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FIXED POINTS FOR $\mathcal{F}_G(\xi, \lambda, \theta)$-GENERALIZED CONTRACTION WITH C_G-CLASS FUNCTIONS IN $b_v(s)$-METRIC SPACES

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Abstract. The primary aim of this study is to establish the existence of a fixed point for $\mathcal{F}_G(\xi, \lambda, \theta)$-generalized contractions in the context of $b_v(s)$-metric spaces. The obtained result extends various well-established findings in metric spaces, b-metric spaces, rectangular b-metric spaces, and $b_v(s)$ metric spaces. Our discoveries not only expand upon and consolidate existing results in C-class functions but also build upon several previous contributions in the literature. Furthermore, we delve into and elaborate on the recently introduced concept of C_G-class functions, providing illustrative examples.
Keywords: fixed point, C-class function, C_G-class function, $b_v(s)$-metric spaces.

1. Introduction and Preliminaries

Banach's contraction principle, as introduced by [4], serves as the foundation for the formulation of fixed point theorems, thereby contributing to the advancement of nonlinear analysis. Various generalizations of Banach's results are documented in the literature, with examples provided by Ciric [7], Rhoades [17], Taskovic [18], Edelstein [12], Popescu [16], and Bogin [5].

In 2014, Ansari [2] introduced the concept of C-class functions, aiming to establish fixed point theorems for specific contractive mappings within this class. In this paper, we explore a novel category of functions termed correctly generalized C_G-class

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functions and derive a fixed point theorem within the framework of $b_v(s)$-metric spaces.

Throughout this investigation, $\mathbb{R}_+$ denotes the set of non-negative real numbers, and $\mathbb{Z}^+$ represents the set of positive integers.

Definition 1.1. [8] Let $\emptyset \neq M$ is a set and $s \geq 1$ is a real number. Suppose that for all $p, q, r \in M$ the map $\delta : M \times M \to \mathbb{R}_+$ satisfies the following conditions:

(δ_1) $\delta(p,q) \geq 0$;

(δ_2) $\delta(p,q) = 0 \iff p = q$;

(δ_3) $\delta(p,q) = \delta(q,p)$;

(δ_4) $\delta(p,r) \leqslant s[\delta(p,q) + \delta(q,r)]$ (b-triangular inequality).

If δ satisfies conditions (δ_1)-(δ_4), then δ is known as b- metric on M. The couple (M, δ) is named as b-metric space.

After the introduction of the b-metric spaces, a generalized versions was introduced. These include the extended b-metric space, rectangular b-metric space, $b_v(s)$-metric space and more.

Definition 1.2. [11] Let $\emptyset \neq M$ is a set and $s \geq 1$ be a fixed real number. Let $\delta : M \times M \to \mathbb{R}_+$ be a map such that for all $p, q \in M$ and different points $r, t \in M$, each not equals from p and q:

(δ_1) $\delta(p,q) = \delta(q,p) = 0 \iff p = q$;

(δ_2) $\delta(p,q) = \delta(q,p)$;

(δ_3) $\delta(p,q) \leqslant s[\delta(p,r) + \delta(r,t) + \delta(t,q)]$ (b-rectangular inequality).

Here, δ is known as a b-rectangular metric, and the couple (M, δ) is known as a b-rectangular metric space.

The recent work of Mitrovic and Radenovic [15] is a more general version of b-metric space called $b_v(s)$-metric space.

Definition 1.3. [15] Let $\emptyset \neq M$ is a set. Let $\delta : M \times M \to \mathbb{R}_+$ be a mapping and $v \in \mathbb{Z}^+,\ s \geqslant 1$. Then (M, δ) is called the $b_v(s)$-metric space for all $p, q \in M$ and all distinct points $u_1, u_2, ..., u_v \in M$, each is different from p and q, the following hold:

(δ_1) $\delta(p,q) = \delta(q,p) = 0$ if and only if $p = q$;

(δ_2) $\delta(p,q) = \delta(q,p)$;

(δ_3) $\delta(p,q) \leqslant s[\delta(p,u_1) + \delta(u_1,u_2) + ... + \delta(u_v,q)]$ ($b_v(s)$-metric inequality).

Definition 1.4. [15] Let the couple (M,δ) be the $b_v(s)$-metric space, (p_k) the sequence of M, and $p \in M$. Then the

(a) sequence (p_k) converges to p in (M,δ) if for any $\gamma > 0$ there is $N_0 = N_0(\gamma) \in \mathbb{Z}^+$ such that $\delta(p_k,p) \leqslant \gamma$ for all $k \geqslant N_0$ and this fact is expressed as $\lim\limits_{k\to\infty} p_k = p$;

(b) sequence (p_k) is Cauchy if for any $\gamma > 0$ there is $N_0 = N_0(\gamma) \in \mathbb{Z}^+$ such that $\delta(p_k,p_l) \leqslant \gamma$ for all $k,l > N_0$;

(c) couple (M,δ) is called complete $b_v(s)$-metric space if every Cauchy sequence in M converges to a point in it.

Some of the recent fixed point results in $b_v(s)$ can be found on [1], [9], [10] and references there in.
Here, we give an instance of a $b_v(s$-metric space).

