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EXISTENCE AND UNIQUENESS OF SOLUTION OF DIFFERENTIAL EQUATION OF FRACTIONAL ORDER VIA S-ITERATION

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Abstract. In this paper, we study the existence, uniqueness and other properties of solutions of differential equation of fractional order involving the Caputo fractional derivative. The tool employed in the analysis is based on application of $S-$ iteration method. The study of qualitative properties in general required differential and integral inequalities, and here $S-$iteration method itself has equally important contribution to study various properties such as dependence on initial data, closeness of solutions and dependence on parameters and functions involved therein. Finally, we present an example in support of all proved results.
Keywords: Existence and uniqueness, Normal S-iterative method, Fractional derivative, Continuous dependence, Closeness, Parameters

1. Introduction

We consider the following differential equation of fractional order involving the Caputo fractional derivative of the type:

$$\big(D^{\alpha}_{*a}\big)y(t) = \mathcal{F}\Big(t, y(t), y(a), y(b)\Big), \tag{1.1}$$

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for $t \in I = [a, b],\ n-1 < \alpha \le n \ \ (n \in \mathbb{N})$, with the given initial conditions

$$y^{(j)}(a) = c_j,\ j = 0, 1, 2, \cdots, n-1, \tag{1.2}$$

where $\mathcal{F} : I \times X \times X \times X \to X$ is continuous function and $c_j \ \ (j = 0, 1, 2, \ldots, n-1)$ are given elements in X.

Several researchers have introduced many iteration methods for certain classes of operators in the sense of their convergence, equivalence of convergence and rate of convergence etc. (see [1, 3, 4, 5, 6, 8, 9, 18, 19, 20, 21, 22, 23, 24, 31, 32]). The most of iterations devoted for both analytical and numerical approaches. The $S-$iteration method, due to simplicity and fastness, has attracted the attention and hence, it is used in this paper.

The problems of existence, uniqueness and other properties of solutions of special forms of IVP (1.1)-(1.2) and its variants have been studied by several researchers under variety of hypotheses by using different techniques, [2, 7, 10, 11, 12, 13, 14, 15, 16, 26, 27, 29, 30] and some of references cited therein. In recently, S. Soltuz and T. Grosan [33] have studied the special version of equation (1.1) for different qualitative properties of solutions. Authors are motivated by the work of D. R. Sahu [31] and influenced by [5, 33].

The main objective of this paper is to use normal $S-$iteration method to establish the existence and uniqueness of solution of the initial value problem (1.1)-(1.2) and other qualitative properties of solutions.

2. Preliminaries

Before proceeding to the statement of our main results, we shall setforth some preliminaries and hypotheses that will be used in our subsequent discussion.

Let X be a Banach space with norm $\|\cdot\|$ and $I = [a, b]$ denotes an interval of the real line $\mathbb{R}$. We define $B = C^r(I, X)$ (where $r = n$ for $\alpha \in \mathbb{N}$ and $r = n-1$ for $\alpha \notin \mathbb{N}$.) as a Banach space of all r times continuously differentiable functions from I into X, endowed with the norm

$$\|y\|_B = \sup\{\|y(t)\| : y \in B\}, \quad t \in I.$$

Definition 2.1. [28] The Riemann-Liouville fractional integral (left-sided) of a function $h \in C^1[a, b]$ of order $\alpha \in \mathbf{R}_+ = (0, \infty)$ is defined by

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds,$$

where Γ is the Euler gamma function.

Definition 2.2. [28] Let $n-1 < \alpha \leqslant n,\ n \in \mathbb{N}$. Then the expression

$$D_a^{\alpha} h(t) = \frac{d^n}{dt^n}\left[I_a^{n-\alpha} h(t)\right],\ t \in [a, b]$$

is called the (left-sided) Riemann-Liouville derivative of h of order α whenever the expression on the right-hand side is defined.

Definition 2.3. [25] Let $h \in C^n[a, b]$ and $n-1 < \alpha \leqslant n,\ n \in \mathbb{N}$. Then the expression

$$\left(D_{*a}^{\alpha}\right) h(t) = I_a^{n-\alpha} h^{(n)}(t),\ t \in [a, b]$$

is called the (left-sided) Caputo derivative of h of order α.

Lemma 2.1. *[17] If the function $f = (f_1, \cdots, f_n) \in C^1[a, b]$, then the initial value problems*

$$\left(D_{*a}^{\alpha_i}\right) y(t) = f_i(t, y_1, \cdots, y_n),\quad y_i^{(k)}(0) = c^i{}_k,\quad i = 1, 2, \cdots, n,\ k = 1, 2, \cdots, m_i$$

where $m_i < \alpha_i \leqslant m_i + 1$ as equivalent to Volterra integral equations:

$$y_i(t) = \sum_{k=0}^{m_i} c^i{}_k \frac{t^k}{k!} + I_a^{\alpha_i} f_i(t, y_1, \cdots, y_n),\ 1 \leqslant i \leqslant n.$$

As a consequence of the above Lemma, it is easy to observe that if $y \in B$ and $\mathcal{F} \in C^1[a, b]$, then $y(t)$ satisfies the following integral equation which is equivalent to (1.1)-(1.2) is

$$y(t) = \sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, y(s), y(a), y(b)\Big) ds. \tag{2.1}$$

We need the following pair of known results:

Theorem 2.1. *(([31], p.194)) Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction operator with contractivity factor $m \in [0, 1)$ and fixed point x^*. Let α_k and β_k be two real sequences in $[0, 1]$ such that $\alpha \leqslant \alpha_k \leqslant 1$ and $\beta \leqslant \beta_k < 1$ for all $k \in \mathbb{N}$ and for some $\alpha, \beta > 0$. For given $u_1 = v_1 = w_1 \in C$, define sequences u_k, v_k and w_k in C as follows:*

S-iteration process: $\begin{cases} u_{k+1} = (1-\alpha_k) T u_k + \alpha_k T y_k, \\ y_k = (1-\beta_k) u_k + \beta_k T u_k, k \in \mathbb{N}. \end{cases}$

Picard iteration: $v_{k+1} = T v_k, k \in \mathbb{N}.$

Mann iteration process: $w_{k+1} = (1-\beta_k) w_k + \beta_k T w_k, k \in \mathbb{N}.$

Then we have the following:

(a) $\|u_{k+1} - x^*\| \leqslant m^k \Big[1 - (1-m)\alpha\beta\Big]^k \|u_1 - x^*\|$, *for all* $k \in \mathbb{N}$.

