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INTERSECTIONS OF SURFACES OF REVOLUTION

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Abstract. In this paper, we deal with surfaces of revolution and their intersections. We start with the surfaces of revolution RS that have their axis along the x^3–axis and find intersections with a line, a plane, and then intersection of two such RS. Furthermore, we apply formulas for the intersection with a line to determine the visibility of RS. Later we develop formulas for the intersection of two surfaces of revolution that have their axis along different arbitrary straight lines, and, as a special case, the intersections of two spheres and intersections of general surface of revolution with a sphere and a surface given by an equation. We apply our own software to the graphical representation of all the results we present.
Keywords: intersections, surfaces of revolution, visualization, visibility.

1. Introduction

Surfaces of revolution are created by rotating a planar curve about an axis in the plane. They are easy to understand and deal with, so they play important role and are widely used in various fields of mathematics, physics and engineering.

Nevertheless, they are still interesting for research. Some papers that explore their properties are [1, 2, 3, 6]. Some of the papers study the intersections of surfaces of revolution are [8, 9, 21]. Some problems related to finding the intersection of two surfaces are discussed in [7]. Determining surfaces of revolution from some of their properties is considered in the papers [10, 11, 22].

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Visualization strongly supports the understanding of mathematical concepts. We developed our own software for the visualization of mathematical objects and relations between them. We use the approach of vector graphics, perform calculations analytically, display the surfaces by families of curves on them without any approximating body of the polygon mesh type, so we get figures of high precision. We use a small number of curves on the surfaces, so that our figures look clean, without unnecessary details, so the parametrization of surfaces and the lines of intersection can be easily recognized. In order to emphasize the lines of intersection, which are most important in this paper, we emphasize them with thicker lines.

The basic version of our software is covered in detail in the book [13]. Later, the software was much improved and applied for the visualization in various fields of mathematics, for example in topology [18] and functional analysis [15, 16, 17, 20], but also in other sciences such as physics [12] and crystallography [14, 19].

In this paper we study intersections of surfaces of revolution RS, first of those which have their axis along the x^3–axis, and later of those with an arbitrary axis. In Section 2. we start with the intersection of RS with a straight line and apply it to the solution of the visibility problem for RS. In the special cases of a sphere, cylinder and cone, the solution of the visibility problem reduces considerably. The special conditions for the intersection of RS with a plane are determined. This section concludes with the intersection of two RS with their axes along the x^3–axis. Section 3. is dedicated to the surfaces of revolution with arbitrary axes, so we refer to them as general surfaces of revolution. The solutions of the problems of visibility and contour line that arise in the graphical representation can be reduced to the methods described in detail in the previous section. For the intersections, first we study special cases of the intersection of two spheres, then intersections of general RS with a sphere, intersections of RS and a surface given by an equation and, in the end, intersections of two general surfaces of revolution.

We developed our own software to visualize all the presented results. All the geometrical figures in this paper have been created by our software package.

2. Surface of Revolution with Axis Along x^3–axis

A surface of revolution is generated by rotating a planar curve γ about an axis in the plane. As far as the study of geometrical properties is concerned, we may assume that the curve γ is in the x^1x^3–plane and the axis of rotation is the x^3–axis. Later, we will consider surfaces of revolution generated by rotations about arbitrary axes in three–dimensional space.

Let $I \subset \mathbb{R}$ be an interval, $r, h \in C^r(I)$, where $r \in \mathbb{N}$ is chosen to need. We assume that γ is given by a parametric representation

$$\vec{x}(t) = (r(t), 0, h(t)) \text{ for } t \in I \text{ where } r(t) > 0 \text{ and } |r'(t)| + |h'(t)| \neq 0 \text{ on } I.$$

Writing $u^1 = t$ and u^2 for the polar angle in the x^1x^2–plane, we obtain

$$\vec{x}(u^i) = (r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)) \ ((u^1, u^2) \in D \subset I \times (0, 2\pi)) \tag{2.1}$$

as a parametric representation for the surface of revolution $RS(\gamma)$ generated by the curve γ.