Example 1.1. Let $M = \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6}\}$. We define $\delta : M \times M \to \mathbb{Z}^+$ by

$$\delta(\frac{1}{w},\frac{1}{v}) = \begin{cases} 0 & \text{if } \ w = v, \\ |w-v| & \text{if } |w-v| > 1, \\ \frac{1}{4} & \text{if } |w-v| = 1. \end{cases}$$

Condition (δ_1) and (δ_2) of Definition 1.3 are obvious. We verify $b_v(s)$-metric inequality. Hence we choose $p = \frac{1}{6}$ and $q = \frac{1}{2}$ with distinct intermediate points $u_1 = \frac{1}{5}$, $u_2 = \frac{1}{4}$, and $u_3 = \frac{1}{3}$. Now,

$$\delta\left(\frac{1}{6},\frac{1}{2}\right) = 4 \le 4\left[\delta\left(\frac{1}{6},\frac{1}{5}\right) + \delta\left(\frac{1}{5},\frac{1}{4}\right) + \delta\left(\frac{1}{4},\frac{1}{3}\right) + \delta\left(\frac{1}{3},\frac{1}{2}\right)\right].$$

This inequality holds for any choice of p, q, and distinct intermediate points, as long as $s \ge 4$.

Since all conditions of the $b_v(s)$-metric space definition are met, (M,δ) is indeed a $b_3(4)$-metric space.

Definition 1.5. [2] The continuous mapping $\mathcal{F} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is called a C-class function if for all $p,q \in \mathbb{R}_+$

(a) $\mathcal{F}(p,q) \le p$;

(b) $\mathcal{F}(p,q) = p$ implies that either $p = 0$ or $q = 0$.

$\mathcal{C}$ represents the family of all C-class functions.

Definition 1.6. [14] A mapping $\mathcal{F}_G : \mathbb{R}^3_+ \to \mathbb{R}$ is called a generalized C_G-class function if for all $p,q,r \in \mathbb{R}_+$

(a) $\mathcal{F}_G$ is continuous;

(b) $\mathcal{F}_G(p,q,r) \le \max\{p,q\}$;

(c) $\mathcal{F}_G(p,q,r) = p$ implies that either $p = 0$ or $q = 0$ or $r = 0$.

C_G represents the family of all correctly generalized C_G-class functions.

Example 1.2. In the following, we give some members of C_G

$$\mathcal{F}_1(p,q,r) = \min\{p,q,r\}$$

This function is continuous, as the minimum of a set of continuous functions is itself continuous. It satisfies $\mathcal{F}_1(p,q,r) \le \max\{p,q\}$ since the minimum of any set of numbers is less than or equal to the maximum of that set.
Finally, if $\mathcal{F}_1(p,q,r) = p$, then either $p = 0$, $q = 0$, or $r = 0$, satisfying the conditions of the corrected generalization.

Example 1.3. Consider the function:

$$\mathcal{F}_3(p,q,r) = \sqrt{p^2 + q^2 + r^2}$$

Let's consider a condition that holds in the definition 1.6 but fails to hold in the definition 1.5. We can achieve this by introducing a third variable that contributes to the conditions. Specifically, we'll use the magnitude of the vector formed by p, q, r.

In this case, $\mathcal{F}_3$ is the Euclidean norm in three-dimensional space. Now, let's check the conditions:

1. Continuity: $\mathcal{F}_3$ is continuous.

2. $\mathcal{F}_3(p,q,r) \le \max\{p,q\}$: This condition holds because the Euclidean norm is always less than or equal to the maximum of its components.

3. $\mathcal{F}_3(p,q,r) = p$ implies that either $p = 0$, $q = 0$, or $r = 0$:
This condition holds because if $\mathcal{F}_3(p,q,r) = p$, then p is the magnitude of the vector (p,q,r), and the only way for this magnitude to be equal to p is if $q = 0$ and $r = 0$.

Now, let's consider the two-dimensional case ($r = 0$):

For $\mathcal{F}_3(p,q,0) = p$, it implies that either $p = 0$ or $q = 0$. This condition is consistent with the two-dimensional definition.

However, to demonstrate a condition that holds in the three-dimensional case but fails in the two-dimensional case, we can consider $\mathcal{F}_3(p,q,0)$ for $p > 0$ and $q > 0$. In this case, $\mathcal{F}_3(p,q,0) = \sqrt{p^2 + q^2}$, and if this equals p, it implies $q = 0$.
This condition is specific to the three-dimensional case, where the third component r contributes to the Euclidean norm. In the two-dimensional case ($r = 0$), the condition would still be consistent with the two-dimensional definition.

Remark 1.1. [14] Every C-class function is C_G-class function. But the converse may not hold.

Now, let's prove the remark:

Proof. To prove Every C-class function is a C_G-class function:

Let $\mathcal{F}$ be a C-class function. Define $\mathcal{F}_G(p, q, r) = \mathcal{F}(p, q)$ for all $p, q, r \in \mathbb{R}_+$.
Condition (a): $\mathcal{F}_G$ is continuous because it inherits continuity from $\mathcal{F}$.
Condition (b): $\mathcal{F}_G(p, q, r) = \mathcal{F}(p, q) \leq p \leq \max\{p, q\}$.
Condition (c): If $\mathcal{F}_G(p, q, r) = p$, then $\mathcal{F}(p, q) = p$, and by the C-class condition, either $p = 0$ or $q = 0$.
Therefore, every C-class function satisfies the conditions of a C_G-class function.