(b) $\|v_{k+1} - x^*\| \leqslant m^k \|v_1 - x^*\|$, *for all* $k \in \mathbb{N}$.

(c) $\|w_{k+1} - x^*\| \leqslant \Big[1 - (1-m)\beta\Big]^k \|w_1 - x^*\|$, *for all* $k \in \mathbb{N}$.

Moreover, the S*-iteration process is faster than the Picard and Mann iteration processes.*

Definition 2.4. ([31], p.193) In particular, for $\alpha_k = 1,\ k \in \mathbb{N} \cup \{0\}$ in the S-iteration process, then it reduces to as follows:

$$\begin{cases} u_0 \in C, \\ u_{k+1} = Ty_k, \\ y_k = (1-\xi_k)u_k + \xi_k Tu_k,\ k \in \mathbb{N} \cup \{0\}. \end{cases} \tag{2.2}$$

This is called normal $S-$iteration method.

Note: For our convenience, we replaced β_k in the S-iteration process by ξ_k.

Lemma 2.2. *(([33], p.4)) Let* $\{\beta_k\}_{k=0}^{\infty}$ *be a nonnegative sequence for which one assumes there exists* $k_0 \in \mathbb{N}$*, such that for all* $k \geqslant k_0$ *one has satisfied the inequality*

$$\beta_{k+1} \leqslant (1-\mu_k)\beta_k + \mu_k \gamma_k, \tag{2.3}$$

where $\mu_k \in (0,1)$*, for all* $k \in \mathbb{N} \cup \{0\}$, $\sum_{k=0}^{\infty} \mu_k = \infty$ *and* $\gamma_k \geqslant 0,\ \forall k \in \mathbb{N} \cup \{0\}$*. Then the following inequality holds*

$$0 \leqslant \limsup_{k \to \infty} \beta_k \leqslant \limsup_{k \to \infty} \gamma_k. \tag{2.4}$$

3. Existence and Uniqueness of Solutions via $S-$iteration

Now, we are able to state and prove the following main theorem which deals with the existence of solutions of the equations (1.1)-(1.2).

Theorem 3.1. *Assume that there exists a function* $p \in C(I, \mathbb{R}_+)$ *and constants* $\lambda,\ \beta,\ \gamma > 0$ *such that for* $t \in I$,

$$\begin{aligned} &\|\mathcal{F}\Big(t, u_1, u_2, u_3\Big) - \mathcal{F}\Big(t, v_1, v_2, v_3\Big)\| \\ &\leqslant p(t)\Big[\lambda\|u_1 - v_1\| + \beta\|u_2 - v_2\| + \gamma\|u_3 - v_3\|\Big]. \end{aligned} \tag{3.1}$$

If $\Theta = {I_a}^{\alpha} p(t)\big(\lambda + \beta + \gamma\big) < 1 \quad (t \in I)$*, then the iterative sequence* $\{y_k\}_{k=0}^{\infty}$ *generated by normal* $S-$ *iteration method (2.2) with the real control sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in*

$[0,1]$ *satisfying* $\sum_{k=0}^{\infty} \xi_k = \infty$, *converges to a unique point* $y \in B$, *which is the required solution of the equations (1.1)-(1.2) with the following estimate:*

$$\|y_{k+1} - y\|_B \leqslant \frac{\Theta^{k+1}}{e^{\left(1-\Theta\right)\sum_{i=0}^{k}\xi_i}}\|y_0 - y\|_B. \tag{3.2}$$

Proof. Let $y(t) \in B$ and define the operator

$$\begin{aligned}(Ty)(t) &= \sum_{j=0}^{n-1} \frac{c_j}{j!}\,(t-a)^j \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, y(s), y(a), y(b)\Big)ds,\ t \in I.\end{aligned} \tag{3.3}$$

Let $\{y_k\}_{k=0}^{\infty}$ be iterative sequence generated by normal $S-$iteration method (2.2) for the operator given in (3.3) with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$. We will show that $y_k \to y$ as $k \to \infty$. From (2.2), (3.3) and assumption, we obtain

$$\begin{aligned}&\|y_{k+1}(t) - y(t)\| \\ &= \|(Tz_k)(t) - (Ty)(t)\| \\ &= \|\sum_{j=0}^{n-1}\frac{c_j}{j!}\,(t-a)^j + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, z_k(s), z_k(a), z_k(b)\Big)ds \\ &\quad -\sum_{j=0}^{n-1}\frac{c_j}{j!}\,(t-a)^j - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, y(s), y(a), y(b)\Big)ds\| \\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|\mathcal{F}\Big(s, z_k(s), z_k(a), z_k(b)\Big) - \mathcal{F}\Big(s, y(s), y(a), y(b)\Big)\|ds \\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s) \\ &\quad \times\Big[\lambda\|z_k(s) - y(s)\| + \beta\|z_k(a) - y(a)\| + \gamma\|z_k(b) - y(b)\|\Big]ds.\end{aligned} \tag{3.4}$$

Now, we estimate

$$\begin{aligned}\|z_k(t) - y(t)\| &= \Big[(1-\xi_k)\|y_k(t) - y(t)\| + \xi_k\|(Ty_k)(t) - (Ty)(t)\|\Big] \\ &\leqslant (1-\xi_k)\|y_k(t) - y(t)\| + \xi_k\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s) \\ &\quad \times\Big[\lambda\|y_k(s) - y(s)\| + \beta\|y_k(a) - y(a)\| + \gamma\|y_k(b) - y(b)\|\Big]ds.\end{aligned} \tag{3.5}$$

Now, by taking supremum in the inequalities (3.4) and (3.5), we obtain

$$\|y_{k+1} - y\|_B \leqslant \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big(\lambda + \beta + \gamma\Big)\|z_k - y\|_B ds$$

$$\begin{aligned} &\leqslant I_a{}^{\alpha} p(t)\Big(\lambda+\beta+\gamma\Big)\|z_k-y\|_B \\ &= \Theta\|z_k-y\|_B \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \|z_k-y\|_B &\leqslant \Big[1-\xi_k\Big(1-I_a{}^{\alpha} p(t)\Big(\lambda+\beta+\gamma\Big)\Big)\Big]\|y_k-y\|_B \\ &= \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-y\|_B, \end{aligned} \tag{3.7}$$

respectively.
Therefore, using (3.7) in (3.6), we have

$$\|y_{k+1}-y\|_B \leqslant \Theta\Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-y\|_B. \tag{3.8}$$

