FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 4 (2024), 607–619
https://doi.org/10.22190/FUMI220114041H
Original Scientific Paper

ON THE MODULE HOMOMORPHISM AND FACTORIZATION PROPERTIES OF MODULE ACTIONS

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Abstract. In this paper, we study approximate identity properties, some propositions from Baker, Dales and Lau in general cases and we establish some relationships between the topological centers of module actions and factorization properties with some results in group algebras.
Keywords: group algebras, module actions, module homomorphism.

1. Introduction

In this article, our primary objective is to explore the application of Banach algebras to module actions. We aim to generalize recent discussions on Banach algebras to present new insights into group algebras. To achieve this, we build upon the existing work of Baker, Dales, and Lau [3] and extend their results to modular operations. We also demonstrate how these extensions can be applied to address various problems in special group algebras. By considering these extensions, we gain a broader perspective on the problems encountered in Banach algebras. This expanded view encompasses a more comprehensive range of spaces. Furthermore, we proceed to expand the scope of Banach algebra problems to include matrices.

Received January 14, 2022, accepted: April 02, 2024
Communicated by Dragan Djordjević
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2020 *Mathematics Subject Classification.* Primary 46L06; Secondary 46L07, 46L10, 47L25

We investigate the Arens multiplication and utilize the topological center argument within these spaces. Below we give some basic definitions that we will use in this article.

Let X, Y and Z be normed spaces and let $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} that he called m is Arens regular whenever $m^{***} = m^{t***t}$, for more information see [8, 11, 14]. Let A be a Banach algebra, regarding A as a Banach A-bimodule, the operation $\pi : A \times A \longrightarrow A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first (left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. The first (left) and second (right) Arens products of $a'', b'' \in A^{**}$ will be simply indicated by $a''b''$ and $a''ob''$, respectively. Let B be a Banach A-bimodule. Then B is called factors on the left (right) with respect to A, if $B = BA$ ($B = AB$). Thus B factors on both sides, if $B = BA = AB$. Let B be a Banach A-bimodule, and let

$$\pi_\ell : \ A \times B \longrightarrow B \quad \text{and} \quad \pi_r : \ B \times A \longrightarrow B,$$

be the right and left module actions of A on B. By the above notation, the transpose of π_r is denoted by $\pi_r^t : A \times B \to B$. Then

$$\pi_\ell^* : B^* \times \ A \longrightarrow B^* \quad \text{and} \quad \pi_r^{t*t} : A \times B^* \longrightarrow B^*.$$

Thus B^* is a left Banach A-module and a right Banach A-module with respect to the module actions π_r^{t*t} and π_ℓ^*, respectively. The second dual B^{**} is a Banach A^{**}-bimodule with the following module actions

$$\pi_\ell^{***} : \ A^{**} \times B^{**} \longrightarrow B^{**} \quad \text{and} \quad \pi_r^{***} : \ B^{**} \times A^{**} \longrightarrow B^{**},$$

where A^{**} is considered as a Banach algebra with respect to the first Arens product. Similarly, B^{**} is a Banach A^{**}-bimodule with the module actions

$$\pi_\ell^{t***t} : \ A^{**} \times B^{**} \longrightarrow B^{**} \quad \text{and} \quad \pi_r^{t***t} : \ B^{**} \times A^{**} \longrightarrow B^{**},$$

where A^{**} is considered as a Banach algebra with respect to the second Arens product. Let B be a left Banach A-module and e left unit element of A. Then e is left unit (resp. weakly left unit) for B, if $\pi_\ell(e, b) = b$ (resp. $\langle b', \pi_\ell(e, b)\rangle = \langle b', b\rangle$ for all $b' \in B^*$) where $b \in B$. The definition of right unit (resp. weakly right unit) is similar. A Banach A-bimodule B is called unital if B has the same left and right unit. In this way, B is called a unitary Banach A-bimodule. Suppose that A is a Banach algebra and B is a Banach A-bimodule. Since B^{**} is a Banach A^{**}-bimodule, where A^{**} is equipped with the first Arens product, we define the topological center of the right module action of A^{**} on B^{**} as follows:

$$\begin{aligned} Z_{A^{**}}^\ell(B^{**}) = Z(\pi_r) \quad = \quad & \{b'' \in B^{**} : \text{ the map } \ a'' \to \pi_r^{***}(b'', a'') \ : \ A^{**} \to B^{**} \\ & \text{is} \quad \text{weak}^*\text{-weak}^* \text{ continuous}\}. \end{aligned}$$

