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# BOUNDING THE ČEBYŠEV FUNCTIONAL FOR A FUNCTION THAT IS CONVEX IN ABSOLUTE VALUE AND APPLICATIONS

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**Abstract.** Some sharp bounds for the Čebyšev functional of a function that is convex in absolute value and applications for functions of self-adjoint operators in Hilbert spaces via the spectral representation theorem are given.
**Keywords:** Čebyšev functional, self-adjoint operator, spectral representation theorem, integrable function, integral mean.

## 1. Introduction

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$C(f, g) := \frac{1}{b-a}\int_a^b f(t)\, g(t)\, dt - \frac{1}{b-a}\int_a^b f(t)\, dt \cdot \frac{1}{b-a}\int_a^b g(t)\, dt.i$$

In 1934, G. Grüss [14] showed that

$$|C(f, g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided $m, M, n, N$ are real numbers with the property that

$$-\infty < m \le f \le M < \infty, \quad -\infty < n \le g \le N < \infty \quad \text{a.e. on } [a, b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $C(f, g)$ was derived in 1882 by Čebyšev [4] under the assumption that $f', g'$ exist and are continuous on $[a, b]$, and is given by

$$|C(f, g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.3}$$

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where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $f,g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a,b]$.

In 1970, A.M. Ostrowski [18] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|C(f,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.4}$$

provided $f$ is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [16] (see also [17, p. 210]) obtained the following result as well:

$$|C(f,g)| \le \frac{1}{\pi^2}\|f'\|_2\|g'\|_2(b-a), \tag{1.5}$$

provided $f,g$ are absolutely continuous and $f', g' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [2], P. Cerone and S.S. Dragomir proved the following inequalities:

$$|C(f,g)| \le \begin{cases} \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt, \\ \\ \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_q \cdot \frac{1}{b-a}\left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} \\ \qquad \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|C(f,g)| \le \|g\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \le g \le M$ for a.e. $x \in [a,b]$, then $\left\|g - \frac{m+M}{2}\right\|_\infty \le \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [5]

$$|C(f,g)| \le \frac{1}{2}(M-m)\cdot\frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [3].

The following result holds [11].

**Theorem 1.1.** *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|C(f,g)| \le \frac{1}{2} \bigvee_a^b (f) \cdot \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right| dt \tag{1.9}$$

*where $\bigvee_a^b (f)$ denotes the total variation of $f$ on the interval $[a,b]$.*

*The constant $\frac{1}{2}$ is best possible in (1.9).*

We denote the *variance* of the function $f : [a,b] \to \mathbb{C}$ by $D(f)$ and defined as

$$D(f) = \left[C\left(f, \bar{f}\right)\right]^{1/2} \left[\frac{1}{b-a} \int_a^b |f(t)|^2\, dt - \left|\frac{1}{b-a} \int_a^b f(t)\, dt\right|^2\right]^{1/2}, \tag{1.10}$$

where $\bar{f}$ denotes the complex conjugate function of $f$.

We have [11]:

**Corollary 1.1.** *If the function $f : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$D(f) \le \frac{1}{2} \bigvee_a^b (f). \tag{1.11}$$

*The constant $\frac{1}{2}$ is best possible in (1.11).*

Now we can state the following result when both functions are of bounded variation [11]:

**Corollary 1.2.** *If $f, g : [a,b] \to \mathbb{C}$ are of bounded variation on $[a,b]$, then*

$$|C(f,g)| \le \frac{1}{4} \bigvee_a^b (f) \bigvee_a^b (g). \tag{1.12}$$

*The constant $\frac{1}{4}$ is best possible in (1.12).*

**Remark 1.1.** We can consider the following quantity associated with a complex valued function $f : [a,b] \to \mathbb{C}$,

$$E(f) := |C(f,f)|^{1/2} = \left| \frac{1}{b-a} \int_a^b f^2(t)\, dt - \left(\frac{1}{b-a} \int_a^b f(t)\, dt\right)^2 \right|^{1/2}.$$

Utilising the above results we can state that

$$E^2(f) \le \frac{1}{2}\bigvee_a^b(f)\cdot\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \le \frac{1}{2}\bigvee_a^b(f)\,D(f) \le \frac{1}{4}\left[\bigvee_a^b(f)\right]^2. \tag{1.13}$$

If we consider

$$G(f) := \left|C\left(f,|f|\right)\right|^{1/2} = \left|\frac{1}{b-a}\int_a^b f(t)\left|f(t)\right|dt - \frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b\left|f(t)\right|dt\right|^{1/2},$$

then we also have

$$G^2(f) \le \frac{1}{2}\bigvee_a^b(f)\cdot\frac{1}{b-a}\int_a^b\left|\left|f(t)\right|-\frac{1}{b-a}\int_a^b\left|f(s)\right|ds\right|dt \le \frac{1}{2}\bigvee_a^b(f)\,D\left(|f|\right) \le \frac{1}{4}\bigvee_a^b(f)\bigvee_a^b\left(|f|\right) \le \frac{1}{4}\left[\bigvee_a^b(f)\right]^2 \tag{1.14}$$

and

$$G^2(f) \le \frac{1}{2}\bigvee_a^b\left(|f|\right)\cdot\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \le \frac{1}{2}\bigvee_a^b\left(|f|\right)D(f) \le \frac{1}{4}\bigvee_a^b(f)\bigvee_a^b\left(|f|\right) \le \frac{1}{4}\left[\bigvee_a^b(f)\right]^2. \tag{1.15}$$

For recent related results see [1] and [9]-[13].

Motivated by the results presented above, we establish in this paper some new bounds for the magnitude of $C(f,g)$ in the case when one of the complex valued function, say $f$, is convex in absolute value while the other is Lebesgue integrable on $[a,b]$. Applications for functions of self-adjoint operators in Hilbert spaces via the spectral representation theorem are also given.