2.1. The Intersection with a Line

We start by finding the intersection of surface of revolution RS with a parametric representation (2.1) where $D = I_1 \times I_2$ and $I_2 \subset (0, 2\pi)$, and a straight line L, given by a a parametric representation $\vec{y} = \vec{p} + t\vec{v}$ ($t \in \mathbb{R}$), that is, we have to find $(u^1, u^2) \in D$ and $t \in \mathbb{R}$ such that

$$x(u^i) = \vec{p} + t\vec{v}. \tag{2.2}$$

Thus, writing

$$\vec{u} = \vec{u}(u^2) = (\cos u^2, \sin u^2, 0) \text{ and } \vec{e}^{\,3} = (0, 0, 1),$$

we have to find the solutions $(u^1, u^2) \in D = I_1 \times I_2$ and $t \in \mathbb{R}$ of the equations

$$r(u^1)\vec{u}(u^2) + h(u^1)\vec{e}^{\,3} - (\vec{p} + t\vec{v}) = \vec{0}. \tag{2.3}$$

This means in particular

$$h(u^1) - (p^3 + tv^3) = 0. \tag{2.4}$$

Case 1. First we consider the case $v^3 \neq 0$ when L is not orthogonal to the axis of rotation of RS. Then (2.4) implies

$$t = t(u^1) = \frac{h(u^1) - p^3}{v^3}. \tag{2.5}$$

We put

$$\vec{a} = \vec{p} - \frac{p^3}{v^3} \cdot \vec{v},\ \vec{b} = \frac{1}{v^3} \cdot \vec{v} \tag{2.6}$$

and obtain, squaring (2.3) and substituting (2.5),

$$r^2(u^1) + h^2(u^1) = \left(\vec{p} - \frac{p^3}{v^3} \cdot \vec{v} + h(u^1) \cdot \frac{1}{v^3} \cdot \vec{v}\right)^2 = (\vec{a} + h(u^1) \cdot \vec{b})^2.$$

Thus we have to find the zeros $u_0^1 \in I_1$ of

$$f(u^1) = r^2(u^1) + h^2(u^1) - (\vec{a} + h(u^1) \cdot \vec{b})^2 \tag{2.7}$$

For each zero u_0^1 of (2.7), we compute the value $t_0 = t(u_0^1)$ from (2.5) and finally the values $u_0^2 \in I_2$ from

$$\cos u_0^2 = \frac{p^1 + t_0 v^1}{r(u_0^1)} \text{ and } \sin u_0^2 = \frac{p^2 + t_0 v^2}{r(u_0^1)}. \tag{2.8}$$

We remark that since

$$r^2(u^1) = (p^1 + tv^1)^2 + (p^2 + tv^2)^2 \geqslant \max\{|p^1 - tv^1|, |p^2 - tv^2|\},$$

and $r^2(u^1) > 0$ for all u^1, the equations in (2.8) always have a unique solution in the interval $(0, 2\pi)$.

Case 2. Now we consider the case $v^3 = 0$ when L is orthogonal to the axis of rotation of RS. Then it follows from (2.4) that

$$f(u^1) = h(u^1) - p^3 = 0. \tag{2.9}$$

Now we find the solutions $u_0^1 \in I_1$ of (2.9). Furthermore, squaring (2.3) leads to the quadratic equation

$$t^2\vec{v}^2 + 2t\vec{p} \bullet \vec{v} + \vec{p}^2 - (r^2(u_0^1) + h^2(u_0^1)) = 0. \tag{2.10}$$

For each such u_0^1 there are at most two points of intersection with the corresponding u^2–line and we obtain the t–parameters $t_0 = t(u_0^1)$ of these points of intersection from (2.10). Finally we find the values $u_0^2 \in I_2$ in the same way as in the Case 1, from (2.8).

Figure 2.1 shows intersections of a surface of revolution with straight lines; the figure on the right hand side shows its intersection with its axis, a case which mathematically cannot happen since $r(u^1) > 0$.