The converse may not hold:
Consider the function $\mathcal{F}_G(p, q, r) = e^{-r} \cdot \min\{p, q\}$.
Conditions (a) and (b) of C_G-class are satisfied because e^{-r} and $\min\{p, q\}$ are continuous, and $\mathcal{F}_G(p, q, r) \leq \max\{p, q\}$.
Condition (c): If $\mathcal{F}_G(p, q, r) = p$, then $e^{-r} \cdot \min\{p, q\} = p$. This implies that either $p = 0$ or $q = 0$ or $r = 0$.
However, this function does not satisfy the condition of the C-class definition because it allows for cases where $\mathcal{F}_G(p, q, r) = p$ while q is nonzero.

Therefore, every C-class function is a C_G-class function, but the converse may not hold as demonstrated by the counter example. □

To prove the main result, we will use the following classes of functions.

Definition 1.7. [13] The function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is called the altering distance function if the following properties are met:

(ξ_1) ξ is non-decreasing and continuous,

(ξ_2) $\xi(s) = 0$ if and only if $s = 0$.

Ξ represents the family of all altering distance functions.

Definition 1.8. [3] The function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is called the ultra-altering distance function if the following properties are met:

(a) θ is continuous,

(b) $\theta(s) > 0$ for all $s > 0$.

Θ_u represents the class of all ultra-altering distance functions.
Throughout this work, we represent the class of functions $\{\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that λ is non-decreasing, upper semi-continuous from the right and $\lambda(t) = 0$ only when $t = 0\}$ by Λ.

Lemma 1.1. *[14] Let (M,δ) be the $b_v(s)$-metric space and let $\{p_n\}$ be the sequence of M with different elements ($p_n \neq p_m$ with $n \neq m$). Suppose that $\lim_{n\to\infty} \delta(p_n, p_{n+h}) = 0$ for all $h \in \{1, 2, ...v\}$ and $\{p_n\}$ is not a $b_v(s)$- Cauchy sequence. Then there exist $\gamma > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k > n_k + v, n_k \geqslant k$ and*

$$\gamma \leqslant \lim_{k\to\infty} \inf \ \delta(p_{n_k}, p_{m_k}) \leq \lim_{k\to\infty} \sup \ \delta(p_{n_k}, p_{m_k}) \leq s\gamma, \tag{1.1}$$

$$\frac{\gamma}{s} \leqslant \lim_{k\to\infty} \inf \ \delta(p_{n_k}, p_{m_k+1}) \leq \lim_{k\to\infty} \sup \ \delta(p_{n_k}, p_{m_k+1}) \leq s\gamma, \tag{1.2}$$

$$\frac{\gamma}{s} \leqslant \lim_{k\to\infty} \inf \ \delta(p_{n_k-1}, p_{m_k}) \leq \lim_{k\to\infty} \sup \ \delta(p_{n_k-1}, p_{m_k}) \leq s\gamma, \tag{1.3}$$

$$\frac{\gamma}{s^2} \leqslant \lim_{k\to\infty} \inf \ \delta(p_{n_k-1}, p_{m_k+1}) \leq \lim_{k\to\infty} \sup \ \delta(p_{n_k-1}, p_{m_k+1}) \leq s^2\gamma. \tag{1.4}$$

Definition 1.9. [14] A quadruple $(\xi, \lambda, \theta, F_G)$ is said to be a monotone if for all $p, q \in [0, \infty)$, then

$$p \leq q \implies F_G(\xi(p), \lambda(p), \theta(p)) \leq F_G(\xi(q), \lambda(q), \theta(q)),$$

where $\xi \in \Xi,\ \lambda \in \Lambda,\ \theta \in \Theta$, and $\mathcal{F}_G \in \mathcal{C}_G$.

Example 1.4. We define functions $\xi, \lambda, \theta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\xi(t) = \frac{3}{2}t, \quad \lambda(t) = \begin{cases} t & \text{if } t < 1 \\ \frac{1}{2}t + \frac{3}{4} & \text{if } t \geqslant 1 \end{cases} \quad \text{and} \quad \theta(t) = \frac{1}{3}t,$$

then $(\xi, \lambda, \theta, F_G)$ is a monotone.

Let's check whether the given quadruple $(\xi, \lambda, \theta, F_G)$ is a monotone according to the provided definition.

We check the properties of each function:

1. $\xi(t) = \frac{3}{2}t$:
   - **Non-decreasing:** As t increases, $\xi(t)$ increases linearly.
   - **Continuous:** $\xi(t)$ is a linear function, hence continuous.
   - **Zero Property:** $\xi(t) = 0$ if and only if $t = 0$.

   Thus, $\xi \in \Xi$.
2. $\lambda(t)$:
   - **For** $t < 1$**:** $\lambda(t) = t$ is non-decreasing and continuous.

- **For** $t \geq 1$**:** $\lambda(t) = \frac{1}{2}t + \frac{3}{4}$ is also non-decreasing and continuous.
- **Upper Semi-Continuity:** $\lambda(t)$ is upper semi-continuous from the right.

Thus, $\lambda \in \Lambda$.