Thus, by induction, we get

$$\|y_{k+1}-y\|_B \leqslant \Theta^{k+1}\prod_{j=0}^{k}\Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_0-y\|_B. \tag{3.9}$$

Since $\xi_k \in [0,1]$ for all $k \in \mathbb{N}\cup\{0\}$, the definition of Θ and $\xi_k \leqslant 1$ yields,

$$\Rightarrow \xi_k\Theta < \xi_k$$

$$\Rightarrow \xi_k\Big(1-\Theta\Big) < 1,\ \forall\, k \in \mathbb{N}\cup\{0\}. \tag{3.10}$$

From the classical analysis, we know that

$$1-x \leqslant e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots,\ x\in[0,1].$$

Hence by utilizing this fact with (3.10) in (3.9), we obtain

$$\begin{aligned} \|y_{k+1}-y\|_B &\leqslant \Theta^{k+1} e^{-\left(1-\Theta\right)\sum_{j=0}^{k}\xi_j}\|y_0-y\|_B \\ &= \frac{\Theta^{k+1}}{e^{\left(1-\Theta\right)\sum_{i=0}^{k}\xi_i}}\|y_0-y\|_B. \end{aligned} \tag{3.11}$$

Thus, we have proved (3.2). Since $\sum_{k=0}^{\infty}\xi_k=\infty$,

$$e^{-\left(1-\Theta\right)\sum_{j=0}^{k}\xi_j} \to 0 \quad \text{as} \quad k\to\infty. \tag{3.12}$$

Hence using this, the inequality (3.11) implies $\lim_{k\to\infty}\|y_{k+1}-y\|_B=0$ and therefore, we get $y_k \to y$ as $k\to\infty$. □

Remark: It is an interesting to note that the inequality (3.11) gives the bounds in terms of known functions, which majorizes the iterations for solutions of the equations (1.1)-(1.2) for $t \in I$.

4. Continuous dependence via $S-$iteration

In this section, we shall deal with continuous dependence of solution of the problem (1.1) on the initial data, functions involved therein and also on parameters.

4.1. Dependence on initial data

Suppose $y(t)$ and $\overline{y}(t)$ are solutions of (1.1) with initial data

$$y^{(j)}(a) = c_j, \ j = 0, 1, 2, \cdots, n-1, \tag{4.1}$$

and

$$\overline{y}^{(j)}(a) = d_j, \ j = 0, 1, 2, \cdots, n-1, \tag{4.2}$$

respectively, where c_j, d_j are elements of the space X.

Then looking at the steps as in the proof of Theorem 3.1, we define the operator for the equations (1.1)- (4.2)

$$\begin{aligned}(\overline{T}\overline{y})(t) &= \sum_{j=0}^{n-1} \frac{d_j}{j!} (t-a)^j \\ &+\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{y}(s), \overline{y}(a), \overline{y}(b)\Big) ds, \ t \in I. \end{aligned} \tag{4.3}$$

We shall deal with the continuous dependence of solutions of equation (1.1) on initial data.

Theorem 4.1. *Suppose the function $\mathcal{F}$ in equation (1.1) satisfies the condition (3.1). Consider the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (3.3) and $\overline{T}$ in (4.3), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. If the sequence $\{\overline{y}_k\}_{k=0}^{\infty}$ converges to $\overline{y}$, then we have*

$$\|y - \overline{y}\|_B \ \leqslant \ \frac{3M}{\Big(1-\Theta\Big)}, \tag{4.4}$$

where

$$M = \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!} (b-a)^j.$$

Proof. Suppose the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (3.3) and $\overline{T}$ in (4.3), respectively with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. From iteration (2.2) and equations (3.3); (4.3) and assumptions, we obtain

$$\begin{aligned}
&\|y_{k+1}(t) - \overline{y}_{k+1}(t)\| \\
&= \|(Tz_k)(t) - (\overline{T}\overline{z}_k)(t)\| \\
&= \| \sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, z_k(s), z_k(a), z_k(b)\Big) ds \\
&\quad - \sum_{j=0}^{n-1} \frac{d_j}{j!}\ (t-a)^j - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b)\Big) ds\| \\
&\leqslant \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!}\ (b-a)^j \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \|\mathcal{F}\Big(s, z_k(s), z_k(a), z_k(b)\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b)\Big)\| ds \\
&\leqslant M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \\
&\quad \times \Big[\lambda \|z_k(s) - \overline{z}_k(s)\| + \beta \|z_k(a) - \overline{z}_k(a)\| + \gamma \|z_k(b) - \overline{z}_k(b)\|\Big] ds. \qquad (4.5)
\end{aligned}$$

Recalling the equations (3.6) and (3.7), the above inequality becomes

$$\|y_{k+1} - \overline{y}_{k+1}\|_B \ \leqslant\ M + \Theta \|z_k - \overline{z}_k\|_B, \tag{4.6}$$

and similarly, it is seen that

$$\|z_k - \overline{z}_k\|_B \ \leqslant\ \xi_k M + \Big[1 - \xi_k\Big(1-\Theta\Big)\Big] \|y_k - \overline{y}_k\|_B. \tag{4.7}$$

Therefore, using (4.7) in (4.6) and using hypothesis $\Theta < 1$, and $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$\begin{aligned}
\|y_{k+1} - \overline{y}_{k+1}\|_B &\leqslant M + \|z_k - \overline{z}_k\|_B \\
&\leqslant M + \xi_k M + \Big[1 - \xi_k\Big(1-\Theta\Big)\Big] \|y_k - \overline{y}_k\|_B \\
&\leqslant 2\xi_k M + \xi_k M + \Big[1 - \xi_k\Big(1-\Theta\Big)\Big] \|y_k - \overline{y}_k\|_B \\
&\leqslant \Big[1 - \xi_k\Big(1-\Theta\Big)\Big] \|y_k - \overline{y}_k\|_B + \xi_k\Big(1-\Theta\Big) \frac{3M}{\Big(1-\Theta\Big)}. \qquad (4.8)
\end{aligned}$$

