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CERTAIN RESULTS ON η-RICCI SOLITIONS AND ALMOST η-RICCI SOLITONS

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Abstract. We prove that if an η-Einstein para-Kenmotsu manifold admits an η-Ricci soliton then it is Einstein. Next, we proved that a para-Kenmotsu metric as an η-Ricci soliton is Einstein if its potential vector field V is infinitesimal paracontact transformation or collinear with the Reeb vector field. Further, we prove that if a para-Kenmotsu manifold admits a gradient almost η-Ricci soliton and the Reeb vector field leaves the scalar curvature invariant then it is Einstein. We have also constructed an example of para-Kenmotsu manifold that admits η-Ricci soliton and satisfy our results. We also have studied η-Ricci soliton in 3-dimensional normal almost paracontact metric manifolds and we show that if in a 3-dimensional normal almost paracontact metric manifold with $\alpha, \beta =$ constant, the metric is η-Ricci soliton, where potential vector field V is collinear with the characteristic vector field ξ, then the manifold is η-Einstein manifold.
Keywords: η-Ricci Soliton, para-Kenmotsu manifold, Einstein manifold, infinitesimal paracontact transformation, normal almost paracontact metric manifold

1. Introduction

In recent years, geometric flows, in particular, the Ricci flow have been an interesting research topic in differential geometry. The concept of Ricci flow was first introduced by Hamilton and developed to answer Thurston's geometric conjecture. A Ricci soliton can be considered as a fixed point of Hamilton's Ricci flow (see details

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[18]) and a natural generalization of the Einstein metric (i.e., the Ricci tensor Ric is a constant multiple of the pseudo-Riemannian metric g), defined on a pseudo-Riemannian manifold (M, g) by

$$\frac{1}{2}\pounds_V g + Ric = \lambda g,$$

where $\pounds_V$ denotes the Lie-derivative in the direction of $V \in \chi(M)$, Ric is the Ricci tensor of g and λ is a constant. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as λ is negative, zero, and positive respectively. Otherwise, it will be called indefinite. A Ricci soliton is trivial if V is either zero or Killing on M. First, Pigoli et al. [26] assume the soliton constant λ to be a smooth function on M and named as Ricci almost soliton. After that, Barros et al. studied almost Ricci soliton detailed in [1, 2]. Recently, Cho-Kimura [3] generalized the notion of Ricci soliton to η-Ricci soliton and Calin-Crasmareanu [7] studied this in Hopf hypersurfaces of complex space forms. A Riemannian or pseudo-Riemannian metric g, defined on a smooth manifold M^n of dimension n is said to be an η-Ricci soliton if there exists a vector field V and constants λ, μ such that

$$\frac{1}{2}\pounds_V g + Ric + \lambda g + \mu\eta \otimes \eta = 0. \tag{1.1}$$

If $\lambda, \mu : M \to \mathbb{R}$ are smooth functions, then (M, g) is called an almost η-Ricci soliton. If the potential vector field V is a gradient of a smooth function f on M i.e., $V = \nabla f$, then the manifold is called an almost gradient η-Ricci soliton. In this case the equation (1.1) can be exibited as

$$Hess\ f + Ric + \lambda g + \mu\eta \otimes \eta = 0, \tag{1.2}$$

where $Hess\ f$ denotes the Hessian of f. The function f is known as the potential function.

In the literature, many authors studied Ricci soliton and η-Ricci soliton in the framework of contact metric manifolds. For instance, Sharma [35] consider a K-contact and (κ, μ)contact metric as Ricci soliton; Cho-Sharma consider a contact metric as Ricci soliton [10]; η-Einstein almost Kenmotsu metric as Ricci soliton by Wang-Liu [36]. Further, Ghosh consider a non-compact almost contact metric, in particular, a Kenmotsu metric as Ricci soliton (see [15, 17]). The interest in Ricci solitons and η-Ricci solitons have risen among theoretical physicists in relation with String Theory and connection to general relativity and therefore these have been extensively studied in pseudo-Riemannian settings (see [9, 23]). So, several authors studied Ricci soliton and η-Ricci soliton on paracontact metric manifolds, for instance, Patra [23] consider a paracontact metric as a Ricci soliton and Naik et al. [21] consider a para-Sasakian metric as η-Ricci soliton. In [38], Welyczko introduced notion of para-Kenmotsu manifold, which is the analogous of Kenmotsu manifold [19] in paracontact geometry and detailed studied by Zamkovoy [42]. Further, Blaga studied some aspects of η-Ricci solitons on para-Kenmotsu and Lorentzian para-Sasakian manifolds (see [4, 5, 6]). Recently, Patra [23] consider Ricci soliton on para-Kenmotsu manifold and proved that a para-Kenmotsu metric as a Ricci soliton is Einstein if it is η-Einstein or the potential vector field V is infinitesimal paracontact transformation. Further, η-Ricci soliton and its generalizations have been studied on contact and paracontact geometry by (see[11, 14, 27, 28, 29, 30, 31, 32, 33, 34]) Motivated by these results we consider a para-Kenmotsu metric as η-Ricci solitons and η-Ricci almost solitons.