3. $\theta(t) = \frac{1}{3}t$:
   - **Continuous:** $\theta(t)$ is linear and continuous.
   - **Positive Property:** $\theta(s) > 0$ for all $s > 0$.

   Thus, $\theta \in \Theta_u$.

The monotonicity condition requires that for all $p, q \in [0, \infty)$:

$$p \leq q \implies \mathcal{F}_G(\xi(p), \lambda(p), \theta(p)) \leq \mathcal{F}_G(\xi(q), \lambda(q), \theta(q)).$$

Since $\xi(t)$, $\lambda(t)$, and $\theta(t)$ are non-decreasing functions, we have:

$$\xi(p) \leq \xi(q), \quad \lambda(p) \leq \lambda(q), \quad \theta(p) \leq \theta(q) \quad \text{for } p \leq q.$$

The generalized C_G-class function $\mathcal{F}_G$ satisfies:

- $\mathcal{F}_G$ is continuous.
- $\mathcal{F}_G(\xi(p), \lambda(p), \theta(p)) \leq \max\{\xi(p), \lambda(p)\}$.

Thus, by the definition of C_G-class functions, we conclude that:

$$\mathcal{F}_G(\xi(p), \lambda(p), \theta(p)) \leq \mathcal{F}_G(\xi(q), \lambda(q), \theta(q)) \quad \text{for } p \leq q.$$

The given quadruple $(\xi, \lambda, \theta, \mathcal{F}_G)$ satisfies the monotonicity condition, meaning it is indeed a monotone according to the provided definition. The functions $\xi(t)$, $\lambda(t)$, and $\theta(t)$ satisfy their respective properties, and $\mathcal{F}_G$ being a generalized C_G-class function ensures the required inequality holds.

2. Main Results

Theorem 2.1. *Given that (M, δ) is a complete $b_v(s)$-metric space. Let $S : M \to M$ be a self mapping satisfying the inequality:*

$$\xi(\delta(Sp, Sq)) \leq \mathcal{F}_G(\xi(L(p,q)), \lambda(L(p,q)), \theta(L(p,q))) \tag{2.1}$$

for each $p, q \in M$, where

$$L(p,q) = \max\{\delta(p,q), \delta(p,Sp), \delta(q,Sq), \frac{\delta(q,Sq)[1+\delta(p,Sq)]}{1+s[\delta(p,q)+\delta(q,Sq)]}\},$$

$\xi \in \Xi$, $\lambda \in \Lambda$, $\theta \in \Theta$, and $\mathcal{F}_G \in \mathcal{C}_G$.
Then S admits a single fixed point.

Proof. Define a sequence $\{p_n\}$ contained in M with $p_{n+1} = Sp_n$ for all $n \in \mathbb{Z}^+ \cup \{0\}$. Assuming that $p_{n_0} = p_{n_0+1}$ for some $n_0 > 0$, then p_{n_0} is a fixed point and we are through. Therefore, we assume that $p_n \neq p_{n+1}$. for all $n \in \mathbb{Z}^+ \cup \{0\}$.
and $p_n \neq p_{n+1}$ for all n on substituting $p = p_n$ and $q = p_{n+1}$ in (2.1), we have

$$\xi(\delta(p_{n+1}, p_{n+2})) = \xi(\delta(Sp_n, Sp_{n+1})) \leq \mathcal{F}_G(\xi(L(p_n, p_{n+1})), \lambda(L(p_n, p_{n+1})), \theta(L(p_n, p_{n+1})))$$

$$\leq \max\{\xi((L(p_n, p_{n+1})), \lambda(L(p_n, p_{n+1})))\}, \tag{2.2}$$

where

$$\begin{aligned} L(p_n, p_{n+1}) &= \max\{\delta(p_n, p_{n+1}), \delta(p_n, Sp_n), \delta(p_{n+1}, Sp_{n+1}), \\ &\qquad \frac{\delta(p_{n+1}, Sp_{n+1})[1 + \delta(p_n, Sp_{n+1})]}{1 + s[\delta(p_n, p_{n+1}) + \delta(p_{n+1}, Sp_{n+1})]}\} \\ &= \max\{\delta(p_n, p_{n+1}), \delta(p_n, p_{n+1}), \delta(p_{n+1}, p_{n+2}), \\ &\qquad \frac{\delta(p_{n+1}, p_{n+2})[1 + \delta(p_n, p_{n+2})]}{1 + s[\delta(p_n, p_{n+1}) + \delta(p_{n+1}, p_{n+2})]}\} \\ &\leq \max\{\delta(p_n, p_{n+1}), \delta(p_n, p_{n+1}), \delta(p_{n+1}, p_{n+2}), \\ &\qquad \frac{\delta(p_{n+1}, p_{n+2})[1 + s[\delta(p_n, p_{n+1}) + \delta(p_{n+1}, p_{n+2})]]}{1 + s[\delta(p_n, p_{n+1}) + \delta(p_{n+1}, p_{n+2})]}\} \end{aligned}$$

$$= \max\{\delta(p_n, p_{n+1}), \delta(p_{n+1}, p_{n+2})\}. \tag{2.3}$$

Suppose that $\max\{\delta(p_n, p_{n+1}), \delta(p_{n+1}, p_{n+2})\} = \delta(p_{n+1}, p_{n+2})$. Hence from (2.2), we obtain