In this way, we write $Z^{\ell}_{B^{**}}(A^{**}) = Z(\pi_\ell)$, $Z^{r}_{A^{**}}(B^{**}) = Z(\pi_\ell^t)$ and $Z^{r}_{B^{**}}(A^{**}) = Z(\pi_r^t)$, where $\pi_\ell : \ A \times B \to B$ and $\pi_r : \ B \times A \to B$ are the left and right module actions of A on B, for more information related to the Arens regularity of module actions on Banach algebras, see [8, 11]. If we set $B = A$, we write $Z^{\ell}_{A^{**}}(A^{**}) = Z_1(A^{**}) = Z^{\ell}_1(A^{**})$ and $Z^{r}_{A^{**}}(A^{**}) = Z_2(A^{**}) = Z^{r}_2(A^{**})$, for more information see [12].

2. Main Results

Baker, Lau and Pym in [3] proved that for a Banach algebra A with bounded right approximate identity, $(A^*A)^\perp$ is an ideal of right annihilators in A^{**} and

$$A^{**} \cong (A^*A) \oplus (A^*A)^\perp.$$

In the following, for a Banach A-bimodule B, we study this problem in a general situation, that is, we show that

$$B^{**} = (B^*A)^* \oplus (B^*A)^\perp.$$

Theorem 2.1. *[3] Let B be a Banach A-bimodule and A has a BRAI. Then the following assertions hold:*

i) $(B^*A)^\perp = \{b'' \in B^{**} : \ \pi_\ell^{***}(a'', b'') = 0 \ \ \textit{for all } a'' \in A^{**}\}$.

ii) $(B^*A)^*$ *is a bounded linear isomorphism with* $\mathrm{Hom}_A(B^*, A^*)$.

Proof. i) Let $b'' \in (B^*A)^\perp$. Then for all $b' \in B^*$ and $a \in A$, we have

$$\langle \pi_\ell^{**}(b'', b'), a\rangle = \langle b'', \pi_\ell^{*}(b', a)\rangle = 0,$$

it follows that for all $a'' \in A^{**}$,

$$\langle \pi_\ell^{***}(a'', b''), b'\rangle = \langle a'', \pi_\ell^{**}(b'', b')\rangle = 0.$$

Conversely, let $b'' \in B^{**}$ with $\pi_\ell^{***}(a'', b'') = 0$, for all $a'' \in A^{**}$. Then for all $a \in A$ and $b' \in B^*$, we have

$$\langle b'', \pi_\ell^{*}(b', a)\rangle = \langle \pi_\ell^{**}(b'', b'), a\rangle = \langle a, \pi_\ell^{**}(b'', b')\rangle = \langle \pi_\ell^{***}(a, b''), b'\rangle = 0,$$

which implies that $b'' \in (B^*A)^\perp$.

ii) Suppose that $b'' \in B^{**}$. We define $T_{b''} \in \mathrm{Hom}_A(B^*, A^*)$, that is, $T_{b''}b' = \pi_\ell^{**}(b'', b')$. Then $\Lambda : b'' \to T_{b''}$ is a linear continuous map from B^{**} into $\mathrm{Hom}_A(B^*, A^*)$ such that

$$\ker \Lambda = \{b'' \in B^{**} : \ \pi_\ell^{**}(b'', b') = 0 \ \ \text{for all } b' \in B^*\}.$$

Consequently, $b'' \in \ker \Lambda$ if and only if

$$\langle b'', \pi_\ell^*(b', a)\rangle = \langle \pi_\ell^{**}(b'', b'), a\rangle = 0,$$

for all $b' \in B^*$ and $a \in A$. It follows that $b'' \in (B^*A)^\perp$. Since $(B^*A)^* \cong \frac{B^{**}}{(B^*A)^\perp}$, the continuous linear mapping Λ from $(B^*A)^*$ into $\mathrm{Hom}_A(B^*, A^*)$ is injective.

Conversely, suppose that $T \in \mathrm{Hom}_A(B^*, A^*)$ and $e'' \in A^{**}$ is a right identity for A^{**}. We define $b''_T \in B^{**}$ such that, for all b', we have $\langle b''_T, b'\rangle = \langle e'', Tb'\rangle$.