## 2. New Results for Čebyšev Functional

Recall that a function $g:[a,b]\to\mathbb{R}$ is *convex* (*strictly convex*) on the interval $[a,b]$, if

$$g((1-t)x+ty) \le (<)\,(1-t)\,g(x)+tg(y)$$

for any $x,y\in[a,b]$ $(x\neq y)$ and $t\in[0,1]$ $((0,1))$.

We observe that the *constant function* $k(t)=k, t\in[a,b]$ and the *identity function* $e(t)=t, t\in[a,b]$ can then be interpreted as convex functions. However, they are not strictly convex functions on $[a,b]$.

We have the following result:

**Theorem 2.1.** *Let $f:[a,b]\to\mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a,b]$ and $g:[a,b]\to\mathbb{C}$ is a Lebesgue integrable function on $[a,b]$. Then*

$$|C(f,g)|\le\max\{|f(a)|,|f(b)|\}\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt. \tag{2.1}$$

*The inequality (2.1) is sharp.*

*Proof.* We use Sonin's identity

$$C(f,g)=\frac{1}{b-a}\int_a^b(f(t)-\lambda)\left[g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right]dt, \tag{2.2}$$

for $\lambda=0$ to get

$$C(f,g)=\frac{1}{b-a}\int_a^b f(t)\left[g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right]dt. \tag{2.3}$$

Taking the modulus and utilizing the convexity of $|f|$ on $[a,b]$ we have

$$\begin{aligned}|C(f,g)| &\le \tfrac{1}{b-a}\int_a^b|f(t)|\left|g(t)-\tfrac{1}{b-a}\int_a^b g(s)\,ds\right|dt\\ &\le \tfrac{1}{b-a}\int_a^b\left[\tfrac{(b-t)|f(a)|+(t-a)|f(b)|}{b-a}\right]\\ &\quad\times\left|g(t)-\tfrac{1}{b-a}\int_a^b g(s)\,ds\right|dt\end{aligned} \tag{2.4}$$

If we denote the right side of (2.4) by $I$, then we have

$$\begin{aligned}I&\le\sup_{t\in[a,b]}\left[\frac{(b-t)|f(a)|+(t-a)|f(b)|}{b-a}\right]\\&\quad\times\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt\\&=\max\{|f(a)|,|f(b)|\}\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt\end{aligned}$$

and by (2.4) we get (2.1).

Assume that the inequality (2.1) holds with a constant $K>0$, namely

$$|C(f,g)|\le K\max\{|f(a)|,|f(b)|\}\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt. \tag{2.5}$$

Consider the functions $f,g:[a,b]\to\mathbb{R}$ defined by

$$f(t):=\begin{cases}-1, & t\in\left[a,\frac{a+b}{2}\right]\\ 1, & t\in\left(\frac{a+b}{2},b\right]\end{cases}$$

and $g:[a,b]\to\mathbb{R}$, $g(t)=t-\frac{a+b}{2}$.

We have $|f|=1$, which satisfy the convexity condition with equality and

$$C(f,g)=\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{b-a}{4},$$

$$\max\{|f(a)|,|f(b)|\}=1$$

and

$$\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt=\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{b-a}{4}$$

and by (2.5) we have

$$\frac{b-a}{4}\le K\frac{b-a}{4},$$

which shows that $K\ge 1$. □

With the notations from the introduction we have:

**Corollary 2.1.** *Let $f:[a,b]\to\mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a,b]$. Then*

$$\begin{aligned}(2.6)\qquad & D^2(f),E^2(f)\\ &\le\max\{|f(a)|,|f(b)|\}\tfrac{1}{b-a}\int_a^b\left|f(t)-\tfrac{1}{b-a}\int_a^b f(s)\,ds\right|dt,\end{aligned}$$

*and*

$$(2.7)\qquad G^2(f)\le\max\{|f(a)|,|f(b)|\}\frac{1}{b-a}\int_a^b\left||f(t)|-\frac{1}{b-a}\int_a^b|f(s)|\,ds\right|dt.$$

We recall the *p-logarithmic mean* defined by

$$L_p^p(m,n):=\frac{m^{p+1}-n^{p+1}}{(p+1)(M-m)},\ m\ne n$$

where $p\ne-1,0$ and $m,n>0$.

The case of $p$-norm of the deviation

$$\left|f-\frac{1}{b-a}\int_a^b f(s)\,ds\right|$$

is as follows:

**Theorem 2.2.** *Let $f:[a,b]\to\mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a,b]$ and, for $p>1$, $g:[a,b]\to\mathbb{C}$ is in the Lebesgue space $L_p[a,b]$. Then*

$$|C(f,g)| \le L_q(|f(a)|,|f(b)|)\left[\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right]^{1/p}, \tag{2.8}$$

*where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.*

*The inequality (2.8) is sharp.*

*Proof.* Making use of Hölder's inequality, we have

$$\begin{aligned}(b-a)I &= \int_a^b\left[\tfrac{(b-t)|f(a)|+(t-a)|f(b)|}{b-a}\right]\left|g(t)-\tfrac{1}{b-a}\int_a^b g(s)\,ds\right|dt\\ &\le \left(\int_a^b\left[\tfrac{(b-t)|f(a)|+(t-a)|f(b)|}{b-a}\right]^q dt\right)^{1/q}\\ &\quad\times\left(\int_a^b\left|g(t)-\tfrac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right)^{1/p}.\end{aligned} \tag{2.9}$$

Observe that, by changing the variable $u=\frac{t-a}{b-a}$ we have

$$\begin{aligned}&\int_a^b\left[\tfrac{(b-t)|f(a)|+(t-a)|f(b)|}{b-a}\right]^q dt\\ &= (b-a)\int_0^1\left[u|f(b)|+(1-u)|f(a)|\right]^q du.\end{aligned}$$