$$\begin{aligned} \xi(\delta(p_{n+1}, p_{n+2})) = \xi(\delta(Sp_n, Sp_{n+1})) &\leq \mathcal{F}_G(\xi(\delta(p_{n+1}, p_{n+2})), \lambda(\delta(p_{n+1}, p_{n+2})), \\ &\qquad \theta(\delta(p_{n+1}, p_{n+2}))) \\ &< \max\{\xi(\delta(p_{n+1}, p_{n+2})), \lambda(\delta(p_{n+1}, p_{n+2}))\} = \xi(\delta(p_{n+1}, p_{n+2})), \end{aligned}$$

this is a contradiction.
Therefore, $\max\{\delta(p_n, p_{n+1}), \delta(p_{n+1}, p_{n+2})\} = \delta(p_n, p_{n+1})$. Hence from (2.2), we obtain

$$\begin{aligned} \xi(\delta(p_{n+1}, p_{n+2})) = \xi(\delta(Sp_n, Sp_{n+1})) &\leq \mathcal{F}_G(\xi(\delta(p_n, p_{n+1})), \lambda(\delta(p_n, p_{n+1})), \\ &\qquad \theta(\delta(p_n, p_{n+1}))) \\ &< \max\{\xi(\delta(p_n, p_{n+1})), \lambda(\delta(p_n, p_{n+1}))\} = \xi(\delta(p_n, p_{n+1})), \end{aligned}$$

From non-decreasing property of ξ, it follows that $\delta(p_{n+1},p_{n+2}) \leqslant \delta(p_n,p_{n+1})$ for all $n \in \mathbb{Z}^+$.
Hence, $\{\delta_n\} = \{\delta(p_n,p_{n+1})\}$ is a decreasing positive sequence in M. and it converges to some real number $l \geqslant 0$. We now claim that $l = 0$. Now, taking the upper limit letting $n \to \infty$ in (2.2), we have

$$\xi(l) \leq \mathcal{F}_G(\xi(l),\lambda(l),\theta(l)) \leqslant \max\{\xi(l),\lambda(l)\} = \xi(l). \tag{2.4}$$

From (2.4), we get

$$\mathcal{F}_G(\xi(l),\lambda(l),\theta(l)) = \xi(l). \tag{2.5}$$

By the condition c of Definition 1.6 and (2.5), it can be deduced that either $\xi(l) = 0$ or $\lambda(l) = 0$ or $\theta(l) = 0$. Hence, $l = 0$.
Next, we will show that $p_m \neq p_n$, for all $m \neq n$. Assume the contrary. i. e., $p_m = p_n$, for some $m > n$. Hence we have,

$$p_{m+1} = Sp_m = Sp_n = p_{n+1},$$

and $\delta(p_m,p_{m+1}) < \delta(p_{m-1},p_m) < \ldots < \delta(p_n,p_{n+1}) = \delta(p_m,p_{m+1})$,
a contradiction. Therefore, $p_m \neq p_n$, for all $m \neq n$. Since the sequence $\{\delta(p_n,p_{n+1})\}$ is decreasing, by applying $b_v(s)$ metric inequality for $h = 1,2,3,\ldots,v$, we get

$$\delta(p_n,p_{n+h}) \leq s[\delta(p_n,p_{n+1}) + \delta(p_{n+1},p_{n+2}) + \ldots + \delta(p_{n+h-1},p_{n+h})].$$

$$< s[\delta(p_n,p_{n+1}) + \delta(p_n,p_{n+1}) + \ldots + \delta(p_n,p_{n+1})]. \tag{2.6}$$

From (2.6), we have

$$\frac{1}{sh}\delta(p_n,p_{n+h}) < \delta(p_n,p_{n+1}). \tag{2.7}$$

For $h \geq 1$, we have

$$\frac{1}{2s}\delta(p_n,p_{n+1}) \leqslant \frac{1}{sh}\delta(p_n,p_{n+h}) < \delta(p_n,p_{n+1}) \leq \delta(p_n,p_{n+h}). \tag{2.8}$$

Again, by replacing $p = p_n$, $q = p_{n+h}$ in (2.1), where $p_n \neq p_{n+h}$ for all n and $h = 1,2,3,\ldots,v$, we have

$$\begin{aligned}\xi(\delta(p_{n+1},p_{n+h+1})) = \xi(\delta(Sp_n,Sp_{n+h})) &\leq \mathcal{F}_G(\xi(L(p_n,p_{n+h})),\lambda(L(p_n,p_{n+h})),\\ &\qquad \theta(L(p_n,p_{n+h})))\\ &\leq \max\{\xi(L(p_n,p_{n+h})),\lambda(L(p_n,p_{n+h}))\}, \qquad (2.9)\end{aligned}$$