This paper is organized as follows. After collecting some the basic definitions and formulas on para-Kenmotsu manifolds in section 2, we prove in section 3 that, a para-Kenmotsu metric as an η-Ricci soliton is Einstein if it is η-Einstein or the potential vector field V is infinitesimal paracontact transformation or V is collinear with the Reeb vector field ξ. In section 4, we consider an almost η-Ricci soliton on para-Kenmotsu manifolds and find some η-Einstein and Einstein manifolds using almost η-Ricci solitons. We draw an example of a para-Kenmotsu manifold that admits η-Ricci soliton. Section 5 deals with some properties of 3-dimensional normal almost paracontact metric manifolds. In section 6, we prove that if a 3-dimensional non-paracosymplectic normal almost paracontact metric manifold with α, β =constant admits an η-Ricci soliton and V is pointwise collinear with the structure vector field ξ, then V is a constant multiple of ξ, the manifold is an η-Einstein manifold and the η-Ricci soliton is shrinking, expanding and steady according to $\mu \geq 0$, $\mu < -2(\alpha^2 + \beta^2)$ and $2(\alpha^2 + \beta^2) + \mu = 0$. We have proved that the converse is also true.

2. Preliminaries

In this section, we give a brief review of several fundamental notions and formulas which we will need later on. We refer to [8, 9, 41, 42] for more details as well as some examples. A $(2n+1)$-dimensional smooth manifold M^{2n+1} has an almost paracontact structure (φ, ξ, η) if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying the following conditions:

(2.1) (i) $\varphi^2 = I - \eta \circ \xi, \ \ \varphi(\xi) = 0, \ \ \eta \circ \varphi = 0, \ \ \eta(\xi) = 1,$

(ii) there exists a distribution $\mathcal{D} : p \in M \to \mathcal{D}_p \subset T_pM : \mathcal{D}_p = Ker(\eta) = \{x \in T_pM : \eta(x) = 0\}$, called paracontact distribution generated by η. If an almost paracontact manifold M^{2n+1} with an structure (φ, ξ, η) admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y)$$

for all X, $Y \in \chi(M)$, then we say that M has an almost paracontact metric structure and g is called compatible metric. The fundamental 2-form Φ of an almost paracontact metric structure (φ, ξ, η, g) defined by $\Phi(X, Y) = g(X, \varphi Y)$ for all vector fields X, Y on M. If $\Phi = d\eta$, then the manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is called a paracontact metric manifold. In this case, η is a contact form, i.e., $\eta\wedge(d\eta)^n \neq 0$, ξ is its Reeb vector field and M is a contact manifold (see [24]). An almost paracontact metric manifold is said to be para-Kenmotsu manifold (see [42]) if

$$(\nabla_X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{2.2}$$

for any X, $Y \in \chi(M)$. On para-Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, the following formulas hold [42]:

$$\nabla_X\xi = X - \eta(X)\xi, \tag{2.3}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.4}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.5}$$

$$Q\xi = -2n\xi \tag{2.6}$$

for all X, $Y \in \chi(M)$, where ∇, R and Q denotes the Riemannian connection, the curvature tensor and the Ricci operator of g associated with the Ricci tensor given by $Ric(X, Y) = g(QX, Y)$ for all X, $Y \in \chi(M)$ respectively. Now, we have the following lemma on para-Kenmotsu manifold.

Lemma 2.1. *[33] On para-Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ the following formulas hold for any X, $Y \in \chi(M)$,*

$$(\nabla_X Q)\xi = -QX - 2nX, \tag{2.7}$$

$$(\nabla_\xi Q)X = -2QX - 4nX. \tag{2.8}$$

3. On η-Ricci soliton

In this section, we study the η-Ricci solitons on para-Kenmotsu manifolds and find some important conditions so that a para-Kenmotsu metric as an η-Ricci soliton is Einstein. First we recall a definition: a contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be η-Einstein, if the Ricci tensor Ric can be written as

$$Ric(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y), \tag{3.1}$$

where α, β are a smooth functions on M. For an η-Einstein K-contact manifold (see Yano and Kon [40]) and para-Sasakian manifold [41] of dimension > 3, it is well known that the functions α, β are constants, but for an η-Einstein para-Kenmotsu manifold this is not true. So, we continue α, β as functions. In [15], Ghosh studied of 3-dimensional Kenmotsu metric as a Ricci soliton and for higher dimension in [17]. Recently, Patra [23] consider Ricci soliton on para-Kenmotsu manifolds and proved that an η-Einstein para-Kenmotsu metric as a Ricci soliton is Einstein and therefore here we consider η-Einstein para-Kenmotsu metric as an η-Ricci soliton. Before enter our main results first we derive the following lemma.