Changing the variable again

$$v=u|f(b)|+(1-u)|f(a)|$$

we have for $|f(a)|\ne|f(b)|$

$$\begin{aligned}(b-a)\int_0^1\left[u|f(b)|+(1-u)|f(a)|\right]^q du &= \frac{b-a}{|f(b)|-|f(a)|}\int_0^1 v^q du\\ &= (b-a)L_q^q(|f(a)|,|f(b)|).\end{aligned}$$

For $|f(a)|=|f(b)|$ we also have

$$\begin{aligned}(b-a)\int_0^1\left[u|f(b)|+(1-u)|f(a)|\right]^q du &= (b-a)|f(a)|^q\\ &= (b-a)L_q^q(|f(a)|,|f(b)|).\end{aligned}$$

Therefore

$$\begin{aligned}(b-a)I &\le \left((b-a)L_q^q(|f(a)|,|f(b)|)\right)^{1/q}\left(\int_a^b\left|g(t)-\tfrac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right)^{1/p}\\ &= (b-a)^{1/q}L_q(|f(a)|,|f(b)|)\left(\int_a^b\left|g(t)-\tfrac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right)^{1/p},\end{aligned}$$

which implies

$$I \le L_q\left(\left|f(a)\right|,\left|f(b)\right|\right)\left(\frac{1}{b-a}\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right)^{1/p}.$$

Making use of (2.9) we get the desired result (2.8).

Assume that

$$\left|C(f,g)\right| \le K L_q\left(\left|f(a)\right|,\left|f(b)\right|\right) \times\left[\tfrac{1}{b-a}\int_a^b \left|g(t)-\tfrac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right]^{1/p}, \tag{2.10}$$

holds with a constant $K>0$ for any $p>1$ and $f,g$ as above.

Consider the functions $f,g:[a,b]\to\mathbb{R}$ defined by

$$f(t):=\begin{cases} -1, & t\in\left[a,\frac{a+b}{2}\right],\\ 1, & t\in\left(\frac{a+b}{2},b\right]\end{cases}$$

and $g:[a,b]\to\mathbb{R}$, $g(t)=t-\frac{a+b}{2}$.

We have $\left|f\right|=1$, which satisfies the convexity condition with equality and

$$C(f,g)=\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{b-a}{4}.$$

We also have $L_q\left(\left|f(a)\right|,\left|f(b)\right|\right)=1$ and

$$\begin{aligned}&\left(\tfrac{1}{b-a}\int_a^b\left|g(t)-\tfrac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right)^{1/p}\\ &=\left(\tfrac{1}{b-a}\int_a^b\left|t-\tfrac{a+b}{2}\right|^p dt\right)^{1/p}=\left(\tfrac{2}{b-a}\int_{\frac{a+b}{2}}^b\left(t-\tfrac{a+b}{2}\right)^p dt\right)^{1/p}\\ &=\left(\tfrac{2}{b-a}\,\tfrac{\left(\frac{b-a}{2}\right)^{p+1}}{p+1}\right)^{1/p}=\tfrac{b-a}{2(p+1)^{1/p}}.\end{aligned}$$

If we replace these values in (2.10) we get

$$\frac{b-a}{4}\le\frac{K(b-a)}{2(p+1)^{1/p}} \tag{2.11}$$

for any $p>1$.

Now, if we let $p\to 1+$ in (2.11) we get $K\ge 1$, which proves the desired sharpness. □

The case $p=q=2$ is of interest.

**Corollary 2.2.** *Let $f : [a, b] \to \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$ and $g : [a, b] \to \mathbb{C}$ is in the Lebesgue space $L_2 [a, b]$. Then*

$$|C(f, g)| \leq \left( \frac{|f(a)|^2 + |f(a) f(b)| + |f(b)|^2}{3} \right)^{1/2} D(g). \tag{2.12}$$

The following particular cases are of interest as well:

**Corollary 2.3.** *Let $f : [a, b] \to \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$. Then*

$$\begin{aligned} D^2(f), E^2(f) &\leq L_q(|f(a)|, |f(b)|) \\ &\quad \times \left[ \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right|^p dt \right]^{1/p}, \end{aligned} \tag{2.13}$$

*and*

$$\begin{aligned} G^2(f) &\leq L_q(|f(a)|, |f(b)|) \\ &\quad \times \left[ \frac{1}{b-a} \int_a^b \left| |f(t)| - \frac{1}{b-a} \int_a^b |f(s)|\, ds \right|^p dt \right]^{1/p}, \end{aligned} \tag{2.14}$$

*where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

*In particular, we have*

$$D^2(f), E^2(f) \leq \left( \frac{|f(a)|^2 + |f(a) f(b)| + |f(b)|^2}{3} \right)^{1/2} D(f), \tag{2.15}$$

*and*

$$G^2(f) \leq \left( \frac{|f(a)|^2 + |f(a) f(b)| + |f(b)|^2}{3} \right)^{1/2} D(|f|). \tag{2.16}$$

*The first inequality in (2.15) is equivalent to*

$$D(f) \leq \left( \frac{|f(a)|^2 + |f(a) f(b)| + |f(b)|^2}{3} \right)^{1/2}. \tag{2.17}$$

The following result also holds:

**Theorem 2.3.** *Let $f:[a,b]\to\mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a,b]$ and $g:[a,b]\to\mathbb{C}$ is essentially bounded on $[a,b]$. Then*

$$|C(f,g)|\le\frac{1}{2}\left[|f(a)|+|f(b)|\right]\sup_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|. \tag{2.18}$$

*The inequality (2.18) is sharp.*

*Proof.* We have

$$\begin{aligned}I&\le\frac{1}{b-a}\int_a^b\left[\frac{(b-t)|f(a)|+(t-a)|f(b)|}{b-a}\right]\\&\quad\times\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt\\&\le ess\sup_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|\\&\quad\times\frac{1}{b-a}\int_a^b\left[\frac{(b-t)|f(a)|+(t-a)|f(b)|}{b-a}\right]dt\\&=\frac{|f(a)|+|f(b)|}{2}ess\sup_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|\end{aligned}$$

and by (2.4) and (2.19) we get the desired result (2.18).