We denote

$$\begin{aligned}
\beta_k &= \|y_k - \overline{y}_k\|_B \geqslant 0, \\
\mu_k &= \xi_k\Big(1-\Theta\Big) \in (0,1), \\
\gamma_k &= \frac{3M}{\Big(1-\Theta\Big)} \geqslant 0.
\end{aligned}$$

The assumption $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (4.8) satisfies all the conditions of Lemma 2.2 and hence we have

$$
\begin{aligned}
0 &\leqslant \limsup_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \gamma_k \\
\Rightarrow 0 &\leqslant \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leqslant \limsup_{k\to\infty} \frac{3M}{\Big(1-\Theta\Big)} \\
\Rightarrow 0 &\leqslant \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leqslant \frac{3M}{\Big(1-\Theta\Big)}. \qquad (4.9)
\end{aligned}
$$

Using the assumptions $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \overline{y}_k = \overline{y}$, we get from (4.9) that

$$
\|y - \overline{y}\|_B \leqslant \frac{3M}{\Big(1-\Theta\Big)}, \qquad (4.10)
$$

which shows that the dependency of solutions of IVPs (1.1)-(1.2) and (1.1)-(4.2) on given initial data. □

4.2. Closeness of solution via $S-$iteration

Consider the problem (1.1)-(1.2) and the corresponding problem

$$
\big(D^{\alpha}_{*a}\big)\overline{y}(t) = \overline{\mathcal{F}}\Big(t, \overline{y}(t), \overline{y}(a), \overline{y}(b)\Big), \qquad (4.11)
$$

for $t \in I = [a,b],\ n-1 < \alpha \le n \ \ (n \in \mathbb{N})$, with the given initial conditions

$$
\overline{y}^{(j)}(a) = d_j,\ j = 0, 1, 2, \cdots, n-1, \qquad (4.12)
$$

where $\overline{\mathcal{F}}$ is defined as $\mathcal{F}$ and $d_j \quad (j = 0, 1, 2, \ldots, n-1)$ are given elements in X.

Then looking at the steps as in the proof of Theorem 3.1, we define the operator for the equation (4.11)- (4.12)

$$
\begin{aligned}
(\overline{T}\overline{y})(t) &= \sum_{j=0}^{n-1} \frac{d_j}{j!}\,(t-a)^j \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{\mathcal{F}}\Big(s, \overline{y}(s), \overline{y}(a), \overline{y}(b)\Big) ds,\ t \in I. \qquad (4.13)
\end{aligned}
$$

The next theorem deals with the closeness of solutions of the problems (1.1)-(1.2) and (4.11)-(4.12).

Theorem 4.2. *Consider the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (3.3) and $\overline{T}$ in (4.13), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N}\cup\{0\}$. Assume that*

(i) all conditions of Theorem 3.1 hold, and $y(t)$ and $\overline{y}(t)$ are solutions of (1.1)-(1.2) and (4.11)-(4.12) respectively.

(ii) there exists non negative constant ϵ such that

$$\|\mathcal{F}\Big(t,u_1,u_2,u_3\Big) - \overline{\mathcal{F}}\Big(t,u_1,u_2,u_3\Big)\| \leqslant \epsilon,\ \forall\ t \in I. \tag{4.14}$$

If the sequence $\{\overline{y}_k\}_{k=0}^{\infty}$ converges to $\overline{y}$, then we have

$$\|y-\overline{y}\|_B \leqslant \frac{3\Big[M + \frac{\epsilon(b-a)^{\alpha}}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Theta\Big)}. \tag{4.15}$$

Proof. Suppose the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (3.3) and $\overline{T}$ in (4.13), respectively with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N}\cup\{0\}$. From iteration (2.2) and equations (3.3); (4.13) and hypotheses, we obtain

$$\begin{aligned}
&\|y_{k+1}(t) - \overline{y}_{k+1}(t)\| \\
&= \|(Tz_k)(t) - (\overline{T}\overline{z}_k)(t)\| \\
&= \|\sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, z_k(s), z_k(a), z_k(b)\Big)ds \\
&\quad -\sum_{j=0}^{n-1} \frac{d_j}{j!}\ (t-a)^j - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\overline{\mathcal{F}}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b)\Big)ds\| \\
&\leqslant \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!}\ (b-a)^j + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\| \\
&\qquad \times\mathcal{F}\Big(s, z_k(s), z_k(a), z_k(b)\Big) - \overline{\mathcal{F}}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b)\Big)\|ds \\
&\leqslant M + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} \\
&\qquad \times\|\mathcal{F}\Big(s, z_k(s), z_k(a), z_k(b)\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b)\Big)\|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} \\
&\qquad \times\|\mathcal{F}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b)\Big) - \overline{\mathcal{F}}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b)\Big)\|ds
\end{aligned}$$

$$\begin{aligned}
&\leqslant M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \epsilon ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \\
&\quad \times \Big[\lambda \|z_k(s) - \overline{z}_k(s)\| + \beta \|z_k(a) - \overline{z}_k(a)\| + \gamma \|z_k(b) - \overline{z}_k(b)\| \Big] ds \\
&\leqslant M + \frac{\epsilon (t-a)^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \\
&\quad \times \Big[\lambda \|z_k(s) - \overline{z}_k(s)\| + \beta \|z_k(a) - \overline{z}_k(a)\| + \gamma \|z_k(b) - \overline{z}_k(b)\| \Big] ds \\
&\leqslant M + \frac{\epsilon (b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \\
&\quad \times \Big[\lambda \|z_k(s) - \overline{z}_k(s)\| + \beta \|z_k(a) - \overline{z}_k(a)\| + \gamma \|z_k(b) - \overline{z}_k(b)\| \Big] ds. \qquad (4.16)
\end{aligned}$$

Recalling the derivations obtained in equations (3.6) and (3.7), the above inequality becomes

$$\|y_{k+1} - \overline{y}_{k+1}\|_B \leqslant M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} + \Theta \|z_k - \overline{z}_k\|_B, \qquad (4.17)$$

and similarly, it is seen that

$$\|z_k - \overline{z}_k\|_B \leqslant \xi_k \Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \Big] + \Big[1 - \xi_k \Big(1 - \Theta \Big) \Big] \|y_k - \overline{y}_k\|_B. \qquad (4.18)$$