Lemma 3.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If g represents an η-Ricci soliton with potential vector field V then for any $X \in \chi(M)$, we have*

$$(\pounds_V R)(X, \xi)\xi = 0. \tag{3.2}$$

Proof: Taking the covariant derivative of (1.1) along an arbitrary vector field Z on M, we get

$$(\nabla_Z \pounds_V g)(X,Y) = -2(\nabla_Z Ric)(X,Y) - 2\mu\{g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\} \tag{3.3}$$

for any $X, Y \in \chi(M)$. Next, recalling the following commutation formula (see Yano [39], p.23)

$$(\pounds_V \nabla_Z g - \nabla_Z \pounds_V g - \nabla_{[V,Z]} g)(X,Y) = -g((\pounds_V \nabla)(Z,X),Y) - g((\pounds_V \nabla)(Z,Y),X), \tag{3.4}$$

for all X, Y, $Z \in \chi(M)$. In view of the parallel Riemannian metric g, it follows that

$$(\nabla_Z \pounds_V g)(X,Y) = g((\pounds_V \nabla)(Z,X),Y) + g((\pounds_V \nabla)(Z,Y),X), \tag{3.5}$$

for all X, Y, $Z \in \chi(M)$. Plugging it into (3.3) we get

$$g((\pounds_V \nabla)(Z,X),Y) + g((\pounds_V \nabla)(Z,Y),X) = -2(\nabla_Z Ric)(X,Y) -2\mu\{g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}, \tag{3.6}$$

for any X, Y, $Z \in \chi(M)$. Interchanging cyclicly the roles of X, Y, Z in (3.6) we can compute

$$g((\pounds_V \nabla)(X,Y),Z) = (\nabla_Z Ric)(X,Y) - (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(Z,X) - 2\mu\{g(X,Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\}, \tag{3.7}$$

for all Y, $Z \in \chi(M)$. Now, substituting ξ for Y in (3.7) and using (2.8), (2.7) yields gives

$$(\pounds_V \nabla)(X,\xi) = 2QX + 4nX, \tag{3.8}$$

for all $X \in \chi(M)$. Next, using (2.3), (3.8) in the covariant derivative of (3.8) along Y gives

$$(\nabla_Y \pounds_V \nabla)(X,\xi) = (\pounds_V \nabla)(X,Y) + 2(\nabla_Y Q)X + 2\eta(Y)(QX + 2nX),$$

for any $X \in \chi(M)$. Making use of this in the following commutation formula (see Yano [39], p.23)

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z), \tag{3.9}$$

we can derive

$$(\pounds_V R)(X,Y)\xi = 2\{(\nabla_Y Q)X - (\nabla_X Q)Y\} + 2\{\eta(X)QY - \eta(Y)QX\} + 4n\{\eta(X)Y - \eta(Y)X\}, \tag{3.10}$$

for all vector fields $X, Y \in \chi(M)$. Substituting Y by ξ in (3.10) and using (2.6), (2.8) and (2.7) we have the requir result. □

Theorem 3.1. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$, $n>1$, be a η-Einstein para-Kenmotsu manifold. If g represents an η-Ricci soliton with potential vector field V, then g is Einstein with constant scalar curvature $r=-2n(2n+1)$.*

Proof: First, tracing (3.1) gives $r=(2n+1)\alpha+\beta$ and putting $X=Y=\xi$ in (3.1) and using (2.6) we get $\alpha+\beta=-2n$. Therefore, by computation, (3.1) transform into

$$Ric(X,Y)=(1+\frac{r}{2n})g(X,Y)-\{(2n+1)+\frac{r}{2n}\}\eta(X)\eta(Y), \tag{3.11}$$

for all X, Y on M. This gives

$$\begin{aligned}(\nabla_Y Q)X &= \frac{(Yr)}{2n}\{X-\eta(X)\xi\}\\ &+ \{(2n+1)+\frac{r}{2n}\}\{g(X,Y))\xi+\eta(X)(Y-2\eta(Y)\xi)\},\end{aligned} \tag{3.12}$$

for all X, $Y\in\chi(M)$. By virtue of this, (3.10) provides

$$(\pounds_V R)(X,Y)\xi=\frac{1}{n}\{(Xr)(Y-\eta(Y)\xi)-(Yr)(X-\eta(X)\xi)\}, \tag{3.13}$$

for all X, $Y\in\chi(M)$. Setting $Y=\xi$ in (3.13) and using the Lemma 3.1 we get $(\xi r)\varphi^2X=0$ for any $X\in\chi(M)$. Using this in the trace of (2.8) we get $r=-2n(2n+1)$. It follows from (3.11) that M is Einstein and hence the proof. □

Remark 3.1. In [23], Patra proved the Theorem 3.1 for Ricci soliton on para-Kenmotsu manifold. Here using different technique we have proved the Theorem 3.1 in very short and direct way for η-Ricci soliton on para-Kenmotsu manifold. As η-Ricci soliton is the generalization of Ricci soliton, so we can easily derive the above result for Ricci soliton using this technique also.