Assume that the inequality (2.18) holds with a constant $D>0$

$$|C(f,g)|\le D\left[|f(a)|+|f(b)|\right]\sup_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|. \tag{2.19}$$

Consider the functions $f,g:[a,b]\to\mathbb{R}$ defined by

$$f(t)=g(t):=\begin{cases}-1, & t\in\left[a,\frac{a+b}{2}\right],\\[2ex] 1, & t\in\left(\frac{a+b}{2},b\right].\end{cases}$$

We have $|f|=1$, which satisfies the convexity condition with equality and

$$C(f,g):=\frac{1}{b-a}\int_a^b dt=1,\quad |f(a)|=|f(b)|=1$$

while

$$\sup_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|=1.$$

From (2.19) we have $1\le 2D$, i.e. $D\ge\frac{1}{2}$. □

**Corollary 2.4.** *Let $f:[a,b]\to\mathbb{C}$ be a measurable function such that $\left|f\right|$ is convex on $[a,b]$. Then*

$$D^2(f), E^2(f) \le \frac{1}{2}\left[\left|f(a)\right| + \left|f(b)\right|\right] \sup_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \tag{2.20}$$

*and*

$$G^2(f) \le \frac{1}{2}\left[\left|f(a)\right| + \left|f(b)\right|\right] \sup_{t\in[a,b]} \left| \left|f(t)\right| - \frac{1}{b-a}\int_a^b \left|f(s)\right| ds \right|. \tag{2.21}$$

## 3. Application for Riemann-Stieltjes Integral

The following representation is of interest in itself. The result was firstly obtained in [6] (see also [7]). For the sake a completeness we give here a short proof as well.

**Lemma 3.1.** *If $v:[a,b]\to\mathbb{C}$ is continuous (of bounded variation) on $[a,b]$ and $h:[a,b]\to\mathbb{C}$ is of bounded variation (continuous) on $[a,b]$, then we have the identity*

$$\begin{aligned} &\tfrac{v(b)\int_a^b (t-a)dh(t) + v(a)\int_a^b (b-t)dh(t)}{b-a} - \int_a^b v(t)\,dh(t) \\ &= \int_a^b h(t)\,dv(t) - \tfrac{v(b)-v(a)}{b-a}\int_a^b h(t)\,dt. \end{aligned} \tag{3.1}$$

*Proof.* Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned} &\tfrac{v(b)\int_a^b (t-a)dh(t) + v(a)\int_a^b (b-t)dh(t)}{b-a} - \int_a^b v(t)\,dh(t) \\ &= \int_a^b \left[\tfrac{v(b)(t-a)+v(a)(b-t)}{b-a} - v(t)\right] dh(t) \\ &= \left[\tfrac{(t-a)v(b)+(b-t)v(a)}{b-a} - v(t)\right] h(t)\Big|_a^b \\ &\quad - \int_a^b h(t)\,d\left[\tfrac{(t-a)v(b)+(b-t)v(a)}{b-a} - v(t)\right] \\ &= [v(b)-v(b)]\,h(b) - [v(a)-v(a)]\,h(a) \\ &\quad - \int_a^b h(t)\left[\tfrac{v(b)-v(a)}{b-a}dt - dv(t)\right] \\ &= \int_a^b h(t)\,dv(t) - \tfrac{v(b)-v(a)}{b-a}\int_a^b h(t)\,dt \end{aligned} \tag{3.2}$$

and the identity is proven. □

We can provide now the following application for Riemann-Stieltjes integral:

**Proposition 3.1.** *If $v:I\to\mathbb{C}$ is differentiable on the interior of the interval $I$ denoted $\mathring{I}$ and $[a,b]\subset\mathring{I}$, $|v'|$ is convex on $[a,b]$ and $h:[a,b]\to\mathbb{C}$ is integrable on $[a,b]$, then we have* *the inequalities*

$$\left|\frac{v(b)\int_a^b (t-a)dh(t)+v(a)\int_a^b (b-t)dh(t)}{b-a}-\int_a^b v(t)\,dh(t)\right| \tag{3.3}$$

$$\leq \begin{cases} \max\{|v'(a)|,|v'(b)|\}\int_a^b \left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|dt, \\ \\ (b-a)\,L_q(|v'(a)|,|v'(b)|)\left[\frac{1}{b-a}\int_a^b \left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|^p dt\right]^{1/p} \\ \text{where } q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{2}(b-a)\left[|v'(a)|+|v'(b)|\right]\sup_{t\in[a,b]}\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|. \end{cases}$$

*Proof.* From (3.1) we have

$$\frac{v(b)\int_a^b (t-a)dh(t)+v(a)\int_a^b (b-t)dh(t)}{b-a}-\int_a^b v(t)\,dh(t) \tag{3.4}$$

$$=\int_a^b h(t)\,v'(t)\,dt-\frac{v(b)-v(a)}{b-a}\int_a^b h(t)\,dt=(b-a)\,C(v',h).$$

Since $|v'|$ is convex on $[a,b]$, then by applying Theorem 2.1-Theorem 2.3 for $f=v'$ and $g=h$ we deduce the desired result (3.3). □

**Remark 3.1.** If $p=q=2$, then by (3.3) we get

$$\left|\frac{v(b)\int_a^b (t-a)dh(t)+v(a)\int_a^b (b-t)dh(t)}{b-a}-\int_a^b v(t)\,dh(t)\right| \tag{3.5}$$

$$\leq (b-a)\left(\frac{|v'(a)|^2+|v'(a)v'(b)|+|v'(b)|^2}{3}\right)^{1/2}$$

$$\times\left[\frac{1}{b-a}\int_a^b \left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2 dt\right]^{1/2},$$

provided that $|v'|$ is convex on $[a,b]$ and $h:[a,b]\to\mathbb{C}$ is integrable on $[a,b]$.