Therefore, using (4.18) in (4.17) and using hypothesis $\Theta < 1$, and $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$\begin{aligned}
&\|y_{k+1} - \overline{y}_{k+1}\|_B \\
&\leqslant \Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \Big] + \|z_k - \overline{z}_k\|_B \\
&\leqslant \Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \Big] + \xi_k \Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \Big] + \Big[1 - \xi_k \Big(1 - \Theta \Big) \Big] \|y_k - \overline{y}_k\|_B \\
&\leqslant 2\xi_k \Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \Big] + \xi_k \Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \Big] \\
&\quad + \Big[1 - \xi_k \Big(1 - \Theta \Big) \Big] \|y_k - \overline{y}_k\|_B \\
&\leqslant \Big[1 - \xi_k \Big(1 - \Theta \Big) \Big] \|y_k - \overline{y}_k\|_B + \xi_k \Big(1 - \Theta \Big) \frac{3 \Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \Big]}{\Big(1 - \Theta \Big)}. \qquad (4.19)
\end{aligned}$$

We denote

$$\begin{aligned}
\beta_k &= \|y_k - \overline{y}_k\|_B \geqslant 0, \\
\mu_k &= \xi_k \Big(1 - \Theta \Big) \in (0, 1), \\
\gamma_k &= \frac{3 \Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \Big]}{\Big(1 - \Theta \Big)} \geqslant 0.
\end{aligned}$$

The assumption $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (4.19) satisfies all the conditions of Lemma 2.2 and hence we have

$$\begin{aligned} 0 &\leqslant \limsup_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \gamma_k \\ \Rightarrow 0 &\leqslant \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leqslant \limsup_{k\to\infty} \frac{3\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Theta\Big)} \\ \Rightarrow 0 &\leqslant \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leqslant \frac{3\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Theta\Big)}. \end{aligned} \tag{4.20}$$

Using the assumptions $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \overline{y}_k = \overline{y}$, we get from (4.20) that

$$\|y - \overline{y}\|_B \leqslant \frac{3\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Theta\Big)}, \tag{4.21}$$

which shows that the dependency of solutions of IVP (1.1)-(1.2) on the function involved on the right hand side of the given equation. □

Remark: The inequality (4.21) relates the solutions of the problems (1.1)-(1.2) and (4.11)-(4.12) in the sense that, if $\mathcal{F}$ and $\overline{\mathcal{F}}$ are close as $\epsilon \to 0$, then not only the solutions of the problems (1.1)-(1.2) and (4.11)-(4.12) are close to each other (i.e. $\|y-\overline{y}\|_B \to 0$), but also depends continuously on the functions involved therein and initial data.

4.3. Dependence on Parameters

We next consider the following problems

$$\big(D^\alpha_{*a}\big)y(t) = \mathcal{F}\Big(t, y(t), y(a), y(b), \mu_1\Big), \tag{4.22}$$

for $t \in I = [a, b]$, $n-1 < \alpha \le n$ $(n \in \mathbb{N})$, with the given initial conditions

$$y^{(j)}(a) = c_j, \; j = 0, 1, 2, \cdots, n-1, \tag{4.23}$$

and

$$\big(D^\alpha_{*a}\big)\overline{y}(t) = \mathcal{F}\Big(t, \overline{y}(t), \overline{y}(a), \overline{y}(b), \mu_2\Big), \tag{4.24}$$

for $t \in I = [a, b]$, $n-1 < \alpha \le n$ $(n \in \mathbb{N})$, with the given initial conditions

$$\overline{y}^{(j)}(a) = d_j, \; j = 0, 1, 2, \cdots, n-1, \tag{4.25}$$

where $\mathcal{F} : I \times X \times X \times X \times \mathbb{R} \to X$ is continuous function, c_j, d_j $(j = 0, 1, 2, \ldots, n-1)$ are given elements in X and constants μ_1, μ_2 are real parameters.

Let $y(t)$, $\overline{y}(t) \in B$ and following steps from the proof of Theorem 3.1, define the operators for the equations (4.22) and (4.24), respectively

$$(Ty)(t) = \sum_{j=0}^{n-1} \frac{c_j}{j!} (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, y(s), y(a), y(b), \mu_1\Big) ds, \ t \in I. \tag{4.26}$$

and

$$(\overline{T}\overline{y})(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{y}(s), \overline{y}(a), \overline{y}(b), \mu_2\Big) ds, \ t \in I. \tag{4.27}$$

The following theorem proves the continuous dependency of solutions on parameters.

Theorem 4.3. *Consider the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ generated normal $S-$iterative method associated with operators T in (4.26) and $\overline{T}$ in (4.27), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. Assume that*

(i) $y(t)$ and $\overline{y}(t)$ are solutions of (4.22)-(4.23) and (4.24)-(4.25) respectively.

(ii) there exists constants $\overline{\lambda}$, $\overline{\beta}$, $\overline{\gamma} > 0$ such that the function $\mathcal{F}$ satisfy the conditions:

$$\|\mathcal{F}\Big(t, u_1, u_2, u_3, \mu_1\Big) - \mathcal{F}\Big(t, v_1, v_2, v_3, \mu_1\Big)\| \leqslant \overline{p}(t)\Big[\overline{\lambda}\|u_1 - v_1\| + \overline{\beta}\|u_2 - v_2\| + \overline{\gamma}\|u_3 - v_3\|\Big].$$

and

$$\|\mathcal{F}\Big(t, u_1, u_2, u_3, \mu_1\Big) - \mathcal{F}\Big(t, u_1, u_2, u_3, \mu_2\Big)\| \leqslant r(t)\Big|\mu_1 - \mu_2\Big|,$$

where $\overline{p}$, $r \in C(I, \mathbb{R}_+)$.

If the sequence $\{\overline{y}_k\}_{k=0}^{\infty}$ converges to $\overline{y}$, then we have

$$\|y - \overline{y}\|_B \leqslant \frac{3\Big[M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t)\Big]}{\Big(1 - \overline{\Theta}\Big)}, \tag{4.28}$$

where $\overline{\Theta} = {I_a}^{\alpha}\overline{p}(t)\big(\overline{\lambda} + \overline{\beta} + \overline{\gamma}\big) < 1 \quad (t \in I)$.