Now, taking the Lie-derivative of $g(\xi,\xi)=1$ along the potential vector field V and applying (1.1) one can obtain

$$\eta(\pounds_V\xi)=\lambda+\mu-2n. \tag{3.14}$$

Further, from (2.3) we get $R(X,\xi)\xi=-X+\eta(X)\xi$ and the Lie derivative of this along V yields

$$(\pounds_V R)(X,\xi)\xi+R(X,\pounds_V\xi)\xi+R(X,\xi)\pounds_V\xi=\{(\pounds_V\eta)X\}\xi+\eta(X)\pounds_V\xi, \tag{3.15}$$

for any $X\in\chi(M)$. If g represents an η-Ricci soliton with potential vector field V then the Lemma 3.1 holds, i.e., $(\pounds_V R)(X,\xi)\xi=0$. Plugging it into (3.15) and using (2.4) provides

$$(\pounds_V g)(X,\xi)+2\eta(\pounds_V\xi)X=0, \tag{3.16}$$

for any $X \in \chi(M)$. Again, applying (1.1) and (3.14) in (3.16) yields $(2n - \lambda - \mu)\varphi^2 X = 0$ for any $X \in \chi(M)$. Next using (2.1) and then tracing yields $2n(2n - \lambda - \mu) = 0$. This implies that

$$\lambda + \mu = 2n. \tag{3.17}$$

Next we consider a para-Kenmotsu metric as an η-Ricci soliton with non-zero potential vector field V is collinear with ξ and prove the following result.

Theorem 3.2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If g represents an η-Ricci soliton with non-zero potential vector field V is collinear with ξ, then g is Einstein with constant scalar curvature $r = -2n(2n+1)$.*

Proof: Since the potential vector field V is collinear with ξ, i.e., $V = \nu\xi$ for some smooth function ν on M. Making use of (2.3) in the covariant derivative of $V = \nu\xi$ along X yields

$$\nabla_X V = (X\nu)\xi + \nu\{(X - \eta(X)\xi)\}$$

for any $X \in \chi(M)$. By virtue of this, the soliton equation (1.1) reduces to

$$2Ric(X,Y) + (X\nu)\eta(Y) + (Y\nu)\eta(X) + 2(\lambda+\nu)g(X,Y) - 2(\nu-\mu)\eta(X)\eta(Y) = 0 \tag{3.18}$$

for all X, $Y \in \chi(M)$. Setting $X = Y = \xi$ in (3.18) and using (2.6), (3.17) we get $\xi\nu = 0$. It follows from (3.18) that $X\nu = 0$. Putting it into (3.18) provides

$$Ric(X,Y) = -(\nu+\lambda)g(X,Y) + (\nu-\mu)\eta(X)\eta(Y) \tag{3.19}$$

for all X, $Y \in \chi(M)$. This shows that M is an η-Einstein and therefore from Theorem 3.1 we conclude that M is Einstein. Thus, from (3.18) we have $\nu = \mu$ and therefore $\nu + \lambda = 2n$ (follows from (3.17)). Hence we have from (3.19) that $Ric = -2ng$ and therefore $r = -2n(2n+1)$, as required. Hence the proof. □

Remark 3.2. In the Theorem 3.2 we see that $X\nu = 0$ for any $X \in \chi(M)$ and therefore the smooth function ν reduces to a constant and it equal to μ and hence $V = \mu\xi$.

In particular, we can also say that if a para-Kenmotsu manifold admits an η-Ricci soliton with the non-zero potential vector field V is ξ, then it is Einstein with constant scalar curvature $r = -2n(2n+1)$.

On paracontact metric manifold M, a vector field X is said to be infinitesimal paracontact transformation if it preserve the paracontact form η, i.e., there exists a smooth function ρ on M satisfies

$$\pounds_X \eta(Y) = \rho\eta(Y), \tag{3.20}$$

for any $Y \in \chi(M)$ and if $\rho = 0$ then X is said to be strict. Here we consider a para-Kenmotsu metric as a η-Ricci soliton with potential vector field V is infinitesimal paracontact transformation and prove the following.

Theorem 3.3. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$, $n>1$, be a para-Kenmotsu manifold. If g represents an η-Ricci soliton with the potential vector field V is infinitesimal paracontact transformation, then V is strict and g is Einstein with constant scalar curvature $r=-2n(2n+1)$.*

Proof: First, recalling the well known formula (see page no. 23 of [39]):

$$\pounds_V\nabla_X Y-\nabla_X\pounds_V Y-\nabla_{[V,X]}Y=(\pounds_V\nabla)(X,Y), \tag{3.21}$$

for all $X,\ Y\in\chi(M)$. Making use of (3.15), (3.20), Lemma 3.1 in the Lie-derivative of $\eta(X)=g(X,\xi)$ along V we get $\pounds_V\xi=\rho\xi$. Thus, equations (3.14) and (3.17) entails that $\rho=0$ and therefore $\pounds_V\xi=0$ and V is strict. Further equation (3.20) gives $\pounds_V\eta=0$. So, we get that $(\pounds_V\nabla)(X,\xi)=0$ for any $X\in\chi(M)$. Thus, from (3.8) we conclude the rest part of this theorem. Hence the proof. □

4. On almost η-Ricci soliton

In this section, we consider an almost η-Ricci soliton on para-Kenmotsu manifold. It follows from (1.2) that η-Ricci almost soliton is the generalization of Ricci almost soliton because it involve two smooth functions λ and μ. First, we consider an almost gradiant η-Ricci soliton on para-Kenmotsu manifold in order to extend the result of almost gradiant Ricci soliton on para-Kenmotsu manifold [23] taking into account of equations (1.1) and (1.2) hold for smooth functions λ, μ. First we prove the following.