It is clear that the linear mapping $T \to b''_T$ is continuous. For all $a \in A$, we have

$$\begin{aligned}\langle \pi_\ell^{**}(b''_T, b'), a\rangle &= \langle b''_T, \pi_\ell^*(b', a)\rangle = \langle e'', T\pi_\ell^*(b', a)\rangle \\ &= \langle e'', (Tb')a\rangle = \langle ae'', Tb'\rangle \\ &= \langle Tb', a\rangle.\end{aligned}$$

Consequently, $\pi_\ell^{**}(b''_T, b') = Tb'$. It follows that the linear mapping $T \to b''_T \to T_{b''_T}$ is the identity map and consequently the isomorphism between $\mathrm{Hom}_A(B^*, A^*)$ and $(B^*A)^*$ is established. □

Corollary 2.1. *Let B be a Banach A-bimodule and let e'' be a right identity of A^{**}. Then $e''B^{**} \cong (B^*A)^*$ and $(B^*A)^\perp = \{b'' - e''b'' : b'' \in B^{**}\}$. Thus $B^{**} = (B^*A)^* \oplus (B^*A)^\perp$.*

Example 2.1.

1. Let G be a locally compact group. Let $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then by Theorem 2.1, we conclude that

$$(L^p(G) * C_0(G))^\perp = \{b \in L^q(G) : \ a''b = 0 \text{ for every } a'' \in L^\infty(G)\},$$

and

$$(L^p(G) * C_0(G))^* \cong \mathrm{Hom}_{C_0(G)}(L^p(G), L^\infty(G)).$$

2. Let G be a locally compact group. The group algebra $L^1(G)$ is a two sided ideal in $M(G)$. By Theorem 3.2 of [5], $L^1(G)^{**} = L^1(G) \oplus C_0(G)^\perp$. On the other hand, $M(G)$ is a unital Banach algebra and $M(G)^{**}$ has a right identity [6, Proposition 2.9.16 (ii)] respect to the first Arens product. Then by Theorem 2.1 and Corollary 2.1, we have

$$\begin{aligned}L^1(G)^{**} &= (L^\infty(G)M(G))^* \oplus (L^\infty(G)M(G))^\perp \\ &\cong \mathrm{Hom}_{M(G)}(L^\infty(G), M(G)^*) \oplus (L^\infty(G)M(G))^\perp.\end{aligned}$$

3. The existence of a unique right identity implies the existence of identity for the second dual of a Banach algebra [3, Theorem 1.6] and by this fact for discrete commutative semigroup S, the existence of an identity for $\ell^1(S)^{**}$ guarantees the existence of an identity for $\ell^1(S)$ [3, Theorem 4.1]. But, there are many semigroup algebras that the second dual of them have more than (or equal) one right identity. For example, suppose that S is an inverse semigroup such that $\ell^1(S)$ has a bounded right approximate identity but has not right identity. Then $\ell^1(S)^{**}$ has at least one right identity [3, Corollary 4.14]. Then similar to the previous example, we have

$$\ell^1(S)^{**} = (\ell^\infty(S)\ell^1(S))^* \oplus (\ell^\infty(S)\ell^1(S))^\perp.$$

4. Put $A = C_0(G)$ and set $B = L^1(G)$ being acted on by the pointwise multiplication of $C_0(G)$. Then it is relatively easy to compute

$$L^\infty(G)C_0(G) = \overline{\text{span}\{1_K : \ K \text{ is Borel and realtively compact}\}}.$$

Hence we have that $(L^\infty(G)C_0(G))^* \cong Hom_{C_0(G)}(L^\infty(G), M(G))$.

Theorem 2.2. *Assume that B is a left Banach A-module and A has a BAI. If B^* factors on the left, then $(B^*)^\perp = 0$.*

Proof. Let $a \in A$, $b' \in B^*$ and $b'' \in (B^*)^\perp$. Then

$$\langle \pi_\ell^{**}(b'', b'), a\rangle = \langle b'', \pi_\ell^*(b', a)\rangle = 0.$$

Thus, for all $b'' \in A^{**}$, we have

$$\langle \pi_\ell^{***}(a'', b''), b'\rangle = \langle a'', \pi_\ell^{**}(b'', b')\rangle = 0.$$

It follows that $\pi_\ell^{***}(a'', b'') = 0$. Now, let $e'' \in A^{**}$ be as a left unit for B^{**} [11, Theorem 3.6], then

$$b'' = \pi_\ell^{***}(e'', b'') = 0.$$

□

Corollary 2.2. *For a left Banach A-module B, if $\overline{B^*A} = B^*$ and B^{**} has a left unit, then $(B^*)^\perp = 0$.*

Example 2.2. Let G be a locally compact group. By [12], $L^1(G)^*$ factors on the left if and only if factors on the right if and only if G is a discrete group. Theorem 2.2 yields that $\ell^\infty(G)^\perp = 0$.