Proof. Suppose the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (4.26) and $\overline{T}$ in (4.27), respectively with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. From iteration (2.2) and equations (4.26); (4.27) and hypotheses, we obtain

$$
\begin{aligned}
&\|y_{k+1}(t) - \overline{y}_{k+1}(t)\| \\
&= \|(Tz_k)(t) - (\overline{T}\overline{z}_k)(t)\| \\
&= \| \sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, z_k(s), z_k(a), z_k(b), \mu_1\Big) ds \\
&\quad - \sum_{j=0}^{n-1} \frac{d_j}{j!}\ (t-a)^j - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b), \mu_2\Big) ds\| \\
&\leqslant \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!}\ (b-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \\
&\qquad \times \|\mathcal{F}\Big(s, z_k(s), z_k(a), z_k(b), \mu_1\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b), \mu_2\Big)\| ds \\
&\leqslant M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \\
&\qquad \times \|\mathcal{F}\Big(s, z_k(s), z_k(a), z_k(b), \mu_1\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b), \mu_1\Big)\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \\
&\qquad \times \|\mathcal{F}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b), \mu_1\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \overline{z}_k(a), \overline{z}_k(b), \mu_2\Big)\| ds \\
&\leqslant M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} r(s) |\mu_1 - \mu_2| ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{p}(s) \\
&\qquad \times \Big[\overline{\lambda}\|z_k(s) - \overline{z}_k(s)\| + \overline{\beta}\|z_k(a) - \overline{z}_k(a)\| + \overline{\gamma}\|z_k(b) - \overline{z}_k(b)\|\Big] ds \\
&\leqslant M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{p}(s) \\
&\qquad \times \Big[\overline{\lambda}\|z_k(s) - \overline{z}_k(s)\| + \overline{\beta}\|z_k(a) - \overline{z}_k(a)\| + \overline{\gamma}\|z_k(b) - \overline{z}_k(b)\|\Big] ds \\
&\leqslant M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{p}(s) \\
&\qquad \times \Big[\overline{\lambda}\|z_k(s) - \overline{z}_k(s)\| + \overline{\beta}\|z_k(a) - \overline{z}_k(a)\| + \overline{\gamma}\|z_k(b) - \overline{z}_k(b)\|\Big] ds. \qquad (4.29)
\end{aligned}
$$

Recalling the derivations obtained in equations (3.6) and (3.7), the above inequality becomes

$$
\|y_{k+1} - \overline{y}_{k+1}\|_B \leqslant M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t) + \overline{\Theta}\|z_k - \overline{z}_k\|_B, \qquad (4.30)
$$

and similarly, it is seen that

$$\|z_k - \overline{z}_k\|_B \leqslant \xi_k\Big[M + |\mu_1 - \mu_2| I_a{}^{\alpha} r(t)\Big] + \Big[1 - \xi_k\Big(1 - \overline{\Theta}\Big)\Big]\|y_k - \overline{y}_k\|_B. \tag{4.31}$$

Therefore, using (4.31) in (4.30) and using hypothesis $\overline{\Theta} < 1$, and $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$\begin{aligned}
&\|y_{k+1} - \overline{y}_{k+1}\|_B \\
&\leqslant \Big[M + |\mu_1 - \mu_2| I_a{}^{\alpha} r(t)\Big] + \|z_k - \overline{z}_k\|_B \\
&\leqslant \Big[M + |\mu_1 - \mu_2| I_a{}^{\alpha} r(t)\Big] + \xi_k\Big[M + |\mu_1 - \mu_2| I_a{}^{\alpha} r(t)\Big] \\
&\quad + \Big[1 - \xi_k\Big(1 - \overline{\Theta}\Big)\Big]\|y_k - \overline{y}_k\|_B \\
&\leqslant 2\xi_k\Big[M + |\mu_1 - \mu_2| I_a{}^{\alpha} r(t)\Big] + \xi_k\Big[M + |\mu_1 - \mu_2| I_a{}^{\alpha} r(t)\Big] \\
&\quad + \Big[1 - \xi_k\Big(1 - \overline{\Theta}\Big)\Big]\|y_k - \overline{y}_k\|_B \\
&\leqslant \Big[1 - \xi_k\Big(1 - \overline{\Theta}\Big)\Big]\|y_k - \overline{y}_k\|_B + \xi_k\Big(1 - \overline{\Theta}\Big)\frac{3\Big[M + |\mu_1 - \mu_2| I_a{}^{\alpha} r(t)\Big]}{\Big(1 - \overline{\Theta}\Big)}. \qquad (4.32)
\end{aligned}$$

We denote

$$\begin{aligned}
\beta_k &= \|y_k - \overline{y}_k\|_B \geqslant 0, \\
\mu_k &= \xi_k\Big(1 - \overline{\Theta}\Big) \in (0, 1), \\
\gamma_k &= \frac{3\Big[M + |\mu_1 - \mu_2| I_a{}^{\alpha} r(t)\Big]}{\Big(1 - \overline{\Theta}\Big)} \geqslant 0.
\end{aligned}$$

The assumption $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (4.32) satisfies all the conditions of Lemma 2.2 and hence we have

$$\begin{aligned}
0 &\leqslant \limsup_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \gamma_k \\
\Rightarrow 0 &\leqslant \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leqslant \limsup_{k\to\infty} \frac{3\Big[M + |\mu_1 - \mu_2| I_a{}^{\alpha} r(t)\Big]}{\Big(1 - \overline{\Theta}\Big)} \\
\Rightarrow 0 &\leqslant \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leqslant \frac{3\Big[M + |\mu_1 - \mu_2| I_a{}^{\alpha} r(t)\Big]}{\Big(1 - \overline{\Theta}\Big)}. \qquad (4.33)
\end{aligned}$$

Using the assumptions $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \overline{y}_k = \overline{y}$, we get from (4.33) that

$$\|y-\overline{y}\|_B \leqslant \frac{3\Big[M+|\mu_1-\mu_2|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Theta}\Big)}, \tag{4.34}$$

which shows the dependence of solutions of the problem (1.1)-(1.2) on parameters μ_1 and μ_2. $\square$

Remark: The result deals with the property of a solution called "dependence of solutions on parameters". Here the parameters are scalars and also note that the initial conditions do not involve parameters. The dependence on parameters are an important aspect in various physical problems.