Theorem 4.1. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a para-Kenmotsu manifold. If M admits an almost gradient η-Ricci soliton and the Reeb vector field ξ leaves the scalar curvature r invariant, then it is Einstein with constant scalar curvature $-2n(2n+1)$.*

Proof: Equation (1.2) can be exhibited as

$$\nabla_X Df+QX+\lambda X+\mu\eta(X)\xi=0 \tag{4.1}$$

for any $X\in\chi(M)$. Using this in $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$ we can easily obtain the curvature tensor expression in the follwing form

$$\begin{aligned} R(X,Y)Df &= (\nabla_Y Q)X-(\nabla_X Q)Y+(Y\lambda)X-(X\lambda)Y\\ &+ (Y\mu)\eta(X)\xi-(X\mu)\eta(Y)\xi+\mu\{\eta(Y)X-\eta(X)Y\} \end{aligned} \tag{4.2}$$

for all $X,\ Y\in\chi(M)$. Taking contraction of (4.2) over X with respect to an orthonormal basis $\{e_i: i=1,2,...,2n+1\}$, we compute

$$Ric(Y,Df)=-\sum_{i=1}^{2n+1}g((\nabla_{e_i}Q)Y,e_i)+(Yr)+2n(Y\lambda)+(Y\mu)+\{(-\xi\mu)+2n\mu\}\eta(Y)$$

for any $Y \in \chi(M)$. Now, contracting Bianchi's second identity we have

$$\sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)Y, e_i) = \frac{1}{2}(Yr).$$

Plugging it into the previous equation gives

$$Ric(Y, Df) = \frac{1}{2}(Yr) + 2n(Y\lambda) + (Y\mu) + \{(-\xi\mu) + 2n\mu\}\eta(Y), \tag{4.3}$$

for any $Y \in \chi(M)$. Now, substituting ξ for Y in (4.2) and using Lemma 2.1 provides

$$R(X, \xi)Df = -QX - 2nX + (\xi\lambda)X - (X\lambda)\xi - (X\mu)\xi + (\xi\mu)\eta(X)\xi + \mu\varphi^2 X \tag{4.4}$$

for any $X \in \chi(M)$. Next, taking inner product of (4.4) with ξ and using (2.6) we obtain $g(R(X, \xi)Df, \xi) = \xi(\lambda + \mu)\eta(X) - X(\lambda + \mu)$ for any $X \in \chi(M)$. By virtue of (2.5), the preceeding equation reduces to

$$X(f + \lambda + \mu) = \xi(f + \lambda + \mu)\eta(X), \tag{4.5}$$

for any $X \in \chi(M)$. Further, using (2.5) in (4.4) we obtain

$$X(f + \lambda + \mu)\xi = -QX + \{\xi(f + \lambda) + \mu - 2n\}X + \{-\mu + (\xi\mu)\}\eta(X)\xi \tag{4.6}$$

for any $X \in \chi(M)$. By virtue of (4.5), equation (4.6) reduces to

$$QX = \{-\mu + \xi(\lambda + f) - 2n\}X - \{\mu + \xi(\lambda + f)\}\eta(X)\xi, \tag{4.7}$$

for any $X \in \chi(M)$. Thus, M is η-Einstein. Substituting Y by Df in (2.5) and contracting we get $Ric(\xi, Df) = -2n(\xi f)$. Plugging it into (4.3) yields $(\xi r) + 4n\xi(f + \lambda) = -4n\mu$. Using it in the trace of (2.8) we have $\xi(\lambda + f) = -\mu + \{2n + 1 + \frac{r}{2n}\}$. By virtue of this, equation (4.7) transform into

$$QX = (1 + \tfrac{r}{2n})X - \{(2n + 1) + \tfrac{r}{2n}\}\eta(X)\xi, \tag{4.8}$$

for any X on $\in \chi(M)$. By hypothesis: $\xi r = 0$ and therefore, the trace of (2.8) gives $r = -2n(2n + 1)$. It follows from (4.8) that $QX = -2nX$, as required. So, we complete the proof. □

Note that, Theorem 4.1 is more general version where λ, μ are smooth functions on M and therefore it also holds for gradient η-Ricci soliton, where λ, μ are constants.

Corollary 4.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If M admits g represents a gradient η-Ricci soliton and the Reeb vector field ξ leaves the scalar curvature r invariant, then M is Einstein with constant scalar curvature $-2n(2n + 1)$.*

Next, considering a para-Kenmotsu metric as an almost η-Ricci soliton with the potential vector field V is pointwise collinear with the Reeb vector field ξ, we extend the Theorem 4.1 from almost gradient η-Ricci soliton to η-Ricci almost soliton and prove the following.