Lemma 2.1. *[8, Theorem 4.5] Let B be a Banach A-bimodule. Suppose that A has a BAI, $(e_\alpha)_\alpha \subseteq A$. Then*

1. *B factors on the left if and only if $\pi_r(b, e_\alpha) \xrightarrow{w} b$, for every $b \in B$.*
2. *B factors on the right if and only if $\pi_\ell(e_\alpha, b) \xrightarrow{w} b$, for every $b \in B$.*
3. *If B^* factors on the right, then $\pi_r(b, e_\alpha) \xrightarrow{w} b$, for every $b \in B$.*

Proof. 1. Suppose that B factors on the left. Then for every $b \in B$, there are $y \in B$ and $a \in A$ such that $b = ya$. Thus for every $b' \in B^*$, we have

$$\begin{aligned} \langle b', \pi_r(b, e_\alpha)\rangle &= \langle b', \pi_r(ya, e_\alpha)\rangle = \langle b', \pi_r(y, ae_\alpha)\rangle = \langle \pi_r^*(b', y), ae_\alpha\rangle \\ &\longrightarrow \langle \pi_r^*(b', y), a\rangle = \langle b', ya\rangle \\ &= \langle b', b\rangle. \end{aligned}$$

It follows that $\pi_r(b, e_\alpha) \xrightarrow{w} b$.

Conversely, by the Cohen's Factorization Theorem, since BA is a closed subspace of B, the proof holds.

2. Proof similar to (1).

3. Assume that B^* factors on the right with respect to A. Then for every $b' \in B^*$, there are $y' \in B$ and $a \in A$ such that $b' = ay'$. Consequently, for every $b \in B$

$$\begin{aligned} \langle b', \pi_r(b, e_\alpha)\rangle &= \langle ay', \pi_r(b, e_\alpha)\rangle = \langle y', \pi_r(b, e_\alpha)a\rangle \\ &= \langle y', \pi_r(b, e_\alpha a)\rangle = \langle \pi_r^*(y', b), e_\alpha a\rangle \\ &\longrightarrow \langle \pi_r^*(y', b), a\rangle = \langle y', \pi_r(b, a)\rangle = \langle ay', b\rangle \\ &= \langle b', b\rangle. \end{aligned}$$

It follows that $\pi_r(b, e_\alpha) \xrightarrow{\text{w}} b$.

□

In the proceeding Theorem, if we take $B = A$, then [12, Lemma 2.1] holds. Suppose that A is a Banach algebra and B is a Banach A-bimodule. According to [19], B^{**} is a Banach A^{**}-bimodule, where A^{**} is equipped with the first Arens product. We define B^*B as a subspace of A^*, that is, for all $b' \in B^*$ and $b \in B$, we define $\langle b'b, a\rangle = \langle b', ba\rangle$. Similarly, we define $B^{***}B^{**}$ as a subspace of A^{***} and we take $A^{(0)} = A$ and $B^{(0)} = B$.

In the following, the notation WSC is used for weakly sequentially complete Banach space A, that is, A is said to be weakly sequentially complete (WSC), if every weakly Cauchy sequence in A has a weak limit in A.

Theorem 2.3. *Let B be a Banach A-bimodule and A has a sequential WBAI. Then we have the following assertions:*

(i) Let B^* be a WSC and A^* factors on the left. Then

1. if B factors on the right, it follows that B^* factors on the left.

2. if B^* factors on the right, it follows that B factors on the left.

(ii) Let $B^{**}B^* = A^{**}A^*$. Then A^* factors on the left if and only if B^* factors on the left.

(iii) Suppose that A is WSC and B factors on the left (resp. right). If $B^*B = A^*$, then we have the following assertions:

1. A is unital and B has a right (resp. left) unit as Banach A-module.

2. A^* factors on the both side and B^* factors on the right (resp. left).

3. $B^{**} \cong (AB^*)^*$ (resp. $B^{**} \cong (B^*A)^*$).