5. Example

We consider the following problem:

$$\big(D_*^{\alpha}\big)y(t) = \frac{3t}{5}\Big[\frac{t-\sin(y(t))}{2} + \frac{y(0)+y(1)}{3}\Big], \tag{5.1}$$

for $t \in [0,1],\ n-1 < \alpha \leq n\ \ (n \in \mathbb{N})$, with the given initial conditions

$$y^{(j)}(0) = c_j,\ j = 0,1,2,\cdots,n-1, \tag{5.2}$$

Comparing this equation with the equation (1.1), we get $\mathcal{F} \in C(I\times\mathbb{R}^3,\mathbb{R})$, with

$$\mathcal{F}(t,y(t),y(0),y(1)) = \frac{3t}{5}\Big[\frac{t-\sin(y(t))}{2} + \frac{y(0)+y(1)}{3}\Big].$$

Now, we have

$$\begin{aligned}
&\Big|\mathcal{F}(t,y(t),y(0),y(1)) - \mathcal{F}(t,\overline{y}(t),\overline{y}(0),\overline{y}(1))\Big| \\
&\leqslant \Big|\frac{3t}{5}\Big|\Big[\Big|\frac{t-\sin(y(t))}{2} - \frac{t-\sin(\overline{y}(t))}{2}\Big| + \Big|\frac{y(0)+y(1)}{3} - \frac{\overline{y}(0)+\overline{y}(1)}{3}\Big|\Big] \\
&\leqslant \frac{3t}{5}\Big[\frac{1}{2}\Big|\sin(y(t))-\sin(\overline{y}(t))\Big| + \frac{1}{3}\Big|y(0)-\overline{y}(0)\Big| + \frac{1}{3}\Big|y(1)-\overline{y}(1)\Big|\Big].
\end{aligned} \tag{5.3}$$

Taking sup norm, we obtain

$$|\mathcal{F}(t,y(t),y(0),y(1)) - \mathcal{F}(t,\overline{y}(t),\overline{y}(0),\overline{y}(1))| \leqslant \frac{3t}{5}\Big(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\Big)|y-\overline{y}|, \tag{5.4}$$

where $p(t) = \frac{3t}{5}$, $\lambda = \frac{1}{2}$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{3}$ and hence the condition (3.1) holds.

5.1. Existence and Uniqueness

Therefore, we the estimate

$$\begin{aligned}\Theta &= I_a{}^{\alpha} p(t)\big(\lambda+\beta+\gamma\big)\\ &= I_a{}^{\alpha}\frac{3t}{5}\Big(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\Big)\\ &= \frac{3}{5}\Big(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\Big)(I_a{}^{\alpha})(t)\\ &= \frac{7}{10}(I_a{}^{\alpha})(t)\\ &= \frac{7}{10}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\ &\leqslant \frac{7}{10\Gamma(\alpha+2)}, \quad (t\leqslant 1). \end{aligned} \tag{5.5}$$

Therefore, the condition $\Theta < 1$ is satisfied only if $\dfrac{7}{10\Gamma(\alpha+2)} < 1$.
We define the operator $T : B \to B$ by

$$\begin{aligned}(Ty)(t) &= \sum_{j=0}^{n-1}\frac{c_j}{j!}\,t^j\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{3s}{5}\Big[\frac{s-\sin(y(s))}{2}+\frac{y(0)+y(1)}{3}\Big]ds,\ t\in I.\end{aligned} \tag{5.6}$$

Since all conditions of Theorem 3.1 are satisfied and so by its conclusion, the sequence $\{y_k\}$ associated with the normal $S-$iterative method (2.2) for the operator T in (5.6) converges to a unique solution $y \in B$.

5.2. Error Estimate

Further, we also have for any $y_0 \in B$

$$\begin{aligned}\|y_{k+1}-y\|_B &\leqslant \frac{\Theta^{k+1}}{e^{\big(1-\Theta\big)\sum_{i=0}^{k}\xi_i}}\|y_0-y\|_B\\ &\leqslant \frac{\Big[\frac{7}{10\Gamma(\alpha+2)}\Big]^{k+1}}{e^{\Big[1-\frac{7}{10\Gamma(\alpha+2)}\Big]\sum_{i=0}^{k}\xi_i}}\|y_0-y\|\\ &\leqslant \frac{\Big(\frac{7}{10\Gamma(\alpha+2)}\Big)^{k+1}}{e^{\Big(1-\frac{7}{10\Gamma(\alpha+2)}\Big)\sum_{i=0}^{k}\frac{1}{1+i}}}\|y_0-y\|,\end{aligned} \tag{5.7}$$

where we have chosen $\xi_i = \frac{1}{1+i} \in [0,1]$. The estimate obtained in (5.7) is called a bound for the error (due to truncation of computation at the $k-$th iteration).

5.3. Continuous dependence

One can check easily the continuous dependence of solutions of equations (1.1) on initial data. Indeed, we have

$$\begin{aligned} \|y-\overline{y}\|_B &\leqslant \frac{3M}{\Big(1-\Theta\Big)} \\ &\leqslant \frac{3\sum_{j=0}^{n-1}\frac{\|c_j-d_j\|}{j!}\ (b-a)^j}{\Big(1-\Theta\Big)} \\ &\leqslant \frac{3\sum_{j=0}^{n-1}\frac{\|c_j-d_j\|}{j!}}{\Big(1-\frac{7}{10\Gamma(\alpha+2)}\Big)}. \end{aligned} \tag{5.8}$$

5.4. Closeness of Solutions

Next, we consider the perturbed equation:

$$ {}^cD^\alpha\overline{y}(t) = \frac{3t}{5}\Big[\frac{t-\sin(\overline{y}(t))}{2} + \frac{\overline{y}(0)+\overline{y}(1)}{3} - t + \frac{1}{7}\Big], \tag{5.9}$$

for $t \in [0,1],\ n-1 < \alpha \leq n\ \ (n \in \mathbb{N})$, with the given initial conditions

$$\overline{y}^{(j)}(0) = d_j,\ j = 0,1,2,\cdots,n-1, \tag{5.10}$$

Similarly, comparing it with the equation (4.11), we have

$$\overline{\mathcal{F}}(t,\overline{y}(t),\overline{y}(0),\overline{y}(1)) = \frac{3t}{5}\Big[\frac{t-\sin(\overline{y}(t))}{2} + \frac{\overline{y}(0)+\overline{y}(1)}{3} - t + \frac{1}{7}\Big].$$