Theorem 4.2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If M admits an almost η-Ricci soliton with non-zero potential vector field V collinear with ξ, then g is η-Einstein. Moreover, if the Reeb vector field ξ leaves the scalar curvature r invariant, then g is Einstein with constant scalar curvature $-2n(2n+1)$.*

Proof: By hypothesis: $V = \sigma\xi$ for some smooth function σ on M. It follows that

$$(\pounds_V g)(X,Y) = (X\sigma)\eta(Y) + (Y\sigma)\eta(X) + 2\sigma\{g(X,Y) - \eta(X)\eta(Y)\}$$

for all X, $Y \in \chi(M)$. By virtue of this, the soliton equation (1.1) transform into

(4.9)
$$2Ric(X,Y) + (X\sigma)\eta(Y) + (Y\sigma)\eta(X) + 2(\sigma+\lambda)g(X,Y) = 2(\sigma-\mu)\eta(X)\eta(Y)$$

for all X, $Y \in \chi(M)$. Now, putting $X = Y = \xi$ in (4.9) and using (2.6) yields $\xi\sigma = 2n - \lambda - \mu$. Thus, Eq. (4.9) gives $X\sigma = (2n - \lambda - \sigma)\eta(X)$. Making use of this in (4.9) entails that

$$Ric(X,Y) = -(\sigma+\lambda)g(X,Y) - (2n - \lambda - \sigma)\eta(X)\eta(Y) \tag{4.10}$$

for all X, $Y \in \chi(M)$. Hence M is η-Einstein. Moreover, if the Reeb vector field ξ leaves the scalar curvature r invariant i.e., $\xi r = 0$. Now, tracing (2.7) yields $(\xi r) = -2\{r + 2n(2n+1)\}$ and therefore, $r = -2n(2n+1)$. Using this in the trace of (4.10) gives $\lambda - \sigma = 2n$. Thus, from (4.10) we have $QX = -2nX$ and therefore M is Einstein. This complete the proof. □

If we consider $V = \sigma\xi$ for some constant σ instead of a function, then (4.9) holds good and therefore insearting $X = Y = \xi$ in (4.9) and using (2.6) gives $\xi\sigma = 2n - \lambda - \mu$. Using this in (4.9) yields $\lambda - \sigma = 2n$, where we have used σ is a constant. Thus, from (4.10) we can conclude the following corollary.

Corollary 4.2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If M admits a non-trivial almost η-Ricci soliton with $V = \sigma\xi$ for some constant σ, then it is Einstein with constant scalar curvature $r = -2n(2n+1)$.*

Now, we present an example of para-Kenmotsu manifold that admits a gradient η-Ricci soliton.

Example Let (x, y, z) be the standard coordinates in $\mathbb{R}^3$ and $M^3 = \{(x, y, z) \in \mathbb{R}^3\}$ be a 3-dimensional manifold. Now, consider an orthonormal basis $\{e_1, e_2, e_3\}$

of vector fields on M^3, where $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Define $(1,1)$ tensor field φ as follows:

$$\varphi(e_2) = e_1, \quad \varphi(e_1) = e_2, \quad \varphi(e_3) = 0.$$

The pseudo-Riemannian metric is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\eta(X) = g(X, e_5)$ for any $X \in \chi(M)$. Then $\eta(e_3) = 1$, $\varphi^2 = X - \eta(X)e_3$, and $g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y)$ for all X, $Y \in \chi(M)$. Thus, (φ, ξ, η, g) is an almost paracontact structure. The non-zero components of the Levi-Civita connection ∇ (using Koszul's formula) are

$$\nabla_{e_1} e_3 = e_1 \quad \nabla_{e_1} e_1 = -\nabla_{e_2} e_2 = -e_3 \quad \nabla_{e_2} e_3 = e_2. \tag{4.11}$$

By virtue of this we can verify (2.2) and therefore $M^5(\varphi, \xi, \eta, g)$ is a para-Kenmotsu manifold. Using the well knownn expression of curvatute tensor $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we now compute the following non-zero components

$$\begin{array}{ccc} R(e_1,e_2)e_1 = e_2 & R(e_1,e_2)e_2 = e_1 & R(e_1,e_3)e_1 = e_3 \\ R(e_1,e_3)e_3 = -e_1 & R(e_2,e_3)e_2 = -e_3 & R(e_2,e_3)e_3 = -e_2. \end{array}$$

Using these, we compute the components of the Ricci tensor

$$Ric(e_i, e_i) = -2 \ \text{ for } i = 1,3 \qquad Ric(e_2, e_2) = 2.$$

Therefore the Ricci tensor is given by $Ric = -2g$ and the scalar curvature $r = -6$. Hence (M^3, g) is Einstein with conatant scalar curvature $r = -2n(2n+1)$ for $n = 1$. Let us consider a potential vector field $V = (x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ on M^3. Then using (4.11) we obtain

$$\tfrac{1}{2}(\pounds_V g)(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\} \tag{4.12}$$

for all X, $Y \in \chi(M^3)$. If we choose the potential function $f(x,y,z) = \frac{(x-1)^2}{2} + \frac{(y-1)^2}{2} + z$ then from $Ric = -2g$ and (4.12) we can conclude that the metric g is a gradient η-Ricci soliton with constants $\lambda = 1$ and $\mu = 1$.