Proof. (i) (1) Assume that $b'' \in B^{**}$ and $b' \in B^*$. Since A^* factors on the left, there are $a' \in A^*$ and $a \in A$ such that $b''b' = a'a$. Suppose that $(e_n)_n \subseteq A$ is a sequential WBAI for A. Then we have

$$\langle b'', b'e_n\rangle = \langle b''b', e_n\rangle = \langle a'a, e_n\rangle = \langle a', ae_n\rangle \to \langle a', a\rangle.$$

It follows that the sequence $(b'e_n)_n$ is weakly Cauchy sequence in B^*. Since B^* is WSC, there exists $x' \in B^*$ such that $b'e_n \xrightarrow{w} x'$. On the other hand, since B factors on the right, by Lemma 2.1, for each $b \in B$, we have $e_n b \xrightarrow{w} b$. Hence, we have

$$\langle x', b\rangle = \lim_n \langle b'e_n, b\rangle = \lim_n \langle b', e_n b\rangle = \langle b', b\rangle.$$

It follows that $x' = b'$, and so by Lemma 2.8, B^* factors on the left.

(2) Proof is similar to part (1).

(ii) Let $a'' \in A^{**}$ and $a' \in A^*$. Then there are $b'' \in B^{**}$ and $b' \in B^*$ such that $b''b' = a''a'$. Hence,

$$\langle a'', a'e_n\rangle = \langle a''a', e_n\rangle = \langle b''b', e_n\rangle = \langle b'', b'e_n\rangle.$$

Thus, by Cohen's factorization Theorem proof holds.

(iii) (1) Suppose that $(e_k)_k \subseteq A$ is a sequential WBAI for A. Let $a' \in A^*$. $B^*B = A^*$ implies that there are $b' \in B^*$ and $b \in B$ such that $b'b = a'$. Since B factors on the left, there are $y \in B$ and $a \in A$ such that $b = ya$. Then

$$\begin{aligned} \langle a', e_k\rangle &= \langle b'b, e_k\rangle = \langle b', be_k\rangle = \langle b', yae_k\rangle = \langle b'y, ae_k\rangle \\ &\longrightarrow \langle b'y, a\rangle = \langle b', ya\rangle \\ &= \langle b', b\rangle. \end{aligned}$$

This shows that the sequence $(e_k)_k \subseteq A$ is a weekly Cauchy sequence in A. Since A is WSC, it converges weakly to some element e of A. Then, for each $x \in A$

$$xe = x(\mathrm{w} - \lim_k e_k) = \mathrm{w} - \lim_k xe_k = x.$$

It is similar to see that $ex = x$, and so A is unital. Now, let $b \in B$, then

$$\begin{aligned} \langle b', be\rangle &= \langle b'b, e\rangle = \lim_k \langle b'b, e_k\rangle \\ &= \lim_k \langle b', yae_k\rangle = \lim_k \langle b'y, ae_k\rangle \\ &\longrightarrow \langle b'y, a\rangle = \langle b', b\rangle. \end{aligned}$$

Thus $be = b$, for all $b \in B$.

(iii) (2) By using part (1) and [12, Theorem 2.6], it is clear that A^* factors on the both sides. Now let $b' \in B^*$ and $b \in B$. By part (1), set $e \in A$ as a left unit element of B. Then

$$\langle eb', b\rangle = \langle b', be\rangle = \langle b', b\rangle.$$

It follows that $eb' = b'$. Thus B^* factors on the right.

(iii) (3) Now let $b'' \in (AB^*)^{\perp}$. By using part (2), since B^* factors on the right, for every $b' \in B^*$, there are $x' \in B^*$ and $a \in A$ such that $b' = ax'$. Then

$$\langle b'', b'\rangle = \langle b'', ax'\rangle = 0.$$

It follows that $b'' = 0$. This means that $(AB^*)^{\perp} = \{0\}$. Therefore, by Corollary 2.1, we are done. □

Weakly sequentially complete Banach algebra with a BAI investigated by Ülger [18], where he proved that for any weakly sequentially complete Banach algebra with a BAI, Arens regularity implies that the Banach algebra must be unital. Miao [13] proved that for a non-unital Banach algebra A with BAI, there is a non-unital subalgebra of A with a sequential bounded approximate identity. This shows that Ülger [18] obtained result is a consequence of Miao [13]. By Theorem 2.3 (iii), we give another version of [13, Corollary 2.3] as follows:

Corollary 2.3. *Let A be a weakly sequentially complete Banach algebra with a BAI and I be a closed two sided ideal. If $I^*I = A^*$, then A is unital and I^* factors on the left.*

Let G be a σ-compact amenable group that is not compact. The Fourier-Stieltjets algebra $B(G)$ of G is a commutative unital Banach algebra under the pointwise and the Fourier algebra $A(G)$ is a closed ideal of $B(G)$. The dual of $A(G)$ is the group von Neumann $VN(G)$ algebra and it does not factor on the left. Thus, according to the above Corollary, $A(G)^*A(G) \neq B(G)^*$, and equality holds when G is compact.