## 4. Applications for Self-adjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be self-adjoint and let $\varphi_\lambda$ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s):=\begin{cases} 1, & \text{for } -\infty<s\leq\lambda, \\ \\ 0, & \text{for } \lambda<s<+\infty. \end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda:=\varphi_\lambda(A) \tag{4.1}$$

is a projection which reduces $A$.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded self-adjoint operators in Hilbert spaces, see for instance [15, p. 256]:

**Theorem 4.1.** Spectral Representation Theorem *Let $A$ be a bonded self-adjoint operator on the Hilbert space $H$ and let $m = \min\left\{\lambda \left| \lambda \in Sp(A)\right.\right\} =: \min Sp(A)$ and $M = \max\left\{\lambda \left| \lambda \in Sp(A)\right.\right\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of $A$, with the following properties*

*a)* $E_\lambda \leq E_{\lambda'}$ *for* $\lambda \leq \lambda'$;

*b)* $E_{m-0} = 0, E_M = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;

*c) We have the representation*

$$A = \int_{m-0}^{M} \lambda dE_\lambda. \tag{4.2}$$

*More generally, for every continuous complex-valued function $\varphi$ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi\left(\lambda_k'\right)\left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \leq \varepsilon \tag{4.3}$$

*whenever*

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases} \tag{4.4}$$

*this means that*

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{4.5}$$

*where the integral is of Riemann-Stieltjes type.*

**Corollary 4.1.** *With the assumptions of Theorem 4.1 for $A, E_\lambda$ and $\varphi$ we have the representations*

$$\varphi(A)\, x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \text{ for all } x \in H \tag{4.6}$$

*and*

$$\text{(4.7)}\qquad \langle \varphi(A)x,y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x,y\rangle \quad \textit{for all } x,y\in H.$$

*In particular,*

$$\text{(4.8)}\qquad \langle \varphi(A)x,x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x,x\rangle \quad \textit{for all } x\in H.$$

*Moreover, we have the equality*

$$\text{(4.9)}\qquad \left\|\varphi(A)x\right\|^2 = \int_{m-0}^{M} \left|\varphi(\lambda)\right|^2 d\left\|E_\lambda x\right\|^2 \quad \textit{for all } x\in H.$$

The next result shows that it is legitimate to talk about "*the*" spectral family of the bounded self-adjoint operator $A$ since it is uniquely determined by the requirements a), b) and c) in Theorem 4.1, see for instance [15, p. 258]:

**Theorem 4.2.** *Let $A$ be a bonded self-adjoint operator on the Hilbert space $H$ and let $m = \min Sp(A)$ and $M = \max Sp(A)$. If $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a family of projections satisfying the requirements a), b) and c) in Theorem 4.1, then $F_\lambda = E_\lambda$ for all $\lambda\in\mathbb{R}$ where $E_\lambda$ is defined by (4.1).*

By the above two theorems, the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ uniquely determines and in turn is uniquely determined by the bounded self-adjoint operator $A$.

We can state now the following generalized trapezoid inequality for functions of self-adjoint operators:

**Theorem 4.3.** *Let $A$ be a bonded self-adjoint operator on the Hilbert space $H$ and let $m = \min\left\{\lambda \left| \lambda\in Sp(A)\right.\right\} =: \min Sp(A)$ and $M = \max\left\{\lambda \left| \lambda\in Sp(A)\right.\right\} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of $A$.*

*If $f : I\to\mathbb{C}$ is differentiable on the interior of the interval $I$, denoted $\mathring{I}$ and $[m,M]\subset\mathring{I}$, $\left|f'\right|$ is convex on $[m,M]$, then we have the inequalities*

$$\text{(4.10)}\qquad \left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle - \langle f(A)x,y\rangle\right|$$
$$\le \frac{1}{2}\left[\left|f'(m)\right|+\left|f'(M)\right|\right](M-m)$$
$$\times \sup_{t\in[m,M]}\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+\frac{M-t}{M-m}\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$$
$$\le \frac{1}{2}\left[\left|f'(m)\right|+\left|f'(M)\right|\right](M-m)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\le \frac{1}{2}\left[\left|f'(m)\right|+\left|f'(M)\right|\right](M-m)\left\|x\right\|\left\|y\right\|$$

*for any $x,y\in H$.*

*Proof.* Let $x, y \in H$ and consider $h : \mathbb{R} \to \mathbb{C}$, $h(t) := \langle E_t x, y \rangle$. If we use the third inequality in (3.3) for the interval $[m-\varepsilon, M]$ with small $\varepsilon > 0$, we have

$$\begin{aligned}(4.11)\quad & \left| \frac{f(M)\int_{m-\varepsilon}^{M}(t-m+\varepsilon)d\langle E_t x,y\rangle + f(m-\varepsilon)\int_{m-\varepsilon}^{M}(M-t)d\langle E_t x,y\rangle}{M-m+\varepsilon} \right. \\ & \left. - \int_{m-\varepsilon}^{M} f(t)\, d\langle E_t x, y\rangle \right| \\ & \le \frac{1}{2}\left[\left|f'(M)\right| + \left|f'(m-\varepsilon)\right|\right](M-m+\varepsilon) \\ & \times \sup_{t\in[m-\varepsilon,M]} \left| \langle E_t x, y\rangle - \frac{1}{M-m+\varepsilon}\int_{m-\varepsilon}^{M} \langle E_s x, y\rangle ds \right|.\end{aligned}$$