5. On 3-dimensional normal almost paracontact metric manifolds

An almost paracontact metric manifold is said to be normal if [20]

$$N(X,Y) - 2d\eta(X,Y)\xi = 0, \tag{5.1}$$

where N is the Nijenhuis torsion tensor of φ given by

$$N(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] \tag{5.2}$$

for all $X, Y \in \chi(M)$. The normality condition tells that the almost paracomplex structure J on $M^{2n+1} \times R$ is defined by [20]

$$J(X, f\frac{d}{dt}) = (\varphi X + f\xi, \eta(X)\frac{d}{dt}) \tag{5.3}$$

is integrable (paracomplex). The following Proposition presents conditions equivalent to the normality of 3-dimensional almost paracontact metric manifold which we will use in later.

Proposition 5.1. *[37] For a 3-dimensional almost paracontact metric manifold M, the following three conditions are mutually equivalent*
(i) M is normal,
(ii) there exists functions α, β on M such that

$$(\nabla_X\varphi)Y = \beta(g(X, Y)\xi - \eta(Y)X) - \alpha(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{5.4}$$

(iii) there exist functions α, β on M such that

$$\nabla_X\xi = \alpha(X - \eta(X)\xi) + \beta\varphi X. \tag{5.5}$$

Proposition 5.2. *[25] For a 3-dimensional normal almost paracontact metric manifold with $\alpha, \beta = constant$, we have*

$$S(X, Y) = -(\frac{r}{2} + \alpha^2 + \beta^2)g(\varphi X, \varphi Y) + 2(\alpha^2 + \beta^2)\eta(X)\eta(Y), \tag{5.6}$$

$$QX = (r^2 + \alpha^2 + \beta^2)X + (-\frac{r}{2} + \alpha^2 + \beta^2)\eta(X)\xi \tag{5.7}$$

for all $X, Y \in \chi(M)$.

A 3-dimensional normal almost paracontact metric manifold is said to be
• paracosymplectic [12] if $\alpha = \beta = 0$,
• quasi-para-Sasakian [13, 37] if and only if $\alpha = 0$ and $\beta \neq 0$,
• β-para-Sasakian [37, 41] if and only if $\alpha = 0$ and $\beta \neq 0$, and β is constant, in particular, para-Sasakian if $\beta = -1$,
• α-para-Kenmotsu [38] if $\alpha \neq 0$ and α is constant and $\beta = 0$.

6. η-Ricci soliton on 3-dimensional normal almost paracontact metric manifolds

We consider 3-dimensional normal almost paracontact metric manifold admitting η-Ricci soliton which is defined by (1.1). Let V be a pointwise collinear vector field with structure field ξ, i.e., $V = b\xi$, where b is a function on M. Now, from (1.1), we get

$$g(\nabla_X b\xi, Y) + g(X, \nabla_Y b\xi) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0 \tag{6.1}$$

for all $X, Y \in \chi(M)$. Then we have

$$\begin{aligned}(Xb)\eta(Y) &+ bg(\nabla_X\xi, Y) + (Yb)\eta(X) + bg(X, \nabla_Y\xi) + 2S(X,Y)\\ &+ 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.\end{aligned} \tag{6.2}$$

Now using the equation (5.4) in (6.2), we achieve

$$\begin{aligned}(Xb)\eta(Y) &+ 2\alpha bg(X,Y) + (Yb)\eta(X) - 2\alpha b\eta(X)\eta(Y) + 2S(X,Y)\\ &+ 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.\end{aligned} \tag{6.3}$$

Inserting $Y = \xi$ in (6.3) and also using (5.6), we obtain

$$Xb + (\xi b)\eta(X) + 4(\alpha^2 + \beta^2)\eta(X) + 2\lambda\eta(X) + 2\mu\eta(X) = 0 \tag{6.4}$$

Now putting $X = \xi$ in (6.4), we acquire

$$\xi b = -(2(\alpha^2 + \beta^2) + \lambda + \mu). \tag{6.5}$$

We replace (6.5) in (6.4) to infer

$$Xb = -(2(\alpha^2 + \beta^2) + \lambda + \mu)\eta(X), \tag{6.6}$$

which yields

$$db = -(2(\alpha^2 + \beta^2) + \lambda + \mu)\eta. \tag{6.7}$$

Applying d on both sides on (6.7), the preceding equation provides

$$(2(\alpha^2 + \beta^2) + \lambda + \mu)d\eta = 0. \tag{6.8}$$

Now as $d\eta \neq 0$, we get

$$2(\alpha^2 + \beta^2) = -(\lambda + \mu), \tag{6.9}$$

which implies by using (6.7),

$$db = 0 \Rightarrow b \;\; is \;\; constant.$$

Now, as b is constant, so from (6.3), we derive

$$S(X,Y) = -(\lambda + \alpha b)g(X,Y) + (\alpha b - \mu)\eta(X)\eta(Y) \tag{6.10}$$

for all $X, Y \in \chi(M)$. Thus we have the following theorem

Theorem 6.1. *Let M be a 3-dimensional non-paracosymplectic normal almost paracontact metric manifold with $\alpha, \beta = constant$. If M admits an η-Ricci soliton and V is pointwise collinear with the structure vector field ξ, then V is a constant multiple of ξ and M is an η-Einstein manifold.*