One can easily define the mapping $\overline{T}: B \to B$ by

$$\begin{aligned} (\overline{T}\overline{y})(t) &= \sum_{j=0}^{n-1}\frac{d_j}{j!}\ t^j \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{3s}{5}\Big[\frac{s-\sin(\overline{y}(s))}{2} + \frac{\overline{y}(0)+\overline{y}(1)}{3} - s + \frac{1}{7}\Big]ds, \end{aligned} \tag{5.11}$$

for $t \in I$. In perturbed equation, all conditions of Theorem 3.1 are also satisfied and so by its conclusion, the sequence $\{\overline{y}_k\}$ associated with the normal $S-$iterative method (2.2) for the operator $\overline{T}$ in (5.11) converges to a unique solution $\overline{y} \in B$. Now, we have the following estimate:

$$\begin{aligned}
&|\mathcal{F}(t, y(t), y(0), y(1)) - \overline{\mathcal{F}}(t, y(t), y(0), y(1))| \\
&= |\frac{3t}{5}\Big[\frac{t - \sin(y(t)))}{2} + \frac{y(0)+y(1)}{3}\Big] \\
&\quad -\frac{3t}{5}\Big[\frac{t - \sin(y(t))}{2} + \frac{y(0)+y(1)}{3} - t + \frac{1}{7}\Big]| \\
&= |\frac{3t}{5}||t - \frac{1}{7}| \\
&\leqslant |t| + \frac{1}{7} \\
&\leqslant 1 + \frac{1}{7} \qquad (t \leqslant 1) \\
&= \frac{8}{7} = \epsilon. \qquad (5.12)
\end{aligned}$$

Consider the sequences $\{y_k\}_{k=0}^{\infty}$ with $y_k \to y$ as $k \to \infty$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ with $\overline{y}_k \to \overline{y}$ as $k \to \infty$ generated normal $S-$ iterative method associated with operators T in (5.6) and $\overline{T}$ in (5.11), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$ satisfying $\frac{1}{2} \leqslant \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. Then we have from Theorem 4.1 that

$$\begin{aligned}
\|y - \overline{y}\|_B &\leqslant \frac{3\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+2)}\Big]}{\Big(1 - \Theta\Big)} \\
&\leqslant \frac{3\sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!} + 3 \times \frac{8}{7}}{\Big(1 - \frac{7}{10\Gamma(\alpha+2)}\Big)} \\
&= \frac{3\sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!} + \frac{24}{7}}{\Big(1 - \frac{7}{10\Gamma(\alpha+2)}\Big)}. \qquad (5.13)
\end{aligned}$$

This shows that the closeness of solutions and dependency of solutions on functions involved therein.

5.5. Dependence on Parameters

Finally, we shall prove the dependency of solutions on real parameters.
We consider the following integral equations involving real parameters $\mu_1,\ \mu_2$:

$$ {}^cD^\alpha y(t) = \frac{3t}{5}\Big[\frac{t - \sin(y(t))}{2} + \frac{y(0)+y(1)}{3} + \mu_1\Big], \qquad (5.14)$$

and

$$ {}^cD^\alpha \overline{y}(t) = \frac{3t}{5}\Big[\frac{t - \sin(\overline{y}(t))}{2} + \frac{\overline{y}(0)+\overline{y}(1)}{3} + \mu_2\Big], \qquad (5.15)$$

for $t \in [0,1]$, $n-1 < \alpha \leq n$ $(n \in \mathbb{N})$. Based on the above discussion, one can observe that $p(t) = \overline{p}(t) = r(t) = \frac{3t}{5}$ and therefore, we have $\Theta = \overline{\Theta}$. Hence by making similar arguments and from Theorem 4.3, one can has

$$\begin{aligned}
\|y-\overline{y}\|_B &\leqslant \frac{3\Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big]}{\Big(1-\overline{\Theta}\Big)} \\
&\leqslant \frac{3\Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big]}{\Big(1-\frac{7}{10\Gamma(\alpha+2)}\Big)} \\
&\leqslant \frac{3\Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} (\frac{3t}{5})\Big]}{\Big(1-\frac{7}{10\Gamma(\alpha+2)}\Big)} \\
&\leqslant \frac{3\Big[M + |\mu_1-\mu_2| \frac{3}{5}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\Big]}{\Big(1-\frac{7}{10\Gamma(\alpha+2)}\Big)} \\
&\leqslant \frac{3\Big[\sum_{j=0}^{n-1}\frac{\|c_j-d_j\|}{j!} + \frac{3}{5}|\mu_1-\mu_2|\frac{1}{\Gamma(\alpha+2)}\Big]}{\Big(1-\frac{7}{10\Gamma(\alpha+2)}\Big)}. \qquad (5.16)
\end{aligned}$$

In particular, we choose $\alpha = \frac{5}{2}$, then we have $n = [\alpha]+1 = [\frac{5}{2}]+1 = 2+1 = 3$ and

$$\begin{aligned}
\frac{7}{10\Gamma(\alpha+2)} &= \frac{7}{10\Gamma(\frac{5}{2}+2)} \\
&= \frac{7}{10\Gamma(\frac{9}{2})} \\
&= \frac{7}{10 \times \frac{105}{16}\sqrt{\pi}} \\
&= \frac{7 \times 16}{10 \times 105 \times \sqrt{\pi}} \\
&= \frac{112}{1050 \times \sqrt{\pi}} \\
&= \frac{562}{525 \times \sqrt{\pi}} \\
&\simeq 0.0602 \\
&< 1.
\end{aligned}$$

This proves that the T is a contraction with contractivity factor $\frac{56}{150\times\sqrt{\pi}}$. Using this factor and putting particular values of c_j, d_j, $j = 0,1,2$, the estimates in (5.7), (5.8), (5.13) and (5.16) can be simplified further.

6. Conclusions

Using the S-iterative approach, we discussed the first main result, which deals with the existence and uniqueness of the solution to the IVP (1.1)-(1.2). Next, we discussed various properties of solutions like continuous dependence on the initial data, closeness of solutions, and dependence on parameters and functions involve therein. Finally, we provided an appropriate example to support all of the findings.

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