Taking the limit over $\varepsilon \to 0+$ and using the Spectral representation theorem, we have

$$\begin{aligned}(4.12)\quad & \left| \left\langle \left[ \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] x, y \right\rangle - \langle f(A) x, y\rangle \right| \\ & \le \frac{1}{2}\left[\left|f'(m)\right| + \left|f'(M)\right|\right](M-m) \\ & \times \sup_{t\in[m,M]} \left| \langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M} \langle E_s x, y\rangle ds \right|\end{aligned}$$

for any $x, y \in H$.

It is well known that if $p : [a,b] \to \mathbb{C}$ is a bounded function, $v : [a,b] \to \mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists, then the following inequality holds

$$(4.13)\quad \left| \int_a^b p(t)\, dv(t) \right| \le \sup_{t\in[a,b]} \left| p(t) \right| \bigvee_a^b (v),$$

where $\bigvee_a^b (v)$ denotes the total variation of $v$ on $[a,b]$.

Now, a simple integration by parts in the Riemann-Stieltjes integral reveals the following equality of interest

$$\begin{aligned}(4.14)\quad & \langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M} \langle E_s x, y\rangle ds \\ & = \frac{1}{M-m}\left[ \int_{m-0}^{t} (s-m)\, d\langle E_s x, y\rangle + \int_t^M (s-M)\, d\langle E_s x, y\rangle \right]\end{aligned}$$

that holds for any $t \in [m, M]$ and for any $x, y \in H$.

Since the function $v(s) := \langle E_s x, y\rangle$ is of bounded variation on $[m, M]$ for any

$x,y\in H$, then on applying the inequality (4.13), we get

$$\begin{aligned}(4.15)\qquad &\left|\left\langle E_t x,y\right\rangle-\tfrac{1}{M-m}\int_{m-0}^{M}\left\langle E_s x,y\right\rangle ds\right|\\ &\le \tfrac{1}{M-m}\left[\left|\int_{m-0}^{t}(s-m)\,d\left\langle E_s x,y\right\rangle\right|+\left|\int_{t}^{M}(s-M)\,d\left\langle E_s x,y\right\rangle\right|\right]\\ &\le \tfrac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+\tfrac{M-t}{M-m}\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\ &\le \max\left\{\tfrac{t-m}{M-m},\tfrac{M-t}{M-m}\right\}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\ &=\left[\tfrac{1}{2}+\left|t-\tfrac{\frac{m+M}{2}}{M-m}\right|\right]\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\end{aligned}$$

that holds for any $t\in[m,M]$ and for any $x,y\in H$.

This implies that

$$\begin{aligned}(4.16)\qquad &\sup_{t\in[m,M]}\left|\left\langle E_t x,y\right\rangle-\tfrac{1}{M-m}\int_{m-0}^{M}\left\langle E_s x,y\right\rangle ds\right|\\ &\le \sup_{t\in[m,M]}\left[\tfrac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+\tfrac{M-t}{M-m}\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]\\ &\le \sup_{t\in[m,M]}\left[\tfrac{1}{2}+\left|t-\tfrac{\frac{m+M}{2}}{M-m}\right|\right]\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)=\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\end{aligned}$$

for any $x,y\in H$.

The proof of the inequality

$$\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le\|x\|\,\|y\|$$

for any $x,y\in H$, can be found in [13, p. 9]. □

We also have:

**Theorem 4.4.** *Let $A$ be a bonded self-adjoint operator on the Hilbert space $H$ and let $m=\min\left\{\lambda\left|\lambda\in Sp(A)\right.\right\}=:\min Sp(A)$ and $M=\max\left\{\lambda\left|\lambda\in Sp(A)\right.\right\}=:\max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of $A$.*

*If $f:I\to\mathbb{C}$ is differentiable on $\mathring{I}$, $[m,M]\subset\mathring{I}$ and $\left|f'\right|$ is convex on $[m,M]$, then we have the inequalities*

$$\begin{aligned}(4.17)\qquad &\left|\left\langle\left[\tfrac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\left\langle f(A)x,y\right\rangle\right|\\ &\le\max\left\{\left|f'(m)\right|,\left|f'(M)\right|\right\}\int_{m-0}^{M}\left|\left\langle E_t x,y\right\rangle-\tfrac{1}{M-m}\int_{m-0}^{M}\left\langle E_s x,y\right\rangle ds\right|dt\\ &\le\tfrac{1}{2}\max\left\{\left|f'(m)\right|,\left|f'(M)\right|\right\}(M-m)\|x\|\,\|y\|\end{aligned}$$

*for any* $x, y \in H$.