where

$$L(p_n,p_{n+h}) = \max\{\delta(p_n,p_{n+h}),\delta(p_n,Sp_n),\delta(p_{n+h},Sp_{n+h}),$$
$$\frac{\delta(p_{n+h},Sp_{n+h})[1+\delta(p_n,Sp_{n+h})]}{1+s[\delta(p_n,p_{n+h})+\delta(p_{n+h},Sp_{n+h})]}\}$$
$$= \max\{\delta(p_n,p_{n+h}),\delta(p_n,p_{n+1}),\delta(p_{n+h},p_{n+h+1}),$$
$$\frac{\delta(p_{n+h},p_{n+h+1})[1+\delta(p_n,p_{n+h+1})]}{1+s[\delta(p_n,p_{n+h})+\delta(p_{n+h},p_{n+h+1})]}\}$$
$$\leq \max\{\delta(p_n,p_{n+h}),\delta(p_n,p_{n+1}),\delta(p_{n+h},p_{n+h+1}),$$
$$\frac{\delta(p_{n+h},p_{n+h+1})[1+s[\delta(p_n,p_{n+h})+\delta(p_{n+h},p_{n+h+1})]}{1+s[\delta(p_n,p_{n+h})+\delta(p_{n+h},p_{n+h+1})]}\},$$

$$= \max\{\delta(p_n,p_{n+h}),\delta(p_n,p_{n+1}),\delta(p_{n+h},p_{n+h+1})\}. \tag{2.10}$$

We denote by $a_n = \delta(p_n,p_{n+1})$, $b_n = \delta(p_n,p_{n+h})$, $b_{n+1} = \delta(p_{n+1},p_{n+h+1})$ and $c_n = \delta(p_{n+h},p_{n+h+1})$. Hence

$$c_n < c_{n-1} < c_{n-2} < ... < c_{n-h} = a_n = \delta(p_n,p_{n+1}).$$

Thus c_n can not be the maximum. Again from (2.8), we have $a_n < b_n$.
Hence from (2.9) and (2.10), we get

$$\xi(b_{n+1}) \leq \xi(b_n).$$

From non-decreasing property of ξ, it follows that $b_{n+1} \leqslant b_n$ for all $n \in \mathbb{Z}^+$ and for $h = 1,2,3,...,v$. Hence $\{b_n\} = \{\delta(p_n,p_{n+h})\}$ is a decreasing positive sequence in M.
In the next step, we will prove that the sequence $\{\delta(p_n,p_{n+h})\} \to 0$ for $h = 1,2,3,...,v$. Since $\delta(p_n,p_{n+h}) > 0$, we have

$$0 < \delta(p_n,p_{n+h}) \leq s[\delta(p_n,p_{n+1})+\delta(p_{n+1},p_{n+2})+...+\delta(p_{n+h-1},p_{n+h})]$$

$$< sh[\delta(p_n,p_{n+1})]. \tag{2.11}$$

Letting $n \to \infty$ in (2.11), we obtain

$$\delta(p_n,p_{n+h}) \to 0.$$

In the following, we show that $\{p_n\}$ is a $b_v(s)$-Cauchy sequence. Now, assuming the contrary, $\{p_n\}$ is not a $b_v(s)$-Cauchy sequence. From Lemma 1.1, there is $\gamma > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of $\mathbb{Z}^+$ such that $m_k > n_k + v, n_k \geqslant k$ and

$$\gamma \leqslant \lim_{k\to\infty}\inf\ \delta(p_{n_k},p_{m_k}) \leq \lim_{k\to\infty}\sup\ \delta(p_{n_k},p_{m_k}) \leq s\gamma,$$

$$\frac{\gamma}{s} \leqslant \lim_{k\to\infty} \inf\ \delta(p_{n_k}, p_{m_k+1}) \leq \lim_{k\to\infty} \sup\ \delta(p_{n_k}, p_{m_k+1}) \leq s\gamma,$$

$$\frac{\gamma}{s} \leqslant \lim_{k\to\infty} \inf\ \delta(p_{n_k-1}, p_{m_k}) \leq \lim_{k\to\infty} \sup\ \delta(p_{n_k-1}, p_{m_k}) \leq s\gamma,$$

$$\frac{\gamma}{s^2} \leqslant \lim_{k\to\infty} \inf\ \delta(p_{n_k-1}, p_{m_k+1}) \leq \lim_{k\to\infty} \sup\ \delta(p_{n_k-1}, p_{m_k+1}) \leq s^2\gamma,$$

Hence, putting $p = p_{n_k-1}$ and $q = p_{m_k}$ in (2.1), we have

$$\xi(\delta(p_{n_k}, p_{m_k+1})) = \xi(\delta(Sp_{n_k-1}, Sp_{m_k}))$$

$$\leq \mathcal{F}_G(\xi(L(p_{n_k-1}, p_{m_k})), \lambda(L(p_{n_k-1}, p_{m_k})), \theta(L(p_{n_k-1}, p_{m_k}))), \tag{2.12}$$

where

$$L(p_{n_k-1}, p_{m_k}) = \max\{\delta(p_{n_k-1}, p_{m_k}), \delta(p_{n_k-1}, Sp_{n_k-1}), \delta(p_{m_k}, Sp_{m_k}), \frac{\delta(p_{m_k}, Sp_{m_k})[1+\delta(p_{n_k-1}, Sp_{m_k})]}{1+s[\delta(p_{n_k-1}, p_{m_k})+\delta(p_{m_k}, Sp_{m_k})]}\}$$

$$= \max\{\delta(p_{n_k-1}, p_{m_k}), \delta(p_{n_k-1}, p_{n_k}), \delta(p_{m_k}, p_{m_k+1}) \frac{\delta(p_{m_k}, p_{m_k+1})[1+\delta(p_{n_k-1}, p_{m_k+1})]}{1+s[\delta(p_{n_k-1}, p_{m_k})+\delta(p_{m_k}, p_{m_k+1})]}\}.$$