Let A be a Banach algebra and B be a left Banach A-module such that B^* factors on the left. Thus, for every $x' \in X^*$, there are $a \in A$ and $y' \in X^*$ such that $x' = y'a$. Pick $a'' \in A^{**}$ and $x'' \in X^{**}$. Suppose that $(x''_\alpha)_\alpha$ convergens to x'' in $\sigma(X^{**}, X^*)$. If $A^{**}A \subseteq Z_1(\pi_\ell)$, then

$$\begin{aligned}
\lim_\alpha \langle \pi_\ell^{***}(a'', x''_\alpha), x' \rangle &= \lim_\alpha \langle \pi_\ell^{***}(a'', x''_\alpha), y'a \rangle \\
&= \lim_\alpha \langle \pi_\ell^{***}(aa'', x''_\alpha), y' \rangle \\
&= \langle \pi_\ell^{***}(aa'', x''), y' \rangle = \langle \pi_\ell^{***}(a'', x''), x' \rangle.
\end{aligned}$$

It follows that $\pi_\ell^{***}(a'', x''_\alpha) \to \pi_\ell^{***}(a'', x'')$ in $\sigma(X^{**}, X^*)$, and so $a'' \in Z_1(\pi_\ell)$. Thus $Z_1(\pi_\ell) = A^{**}$. Therefore, one can write [2, Proposition 2.1] as follows:

Proposition 2.1. *Let A be a Banach algebra, B a left Banach A-module and let B^* factors on the left. If $A^{**}A \subseteq Z_1(\pi_\ell)$, then $Z_1(\pi_\ell) = A^{**}$.*

Theorem 2.4. *Suppose that B is a left Banach A-module and has a WLBAI $(e_\alpha)_\alpha \subseteq A$. Then we have the following assertions:*

1. *B factors on the left.*

2. *If A^* factors on the left, then B^* factors on the left. Moreover,*

   (i) if $A^{**}A \subseteq Z_1(\pi_\ell)$, then $Z_1(\pi_\ell) = A^{**}$.

   (ii) if B is a Banach A-bimodule and $AB^{**} \subseteq Z_1(\pi_r)$, then $Z^\ell_{A^{**}}(B^{**}) = B^{**}$.

Proof. (1) By Lemma 2.1, proof holds.

(2) Let $b'' \in B^{**}$ and $b' \in B^*$. Since $\pi_\ell^{**}(b'', b') \in A^*$ and A^* factors on the left, there are $a' \in A^*$ and $a \in A$ such that $\pi_\ell^{**}(b'', b') = a'a$. Without loss of generality, we let $e_\alpha \xrightarrow{w^*} e''$, where e'' is left unit for A^{**}. Then for every $b \in B$, we have

$$\langle \pi_\ell^{****}(b', e''), b\rangle = \langle b', \pi_\ell^{***}(e'', b)\rangle = \lim_\alpha \langle b', \pi_\ell(e_\alpha b)\rangle = \langle b', b\rangle.$$

It follows that $\pi_\ell^{****}(b', e'') = b'$. Then

$$\begin{aligned} \langle b'', \pi_\ell^{*}(b', e_\alpha) - b'\rangle &= \langle b'', \pi_\ell^{****}(b', (e_\alpha - e''))\rangle \\ &= \langle \pi_\ell^{*****}(b'', b'), (e_\alpha - e'')\rangle = \langle \pi_\ell^{**}(b'', b'), (e_\alpha - e'')\rangle \\ &= \langle a'a, (e_\alpha - e'')\rangle = \langle a', ae_\alpha - ae''\rangle \\ &= \langle a', ae_\alpha - a\rangle \\ &\longrightarrow 0. \end{aligned}$$

It follows that $\pi_\ell^{*}(b', e_\alpha) \xrightarrow{w} b'$, and so by the Cohen's Factorization Theorem, we are done.