Let we discuss the converse of the above theorem, that is, let M be a 3-dimensional η-Einstein normal almost paracontact metric manifold with $\alpha, \beta = constant$ and $V = \xi$. Then we get,

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{6.11}$$

where a, b are scalars and $X, Y \in \chi(M)$. From (5.5), we have

$$\begin{aligned} (\pounds_\xi g)(X,Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= 2\alpha g(X,Y) - 2\alpha\eta(X)\eta(Y), \end{aligned} \tag{6.12}$$

which implies

$$\begin{aligned} &(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) \\ &= 2(\alpha + a + \lambda)g(X,Y) - 2(\alpha - (b + \mu))\eta(X)\eta(Y). \end{aligned} \tag{6.13}$$

Now equation (6.13) becomes η-Ricci soliton if

$$\alpha + a + \lambda = 0$$

and

$$b + \mu = constant,$$

i.e.,

$$b = constant.$$

Now we equate the R.H.S of the equation (5.6) and (6.11) and taking $X = Y = \xi$ which yields

$$2(\alpha^2 + \beta^2) = a + b,$$

i. e.,

$$a = 2(\alpha^2 + \beta^2) - b = constant.$$

Theorem 6.2. *Let M be a 3-dimensional non-paracosymplectic normal almost paracontact metric manifold with $\alpha, \beta = constant$. If M is an η-Einstein manifold with $S = ag + b\eta\otimes\eta$, then the manifold admits an η-Ricci soliton $(g, \xi, -(a+\alpha), (\alpha - b))$.*

Now we plugging $V = \xi$ into the identity (1.1) to achieve

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + \mu\eta(X)\eta(Y) = 0 \tag{6.14}$$

for all $X, Y \in \chi(M)$. Now using (5.6) and (6.12), we obtain

$$\begin{aligned} (\pounds_\xi g)(X,Y) + 2S(X,Y) &= \{r + 2(\alpha^2 + \beta^2)\}g(X,Y) \\ &- \{r - 2(\alpha^2 + \beta^2 - \alpha)\}\eta(X)\eta(Y) \end{aligned} \tag{6.15}$$

Replacing (6.15) in (6.14), we get

$$\{r + 2(\alpha^2 + \beta^2) + 2\alpha + 2\lambda\}g(X,Y) - \{r - 2(\alpha^2 + \beta^2) + 2\alpha - 2\mu\}\eta(X)\eta(Y) = 0. \tag{6.16}$$

Now we insert $X = Y = \xi$ into (6.16) to infer

$$4(\alpha^2 + \beta^2) + 2(\lambda + \mu) = 0,$$

eqivalently

$$\lambda = -\{2(\alpha^2 + \beta^2) + \mu\}.$$

Thus we have

Theorem 6.3. *If a 3-dimensional non-paracosymplectic normal almost paracontact metric manifold with $\alpha, \beta = constant$ admits a η-Ricci soliton (g, ξ, λ) then the η-Ricci soliton is shrinking if $\mu \geq 0$ and expanding if $\mu < -2(\alpha^2 + \beta^2)$ and steady if $2(\alpha^2 + \beta^2) + \mu = 0$.*

7. Conclusion

In this article, we have used the methods of local Riemannian or semi-Riemannian geometry to interpret the solutions of (1.2) and impregnate Einstein metrics in a large class of metrics of η-Ricci solitons and almost η-Ricci solitons on paracontact geometry, specially on para-Kenmotsu and para-cosymplectic manifold. Our results will not only play an indispensable and incitement role in paracontact geometry but also they have significant and motivational contribution in the area of further research on complex geometry, specially on Kähler and para-Kähler manifold etc. Moreover, we can think about the physical interpretation of conformal Ricci solitons and $*$-conformal Ricci solitons in differential geometry. There are some questions that have arisen from our article and are a potential study for further research.
(i)Are Theorem 3.1 true without assuming η-Einstein condition?
(ii)If we consider non-zero vector field V is not collinear with ξ or without assuming infinitesimal paracontact transformation condition, then are Theorem 3.2 and Theorem 3.3 true?
(iii)Are Theorem 4.1 true if we assume Reeb vector field ξ leaves the scalar curvature r invariant?
(iv) Are Theorem 6.1 to 6.3 true if we consider dimension of the manifold is more than 3 or α, β are not constant?
(v) Are the results of our paper also true in nearly Kenmotsu manifolds or f-Kenmotsu manifolds or f-cosymplectic manifolds?

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