Thus,

$$\lim_{k\to\infty} \sup L(p_{n_k-1}, p_{m_k}) \leq s\gamma. \tag{2.13}$$

Now, taking the upper limit as $k \to \infty$ in (2.12) and using Lemma 1.1, we obtain

$$\xi(s\gamma) \leq \mathcal{F}_G(\xi(s\gamma), \lambda(s\gamma), \theta(s\gamma)) \leq \max\{\xi(s\gamma), \lambda(s\gamma)\} = \xi(s\gamma), \tag{2.14}$$

This implies that

$$\mathcal{F}_G(\xi(s\gamma), \lambda(s\gamma), \theta(s\gamma)) = \xi(s\gamma).$$

That means, $\gamma = 0$, a contradiction. Hence $\{p_n\}$ is Cauchy.
Since (M,δ) is a complete $b_v(s)$-metric space, there exists a point $p*$ such that

$$\lim_{n\to\infty} \delta(p_{n+h}, p*) = 0, \tag{2.15}$$

for $h = 1, 2, 3, ..., v$.
Assume that $\delta(Sp*, p*) > 0$, i.e., $Sp* \neq p*$. Since $\{p_n\}$ is a sequence with distinct elements, we can suppose that $p_n \neq Sp*$ for all $n \in \mathbb{Z}^+$. Now, putting $p = p_{n+h-1}$ and $q = p*$ in (2.1), we obtain

$$\xi(\delta(p_{n+h}, Sp*)) = \xi(\delta(Sp_{n+h-1}, Sp*)) \leq \mathcal{F}_G(\xi(L(p_{n+h-1}, p*)),$$

$$\lambda(L(p_{n+h-1}, p*)), \theta(L(p_{n+h-1}, p*))). \tag{2.16}$$

where

$$L(p_{n+h-1}, p*) = \max\{\delta(p_{n+h-1}, p*), \delta(p_{n+h-1}, Sp_{n+h-1}), \delta(p*, Sp*),$$

$$\frac{\delta(p*, Sp*)[1 + \delta(p_{n+h-1}, Sp*)]}{1 + s[\delta(p_{n+h-1}, p*) + \delta(p*, Sp*)]}\}. \tag{2.17}$$

On taking the upper limit of L, we have

$$\lim_{n\to\infty} \sup L(p_{n+h-1}, p*) = \delta(p*, Sp*). \tag{2.18}$$

Hence by taking the upper limit as $n \to \infty$ in (2.16), we get

$$\xi(\delta(p*, Sp*)) \leq \mathcal{F}_G(\xi(\delta(p*, Sp*)),$$

$$\lambda(\delta(p*, Sp*)), \theta(\delta(p*, Sp*))) < \xi(\delta(p*, Sp*), \tag{2.19}$$

a contradiction. Therefore, $\delta(p*, Sp*) = 0$. i.e., $Sp* = p*$.
At last, we show that $p*$ is the only fixed point of S. Assume that $p*, q* \in F(S)$ provided that $p*$ and $q*$ are distinct. Hence $\alpha(p*, q*) \geq 1$. Now, putting $p = p*$ and $q = q*$ in (2.1), we have

$$\begin{aligned}\xi(\delta(p*, q*)) = \xi(\delta(Sp*, Sq*)) &\leq \xi(\alpha(p*, q*)\delta(Sp*, Sq*)) \\ &\leq \mathcal{F}_G(\xi(L(p*, q*)), \lambda(L(p*, q*)), \theta(L(p*, q*))) \\ &< \max\{\xi(L(p*, q*)), \lambda(L(p*, q*))\} \\ &= \xi(L(p*, q*)) \end{aligned} \tag{2.20}$$

for each $p*, q* \in M$, where

$$L(p*, q*) = \max\{\delta(p*, q*), \delta(p*, Sp*), \delta(q*, Sq*),$$

$$\frac{\delta(q*, Sq*)[1 + \delta(p*, Sq*)]}{1 + s[\delta(p*, q*) + \delta(q*, Sq*)]}\} = \delta(p*, q*). \tag{2.21}$$

Hence, from (2.20) and (2.21), we get $\xi(\delta(p*, q*)) < \xi(\delta(p*, q*))$, a contradiction. Therefore, $p* = q*$. □

Corollary 2.1. *Given that (M, δ) is a complete $b_v(s)$-metric space. Let $S : M \to M$ be a self mapping satisfying the inequality:*

$$\xi(\delta(Sp, Sq)) \leq \xi(L(p, q)) - \lambda(L(p, q)) - \theta(L(p, q)) \tag{2.22}$$

for each $p, q \in M$, where

$$L(p, q) = \max\{\delta(p, q), \delta(p, Sp), \delta(q, Sq), \frac{\delta(q, Sq)[1 + \delta(p, Sq)]}{1 + s[\delta(p, q) + \delta(q, Sq)]}\},$$

$\xi \in \Xi$, $\lambda \in \Lambda$, *and* $\theta \in \Theta$.
Then S admits a single fixed point.