The cases (i) and (ii) are the immediate results of [2, Proposition 2.1, Proposition 2.2]. □

Example 2.3. Let G be a locally compact group and $S^1(G)$ be a Segal algebra with respect to $L^1(G)$. If G is a discrete group, then Theorem 2.4 implies that $S^1(G)^*$ is factors on the left. Also, by Theorem 2.2, we have $(S^1(G)^*)^\perp = 0$.

Theorem 2.5. *Suppose that B is a right Banach A-module and has a RBAI $(e_\alpha)_\alpha \subseteq A$. Then we have the following assertions:*

1. *B factors on the right.*
2. *If A^* factors on the right and $Z^\ell_{A^{**}}(B^{**}) = B^{**}$, then B^* factors on the right.*

Proof. (1) By Lemma 2.1, proof holds.

(2) Let $b'' \in B^{**}$ and $b' \in B^*$. First, we show that $\pi_r^{****}(b', b'') \in A^*$. Suppose that $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$. Since $Z^\ell_{A^{**}}(B^{**}) = B^{**}$, for each $b'' \in B^{**}$, we have $\pi_r^{***}(b'', a''_\alpha) \xrightarrow{w^*} \pi_r^{***}(b'', a'')$. Then

$$\begin{aligned} \langle \pi_r^{****}(b', b''), a''_\alpha\rangle &= \langle \pi_r^{***}(b'', a''_\alpha), b'\rangle \to \langle \pi_r^{***}(b'', a''), b'\rangle \\ &= \langle \pi_r^{****}(b', b''), a''\rangle. \end{aligned}$$

Consequently, $\pi_r^{****}(b', b'') \in (A^{**}, \text{weak}^*)^* = A^*$. Since A^* factors on the right, there are $a' \in A^*$ and $a \in A$ such that $\pi_r^{****}(b', b'') = a'a$. Without loss of generality, we let $e_\alpha \xrightarrow{w^*} e''$, where e'' right unit for A^{**}. Then for each $b \in B$, we have

$$\langle \pi_r^{**}(e'', b'), b\rangle = \langle b', \pi_r^{***}(b, e'')\rangle = \lim_\alpha \langle b', \pi_r(b, e_\alpha)\rangle = \langle b', b\rangle.$$

It follows that $\pi_r^{**}(e'', b') = b'$. Hence,

$$\begin{aligned}\langle b'', \pi_r^{**}(e_\alpha, b') - b'\rangle &= \langle b'', \pi_r^{**}(e_\alpha, b') - \pi_r^{**}(e'', b')\rangle \\ &= \langle \pi_r^{****}(b', b''), e_\alpha - e''\rangle = \langle a'a, e_\alpha - e''\rangle \\ &= \langle a', ae_\alpha - ae''\rangle = \langle a', ae_\alpha - a\rangle \\ &\longrightarrow 0.\end{aligned}$$

Thus, $\pi_r^{**}(e_\alpha, b') \xrightarrow{w} b'$. Consequently, by Cohen's factorization, we are done. □

By Theorems 2.4 and 2.5, we have the following result:

Corollary 2.4. *Suppose that B is a Banach A-bimodule and has a BAI $(e_\alpha)_\alpha \subseteq A$. Then we have the following assertions:*

1. *B factors.*
2. *If A^* factors on the both sides and $AB^{**} \subseteq Z_1(\pi_r)$, then B^* factors on the both sides.*

Example 2.4. Assume that G is a locally compact group. We know that $L^1(G)$ is a $M(G)$-bimodule. Since $M(G)L^1(G) \neq M(G)$ and $L^1(G)M(G) \neq M(G)$, by Theorems 2.4 and 2.5, we conclude that every LBAI or RBAI for $L^1(G)$ is not LBAI or RBAI for $M(G)$, respectively.

By the following Example, we show that the converse of the case (2) of Corollary 2.4 true even the condition $AB^{**} \subseteq Z_1(\pi_r)$ does not hold.

Example 2.5. Assume that G is an infinite discrete group. Then $\ell^1(G)^*$ factors on the both sides and $\ell^1(G)\ell^1(G)^{**} \nsubseteq Z_1(\ell^1(G)^{**})$. If $\ell^1(G)\ell^1(G)^{**} \subseteq Z_1(\ell^1(G)^{**})$, then G is finite [2, Corollary 2.4], a contradiction.