*Proof.* Let $x, y \in H$ and consider $h : \mathbb{R} \to \mathbb{C}$, $h(t) := \langle E_t x, y \rangle$. If we use the first inequality in (3.3) for the interval $[m - \varepsilon, M]$ with small $\varepsilon > 0$, we have

$$\begin{aligned} &\left| \frac{f(M) \int_{m-\varepsilon}^{M} (t-m+\varepsilon) d\langle E_t x, y\rangle + f(m-\varepsilon) \int_{m-\varepsilon}^{M} (M-t) d\langle E_t x, y\rangle}{M-m+\varepsilon} \right. \\ &\qquad \left. - \int_{m-\varepsilon}^{M} f(t)\, d\langle E_t x, y\rangle \right| \\ &\leq \max\left\{ \left| f'(M) \right|, \left| f'(m-\varepsilon) \right| \right\} (M - m + \varepsilon) \\ &\times \frac{1}{M-m+\varepsilon} \int_{m-\varepsilon}^{M} \left| \langle E_t x, y\rangle - \frac{1}{M-m+\varepsilon} \int_{m-\varepsilon}^{M} \langle E_s x, y\rangle\, ds \right| dt. \end{aligned} \tag{4.18}$$

Taking the limit over $\varepsilon \to 0+$ and using the Spectral representation theorem, we have

$$\begin{aligned} &\left| \left\langle \left[ \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] x, y \right\rangle - \langle f(A)\, x, y\rangle \right| \\ &\leq \max\left\{ \left| f'(m) \right|, \left| f'(M) \right| \right\} \int_{m-0}^{M} \left| \langle E_t x, y\rangle - \frac{1}{M-m} \int_{m-0}^{M} \langle E_s x, y\rangle\, ds \right| dt \end{aligned} \tag{4.19}$$

for any $x, y \in H$.

By the Schwarz inequality in $H$ we have that

$$\begin{aligned} &\int_{m-0}^{M} \left| \langle E_t x, y\rangle - \frac{1}{M-m} \int_{m-0}^{M} \langle E_s x, y\rangle\, ds \right| dt \\ &= \int_{m-0}^{M} \left| \left\langle \left[ E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right], y \right\rangle \right| dt \\ &\leq \|y\| \int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\| dt \end{aligned} \tag{4.20}$$

for any $x, y \in H$.

On utilizing the Cauchy-Buniakovski-Schwarz integral inequality we may state that

$$\begin{aligned} &\int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\| dt \\ &\leq (M-m)^{1/2} \left( \int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\|^2 dt \right)^{1/2} \end{aligned} \tag{4.21}$$

for any $x \in H$.

Observe that the following equalities of interest hold and they can be easily proved by direct calculations

$$\begin{aligned} &\frac{1}{M-m} \int_{m-0}^{M} \left\| E_t x - \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\|^2 dt \\ &= \frac{1}{M-m} \int_{m-0}^{M} \|E_t x\|^2 dt - \left\| \frac{1}{M-m} \int_{m-0}^{M} E_s x ds \right\|^2 \end{aligned} \tag{4.22}$$

and

$$\frac{1}{M-m}\int_{m-0}^{M}\|E_t x\|^2 dt - \left\|\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2 = \frac{1}{M-m}\int_{m-0}^{M}\left\langle E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x - \frac{1}{2}x\right\rangle dt \tag{4.23}$$

for any $x \in H$.

By (4.21), (4.22) and (4.23) we get

$$\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \leq (M-m)^{1/2}\left(\int_{m-0}^{M}\left\langle E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x - \frac{1}{2}x\right\rangle dt\right)^{1/2} \tag{4.24}$$

for any $x \in H$.

On making use of the Schwarz inequality in $H$ we also have

$$\begin{aligned}\int_{m-0}^{M}&\left\langle E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x - \frac{1}{2}x\right\rangle dt \\ &\leq \int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|\left\|E_t x - \frac{1}{2}x\right\| dt \\ &= \frac{1}{2}\|x\|\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt,\end{aligned} \tag{4.25}$$

where we used the fact that $E_t$ are projectors, and in this case we have

$$\left\|E_t x - \frac{1}{2}x\right\|^2 = \|E_t x\|^2 - \langle E_t x, x\rangle + \frac{1}{4}\|x\|^2 = \frac{1}{4}\|x\|^2$$

for any $t \in [m, M]$ for any $x \in H$.

From (4.24) and (4.25) we get

$$\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \leq (M-m)^{1/2}\left(\frac{1}{2}\|x\|\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt\right)^{1/2}, \tag{4.26}$$

which is clearly equivalent with the following inequality of interest in itself

$$\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \leq \frac{1}{2}\|x\|(M-m) \tag{4.27}$$

for any $x \in H$.

From (4.20) we then get

$$\frac{1}{M-m}\int_{m-0}^{M}\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x, y\rangle ds\right| dt \leq \frac{1}{2}\|x\|\|y\|$$

for any $x, y \in H$. □

Finally, we also have:

**Theorem 4.5.** *Let $A$ be a bonded self-adjoint operator on the Hilbert space $H$ and let $m = \min\left\{\lambda \,\middle|\, \lambda \in Sp(A)\right\} =: \min Sp(A)$ and $M = \max\left\{\lambda \,\middle|\, \lambda \in Sp(A)\right\} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of $A$.*

*If $f : I \to \mathbb{C}$ is differentiable on $\mathring{I}$, $[m, M] \subset \mathring{I}$ and $\left|f'\right|$ is convex on $[m, M]$, then we have the inequalities*

$$\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x, y\right\rangle - \langle f(A)x, y\rangle\right| \tag{4.28}$$
$$\le \left(\frac{\left|f'(M)\right|^2+\left|f'(M)f'(m)\right|+\left|f'(m)\right|^2}{3}\right)^{1/2}(M-m)$$
$$\times\left(\frac{1}{M-m}\int_{m-0}^{M}\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x, y\rangle ds\right|^2 dt\right)^{1/2}$$
$$\le \frac{1}{2}\left(\frac{\left|f'(M)\right|^2+\left|f'(M)f'(m)\right|+\left|f'(m)\right|^2}{3}\right)^{1/2}(M-m)\|x\|\left\|y\right\|$$

*for any $x, y \in H$.*

*Proof.* Utilising the inequality (3.5) we can prove in a similar manner as above the first inequality in (4.28).