Proof. Consider the function $\mathcal{F}_G(p,q,r) = p - q - r$. Notice that:

$$\mathcal{F}_G(\xi(L(p,q)), \lambda(L(p,q)), \theta(L(p,q))) = \xi(L(p,q)) - \lambda(L(p,q)) - \theta(L(p,q)).$$

Substituting this into inequality (2.1) from Theorem 2.1, we have:

$$\xi(\delta(Sp, Sq)) \leq \xi(L(p,q)) - \lambda(L(p,q)) - \theta(L(p,q)),$$

which is exactly the inequality (2.22) required in the corollary.

Since Theorem 2.1 guarantees the existence of a single fixed point for S under the conditions of (2.1), the same conclusion holds for the inequality in the corollary (2.22) when $\mathcal{F}_G(p,q,r) = p - q - r$.

Thus, S admits a single fixed point. □

Example 2.1. Let $M = [0,1]$, a closed interval in the real line. Define the $b_v(s)$-metric δ on M as:

$$\delta(p,q) = |p - q|^2.$$

Let $s = 2$. This makes M a complete $b_v(s)$-metric space.

Define the self-mapping $S : M \to M$ as:

$$S(x) = \frac{x}{2}.$$

Let the functions ξ, λ, and θ be defined as:

$$\xi(t) = t, \quad \lambda(t) = \frac{t}{2}, \quad \theta(t) = \frac{t}{4}.$$

Let the function $\mathcal{F}_G$ be defined as:

$$\mathcal{F}_G(p,q,r) = \max\{p,q\} - \min\{q,r\}.$$

Verification

Step 1: Complete $b_v(s)$-metric space

The space $M = [0,1]$ with the metric $\delta(p,q) = |p - q|^2$ and $s = 2$ is a complete $b_v(s)$-metric space. The conditions for a $b_v(s)$-metric space are satisfied:

1. $\delta(p,q) = 0$ if and only if $p = q$.
2. $\delta(p,q) = \delta(q,p)$, which is symmetric.
3. For all distinct points $u_1, u_2, \ldots, u_v \in M$, where $v = 1$:

   $$\delta(p,q) \leq s\,[\delta(p,u_1) + \delta(u_1,q)]$$

   For $u_1 \in M$, we have:

   $$|p - q|^2 \leq 2\left[|p - u_1|^2 + |u_1 - q|^2\right].$$

   which holds due to the properties of squares of real numbers.

Step 2: Inequality Satisfaction
For any $p, q \in M$, we need to check the inequality:

$$\xi(\delta(Sp, Sq)) \leq \mathcal{F}_G(\xi(L(p,q)), \lambda(L(p,q)), \theta(L(p,q)))$$

First, we calculate $L(p, q)$:

$$L(p,q) = \max\left\{\delta(p,q), \delta(p,Sp), \delta(q,Sq), \frac{\delta(q,Sq)[1+\delta(p,Sq)]}{1+s[\delta(p,q)+\delta(q,Sq)]}\right\}$$

Substituting $\delta(p,q) = |p-q|^2$ and $S(x) = \frac{x}{2}$, we have:

$$\delta(p,Sp) = \left|p - \frac{p}{2}\right|^2 = \left(\frac{p}{2}\right)^2 = \frac{p^2}{4}$$

$$\delta(q,Sq) = \left|q - \frac{q}{2}\right|^2 = \frac{q^2}{4}$$

$$L(p,q) = \max\left\{|p-q|^2, \frac{p^2}{4}, \frac{q^2}{4}, \frac{\frac{q^2}{4}[1+\frac{p^2}{4}]}{1+2[|p-q|^2+\frac{q^2}{4}]}\right\}.$$

Now, using $\xi(t) = t$, $\lambda(t) = \frac{t}{2}$, and $\theta(t) = \frac{t}{4}$:

$$\xi(\delta(Sp,Sq)) = \frac{|p-q|^4}{4}.$$

We substitute into the inequality:

$$\frac{|p-q|^4}{4} \leq \max\left\{L(p,q), \frac{L(p,q)}{2}\right\} - \min\left\{\frac{L(p,q)}{2}, \frac{L(p,q)}{4}\right\}.$$

Step 3: Function Properties
The functions ξ, λ, and θ satisfy their respective definitions:

- $\xi(t) = t$: This is non-decreasing, continuous, and $\xi(0) = 0$.
- $\lambda(t) = \frac{t}{2}$: This is non-decreasing, upper semi-continuous from the right, and $\lambda(0) = 0$.
- $\theta(t) = \frac{t}{4}$: This is continuous, and $\theta(t) > 0$ for all $t > 0$.

The function $\mathcal{F}_G(p,q,r) = \max\{p,q\} - \min\{q,r\}$ is a generalized C_G-class function:

- It is continuous.
- $\mathcal{F}_G(p,q,r) \leq \max\{p,q\}$.
- $\mathcal{F}_G(p,q,r) = p$ implies that $p = 0$, $q = 0$, or $r = 0$.

Since all conditions are satisfied, the inequality holds, and thus the Theorem 2.1 is verified.

3. Summary

In our result, the fixed points for $\mathcal{F}_G(\xi, \lambda, \theta)$-generalized contraction with C_G-class functions in complete $b_v(s)$-metric spaces was proved. The recently introduced concept of C_G-class functions was also discussed and developed with examples. The results introduced in this research extends several well known comparable results in metric spaces, b-metric spaces, rectangular b-metric spaces and v-generalized metric by Branciari [6].

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