Let A and B be Banach algebras. Suppose that $\mathcal{M}$ is a left Banach A-module and right Banach B-module. The triangular Banach algebra is

$$\mathcal{T} = \begin{bmatrix} A & \mathcal{M} \\ & B \end{bmatrix},$$

with the sum and product being given by the usual 2×2 matrix operations and internal module actions. The norm on $\mathcal{T}$ is

$$\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \| = \|a\|_A + \|m\|_{\mathcal{M}} + \|b\|_B.$$

The Banach algebra $\mathcal{T}$ as a Banach space is isomorphic to the ℓ^1-direct sum of A, B and $\mathcal{M}$. Forrest and Marcoux have studied the Arens regularity of triangular Banach algebras in [9] and some results regarding the module Arens regularity are given in [4]. Moreover, the topological center of these algebras is characterized by

Eshaghi Gordji and Filali in [8]. We extend the actions of A on M and of B on $\mathcal{M}$ to actions of A^{**} and B^{**} on $\mathcal{M}^{**}$ via

$$\Gamma\Box\Pi = w^* - \lim_i \lim_k a_i \cdot x_k, \qquad \text{and} \qquad \Pi\Box\Psi = w^* - \lim_k \lim_j x_k \cdot b_j,$$

where $\Gamma = w^* - \lim_i a_i$, $\Psi = w^* - \lim_j b_j$, and $\Pi = w^* - \lim_k x_k$. Let $T_1 = \begin{bmatrix} \Gamma_1 & \Pi_1 \\ & \Psi_1 \end{bmatrix}, T_2 = \begin{bmatrix} \Gamma_2 & \Pi_2 \\ & \Psi_2 \end{bmatrix} \in \mathcal{T}^{**}$. The first and second Arens products on $\mathcal{T}^{**}$ are defined as follows

$$T_1\Box T_2 = \begin{bmatrix} \Gamma_1\Box\Gamma_2 & \Gamma_1\Box\Pi_2 + \Pi_1\Box\Psi_2 \\ & \Psi_1\Box\Psi_2 \end{bmatrix}, \tag{2.1}$$

and

$$T_1 \diamond T_2 = \begin{bmatrix} \Gamma_1 \diamond \Gamma_2 & \Gamma_1 \diamond \Pi_2 + \Pi_1 \diamond \Psi_2 \\ & \Psi_1 \diamond \Psi_2 \end{bmatrix}. \tag{2.2}$$

The Banach algebras A and B act on $\mathcal{M}$ regularly if $\Gamma\Box\Pi = \Gamma \diamond \Pi$ and $\Pi\Box\Psi = \Pi\diamond\Psi$, for all $\Gamma \in A^{**}$, $\Psi \in B^{**}$ and $\Pi \in \mathcal{M}^{**}$. Triangular Banach algebras are good tools for giving counter-examples for some concepts related to Banach algebras, for example, for the Arens regularity, see [8].

Corollary 2.5. *Let A and B have LBAI and RBAI, respectively, and $\mathcal{M}$ be as the above. Then*

1. *$\mathcal{M}$ factors on the left and the right respect to A and B, respectively.*
2. *if A^* factors on the left, B^* factors on the right and $B\mathcal{M}^{**} \subseteq Z_1(\pi_r)$, then $\mathcal{T}^*$ factors on the left respect to A and factors on the right respect to B.*
3. *if A^* and B^* both factor on the both sides and $B\mathcal{M}^{**} \subseteq Z_1(\pi_r)$, then $\mathcal{T}^*$ factors on the both sides.*

The converse of the case (3) of the above Theorem may be not true. For example, similar to Example 2.5, if G is an infinite discrete group and

$$\mathcal{T} = \begin{bmatrix} \ell^1(G) & \ell^1(G) \\ & \ell^1(G) \end{bmatrix}.$$

Then $\mathcal{T}^*$ factors on the both sides and $\ell^1(G)\ell^1(G)^{**} \nsubseteq Z_1(\pi_r)$. Another example of this argument is an infinite dimensional unital C^*-algebra. Let A be an infinite dimensional unital C^*-algebra. Let

$$\mathcal{T} = \begin{bmatrix} A & A^* \\ & A \end{bmatrix}.$$

Then $\mathcal{T}^*$ factors on the both sides and $AA^{***} \nsubseteq Z_1(\pi_r)$, because, $AA^{***} \subseteq Z_1(\pi_r)$ implies that $Z_1(\pi_r) = A^{***}$ [2, Proposition 2.2]. This is equivalent to that A is of finite dimension [8, Corollary 2.3], a contradiction.

Acknowledgements

The authors thank the anonymous referees for their thorough review.

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