By the Schwarz inequality in $H$ we have that

$$\frac{1}{M-m}\int_{m-0}^{M}\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x, y\rangle ds\right|^2 dt \tag{4.29}$$
$$= \frac{1}{M-m}\int_{m-0}^{M}\left|\left\langle\left[E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right], y\right\rangle\right|^2 dt$$
$$\le \left\|y\right\|^2 \frac{1}{M-m}\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2 dt$$

for any $x, y \in H$.

As in the proof of Theorem 4.4 we also have

$$\frac{1}{M-m}\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|^2 dt \tag{4.30}$$
$$\le \frac{1}{2}\|x\|\frac{1}{M-m}\int_{m-0}^{M}\left\|E_t x - \frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\| dt \le \frac{1}{4}\|x\|^2.$$

By (4.29) and (4.30) we then get

$$\frac{1}{M-m}\int_{m-0}^{M}\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x, y\rangle ds\right|^2 dt \le \frac{1}{4}\|x\|^2\left\|y\right\|^2,$$

namely

$$\left[\frac{1}{M-m}\int_{m-0}^{M}\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x, y\rangle ds\right|^2 dt\right]^{1/2} \le \frac{1}{2}\|x\|\left\|y\right\|,$$

for any $x, y \in H$.

This proves the last part of (4.28). $\square$

**Example 4.1.** a) Let $A$ be a bonded self-adjoint operator on the Hilbert space $H$ and let $m = \min\left\{\lambda \left| \lambda \in Sp(A)\right.\right\} =: \min Sp(A) \geq 0$ and $M = \max\left\{\lambda \left| \lambda \in Sp(A)\right.\right\} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of $A$. Then by Theorem 4.3-4.5 we have for $f(t) = t^p$, $p \geq 2$ that

$$\left|\left\langle\left[\frac{m^p(M1_H-A)+M^p(A-m1_H)}{M-m}\right]x, y\right\rangle - \langle A^p x, y\rangle\right| \tag{4.31}$$
$$\leq \frac{1}{2}p\left(m^{p-1}+M^{p-1}\right)(M-m)$$
$$\times \sup_{t\in[m,M]}\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}(\langle E_{(\cdot)}x, y\rangle) + \frac{M-t}{M-m}\bigvee_{t}^{M}(\langle E_{(\cdot)}x, y\rangle)\right]$$
$$\leq \frac{1}{2}p\left(m^{p-1}+M^{p-1}\right)(M-m)\bigvee_{m-0}^{M}(\langle E_{(\cdot)}x, y\rangle)$$
$$\leq \frac{1}{2}p\left(m^{p-1}+M^{p-1}\right)(M-m)\|x\|\left\|y\right\|,$$

$$\left|\left\langle\left[\frac{m^p(M1_H-A)+M^p(A-m1_H)}{M-m}\right]x, y\right\rangle - \langle A^p x, y\rangle\right| \tag{4.32}$$
$$\leq pM^{p-1}\int_{m-0}^{M}\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x, y\rangle\, ds\right| dt$$
$$\leq \frac{1}{2}pM^{p-1}(M-m)\|x\|\left\|y\right\|$$

and

$$\left|\left\langle\left[\frac{m^p(M1_H-A)+M^p(A-m1_H)}{M-m}\right]x, y\right\rangle - \langle A^p x, y\rangle\right| \tag{4.33}$$
$$\leq p\left(\frac{M^{2(p-1)}+(Mm)^{p-1}+m^{2(p-1)}}{3}\right)^{1/2}(M-m)$$
$$\times\left(\frac{1}{M-m}\int_{m-0}^{M}\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^{M}\langle E_s x, y\rangle\, ds\right|^2 dt\right)^{1/2}$$
$$\leq \frac{1}{2}\left(\frac{M^{2(p-1)}+(Mm)^{p-1}+m^{2(p-1)}}{3}\right)^{1/2}(M-m)\|x\|\left\|y\right\|$$

for any $x, y \in H$.

b) With the assumptions of a) and if $m > 0$, then by Theorem 4.3-4.5 we have for $f(t) = \ln t$, that

$$\left|\left\langle\left[\frac{\ln m(M1_H-A)+\ln M(A-m1_H)}{M-m}\right]x, y\right\rangle - \langle \ln Ax, y\rangle\right| \tag{4.34}$$
$$\leq \frac{m+M}{2mM}(M-m)$$
$$\times \sup_{t\in[m,M]}\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}(\langle E_{(\cdot)}x, y\rangle) + \frac{M-t}{M-m}\bigvee_{t}^{M}(\langle E_{(\cdot)}x, y\rangle)\right]$$
$$\leq \frac{m+M}{2mM}(M-m)\bigvee_{m-0}^{M}(\langle E_{(\cdot)}x, y\rangle) \leq \frac{m+M}{2mM}(M-m)\|x\|\left\|y\right\|,$$

$$\left|\left\langle\left[\frac{\ln m(M1_H-A)+\ln M(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle\ln Ax,y\rangle\right| \tag{4.35}$$
$$\le\frac{1}{m}\int_{m-0}^{M}\left|\langle E_tx,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_sx,y\rangle ds\right|dt$$
$$\le\frac{1}{2m}\|x\|\left\|y\right\|(M-m)$$

and

$$\left|\left\langle\left[\frac{\ln m(M1_H-A)+\ln M(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle\ln Ax,y\rangle\right| \tag{4.36}$$
$$\le\left(\frac{M^2+mM+m^2}{3m^2M^2}\right)^{1/2}(M-m)$$
$$\times\left(\frac{1}{M-m}\int_{m-0}^{M}\left|\langle E_tx,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_sx,y\rangle ds\right|^2dt\right)^{1/2}$$
$$\le\frac{1}{2mM}\left(\frac{M^2+mM+m^2}{3}\right)^{1/2}(M-m)\|x\|\left\|y\right\|$$

for any $x, y \in H$.

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