u \in p_y$, then the equation of the circle k_1 in β_1 is given by*

$$2\cos\varphi x^2 + 2\cos\varphi y^2 = a, \quad \varphi \neq \frac{\pi}{2}. \tag{5.2}$$

Proof. The radius vector v of an arbitrary point on β_1 is expressed by

$$v = xu + yS^2u, \tag{5.3}$$

which implies $S^2v = -yu + xS^2u$. Then from (2.4), (5.1) and (5.3) it follows (5.2). □

Corollary 5.1. *Let k_1 be the curve determined by* (5.2)*. Then k_1 is a circle in case when $a > 0$ and $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, or in case when $a < 0$ and $\varphi \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. The curve k_1 degenerates into the point p in case $a = 0$.*

We note that a 2-plane $\delta = \{u, Ju\}$, where $\delta = J\delta$, is known as J-invariant section of T_pM on an almost Hermitian manifold (M, g, J). Therefore, the 2-plane $\beta_1 = \{u, S^2u\}$ is a J-invariant section of T_pM on the manifold (M, g, J), $J = S^2$.

5.2. Circles in the 2-plane β_2

Lemma 5.2. *Let β_2 be the 2-plane spanned by unit vectors u and Su. The system of vectors $\{e_1, e_2\}$, determined by the equalities*

$$e_1 = \frac{1}{\sqrt{2(1+\cos\varphi)}}(u + Su), \quad e_2 = \frac{1}{\sqrt{2(1-\cos\varphi)}}(-u + Su), \tag{5.4}$$

where $\varphi = \angle(u, Su)$, is an orthonormal basis of β_2.

Proof. Using (2.7) and (5.4), we calculate $g(e_1, e_2) = 0$ and $g(e_1, e_1) = g(e_2, e_2) = 1$. □

We construct a coordinate system p_{xy} on β_2, such that e_1 is on the axis p_x and e_2 is on the axis p_y, i.e.p_{xy} is orthonormal.

Lemma 5.3. *The system $\{e_1, e_2\}$ satisfies the following equalities:*

$$\tilde{g}(e_1, e_1) = \frac{2\cos\varphi + 1}{1+\cos\varphi}, \quad \tilde{g}(e_2, e_2) = \frac{2\cos\varphi - 1}{1-\cos\varphi}, \quad \tilde{g}(e_1, e_2) = 0. \tag{5.5}$$

Proof. Using (2.2), (2.3) and (2.7) we get (5.5) by direct calculations. □

A circle k_2 in β_2 centered at the origin p, with respect to $\tilde{g}$ on (M, g, S), is defined by (2.4). Further we obtain the equation of k_2 with respect to p_{xy}.

Theorem 5.2. *Let $\tilde{g}$ be the associated metric on (M, g, S) and let $\beta_2 = \{u, Su\}$ be a 2-plane in T_pM with an orthonormal basis* (5.4)*. If p_{xy} is a coordinate system such that $e_1 \in p_x$, $e_2 \in p_y$, then the equation of the circle k_2 in β_2 is given by*

$$\frac{2\cos\varphi + 1}{1+\cos\varphi}x^2 + \frac{2\cos\varphi - 1}{1-\cos\varphi}y^2 = a. \tag{5.6}$$

Proof. The radius vector v of an arbitrary point on β_2 is $v = xe_1 + ye_2$. Using the latter equality, from (2.4) we get

$$\tilde{g}(v, v) = \tilde{g}(e_1, e_1)x^2 + 2\tilde{g}(e_1, e_2)xy + \tilde{g}(e_2, e_2)y^2 = a.$$

Applying (5.5) into the above equation we find (5.6). □

According to the parameters a and φ the equation (5.6) describes different quadratic curves. All possible values of these parameters and the corresponding types of the curve (5.6) are studied in the Table 5.1.

5.3. Circles in the 2-plane β_3

Lemma 5.4. *Let β_3 be the 2-plane spanned by unit vectors u and S^3u. The system of vectors $\{e_1, e_2\}$, determined by the equalities*

$$e_1 = \frac{1}{\sqrt{2(1 - \cos\varphi)}}(u + S^3u), \quad e_2 = \frac{1}{\sqrt{2(1 + \cos\varphi)}}(-u + S^3u), \tag{5.7}$$

where $\varphi = \angle(u, Su)$, is an orthonormal basis of β_3.

Proof. Using (2.7) and (5.7), we calculate $g(e_1, e_2) = 0$ and $g(e_1, e_1) = g(e_2, e_2) = 1$. □

We construct a coordinate system p_{xy} on β_3, such that $e_1 \in p_x$ and $e_2 \in p_y$, i.e.p_{xy} is orthonormal.

Lemma 5.5. *The system $\{e_1, e_2\}$ satisfies the following equalities:*

$$\tilde{g}(e_1, e_1) = \frac{2\cos\varphi - 1}{1 - \cos\varphi}, \quad \tilde{g}(e_2, e_2) = \frac{2\cos\varphi + 1}{1 + \cos\varphi}, \quad \tilde{g}(e_1, e_2) = 0. \tag{5.8}$$

Proof. Using (2.1), (2.2), (2.3) and (2.7) we get (5.8) by direct calculations. □

A circle k_3 in β_3 centered at the origin p, with respect to $\tilde{g}$ on (M, g, S), is defined by (2.4). In the next statement we obtain the equation of k_3 with respect to p_{xy}.

Theorem 5.3. *Let $\tilde{g}$ be the associated metric on (M, g, S) and let $\beta_3 = \{u, S^3u\}$ be a 2-plane in T_pM with an orthonormal basis (5.7). If p_{xy} is a coordinate system such that $e_1 \in p_x$, $e_2 \in p_y$, then the equation of a circle k_3 in β_3 is given by*

$$\frac{2\cos\varphi - 1}{1 - \cos\varphi}x^2 + \frac{2\cos\varphi + 1}{1 + \cos\varphi}y^2 = a. \tag{5.9}$$

Proof. The radius vector v of an arbitrary point on β_3 is $v = xe_1 + ye_2$. Then (2.4) imply

$$\tilde{g}(v, v) = \tilde{g}(e_1, e_1)x^2 + 2\tilde{g}(e_1, e_2)xy + \tilde{g}(e_2, e_2)y^2 = a. \tag{5.10}$$

We apply (5.8) into (5.10) and we find (5.9). □

The equation (5.9) determines curves which are the same as the obtained ones by (5.6). They are described in the Table 5.1.

Table 5.1: Curves k_2 and k_3

φ	a	k_2, k_3
$(\frac{\pi}{4}, \frac{\pi}{3})$	$a > 0$	an ellipse
	$a = 0$	the point p
	$a < 0$	the empty set
$\frac{\pi}{3}$	$a > 0$	two lines $x = \pm\frac{\sqrt{3a}}{2}$
	$a = 0$	the line x=0
	$a < 0$	the empty set
$(\frac{\pi}{3}, \frac{2\pi}{3})$	$a > 0$	a hyperbola
	$a = 0$	two lines $y = \pm cx$, c is a constant
	$a < 0$	a hyperbola
$\frac{2\pi}{3}$	$a > 0$	the empty set
	$a = 0$	the line y=0
	$a < 0$	two lines $y = \pm\frac{\sqrt{-3a}}{2}$
$(\frac{2\pi}{3}, \frac{3\pi}{4})$	$a > 0$	the empty set
	$a = 0$	the point p
	$a < 0$	an ellipse

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