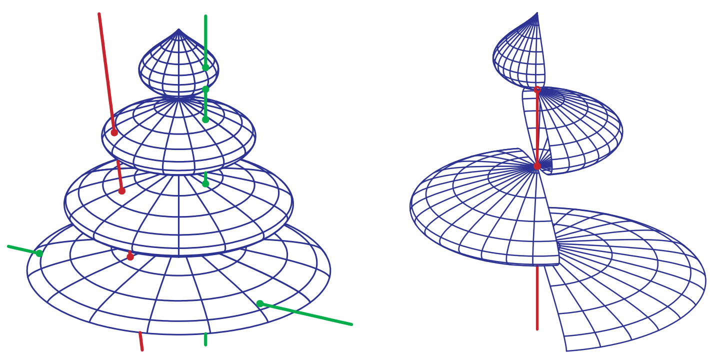

Fig. 2.1: Intersections of a surface of revolution and straight lines

2.1.1. Visibility of Surfaces of Revolution

The visibility of points on a surface of revolution is determined analytically. To check the visibility of a point P we choose the straight line L to be the projection ray. Let C be the centre of projection and $\vec{p}$ denote the position vector of a point P then we put $\vec{v} = \overrightarrow{PC}$ in the equations above. Now P is hidden by RS if and only if there is a solution $u_0^1 \in I_1$, for $v^3 \neq 0$ of (2.7) with corresponding t_0 from (2.5), or for $v^3 = 0$ of (2.9) with corresponding $t_0 > 0$ from (2.10), and $u_0^2 \in I_2$ from (2.8). The same argument applies for the visibility of a point P on RS with respect to RS itself; now we observe that $P \in RS$ implies $\vec{p}^{\,2} - (r^2(u^1) + h^2(u^1)) = 0$ and the quadratic equation (2.10) reduces to $t = -(2\vec{p} \bullet \vec{v})/\vec{v}^{\,2}$.

We have seen above that intersecting a surface of revolution and a straight line involves finding the zeros of the real valued function f in (2.7) or (2.9). An algorithm for this and its implementation can be found in [13, Section 6.1, pp. 502–511].

2.1.2. Visibility in Special Cases: Sphere, Cylinder and Cone

Finally we consider the special cases when the surface of revolution is a sphere, cylinder or cone. Then the solution of the visibility problem reduces considerably.

A sphere Sph with radius r and its centre in the origin has a parametric representation (2.1) with $r(u^1) = r \cos u^1$ and $h(u^1) = r \sin u^1$, and is given by the equation

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = r^2. \tag{2.11}$$

Substituting the parametric representation (2.2) of a straight line L in (2.11), we obtain

$$(t\vec{v} + \vec{p})^2 = t^2\vec{v}^{\,2} + 2t\vec{v} \bullet \vec{p} + \vec{p}^{\,2} = r^2.$$

Thus the t–parameters along L of the points of intersection are the solutions of the quadratic equation

$$at^2 + bt + c = 0 \text{ with } a = \vec{v}^{\,2},\ b = 2\vec{v} \bullet \vec{p} \text{ and } c = \vec{p}^{\,2} - r^2. \tag{2.12}$$

We observe that if we check the visibility of a point P on Sph with respect to Sph itself then $\vec{p} = r^2$, and the quadratic equation (2.12) reduces to $at = b$.

A circular cylinder Cyl with radius r and its axis along the x^3–axis has a parametric representation (2.1) with $r(u^1) = r$ and $h(u^1) = u^1$, and is given by the equation

$$(x^1)^2 + (x^2)^2 = r^2. \tag{2.13}$$

Now the t–parameters along L of the points of intersection are the solutions of the quadratic equation (2.12) with

$$a = (v^1)^2 + (v^2)^2,\ b = 2(v^1p^1 + v^2p^2) \text{ and } c = (p^1)^2 + (p^2)^2 - r^2$$

which again reduces to $at = b$ when we check the visibility of a point P on Cyl with respect to Cyl itself.

A cone $Cone$ with its axis along the x^3-axis, its vertex in the origin and an angle of $2\beta \in (0, \pi)$ at its vertex has a parametric representation (2.1) with $r(u^1) = u^1 \sin\beta$ and $h(u^1) = u^1 \cos\beta$, and is given by the equation

$$(x^1)^2 + (x^2)^2 - \tan^2\beta = 0. \tag{2.14}$$

Now the t–parameters along L of the points of intersection are the solutions of the quadratic equation (2.12) with

$$a = (v^1)^2 + (v^2)^2 - (v^3 \tan\beta)^2,\ b = 2\left(v^1 p^1 + v^2 p^2 - \beta v^3 p^3 \tan\beta\right)$$
$$\text{and } c = (p^1)^2 + (p^2)^2 - (p^3 \tan\beta)^2$$

which again reduces to $at = b$ when we check the visibility of a point P on $Cone$ with respect to $Cone$ itself.

2.2. The Intersections of Surfaces of Revolution and Planes

Let RS be a surface of revolution given by a parametric representation (2.1) and Pl be a plane through a point P and orthogonal to a vector $\vec{N}_{Pl} = \{n^1_{Pl}, n^2_{Pl}, n^3_{Pl}\}$. Then the intersection $IS = RS \cap Pl$ of RS and Pl is given by the solution of

$$\left(r(u^1)\vec{u}(u^2) + h(u^1)\vec{e}^{\,3} - \vec{p}\right) \bullet \vec{N}_{Pl} = 0.$$

In view of the symmetry of rotation, we may assume $n^2_{Pl} = 0$ and apply the same argument as in Subsection 2.1.2. to treat the general case. Writing $a_0 = \vec{p} \bullet \vec{N}_{Pl}$, we have to solve

$$n^1_{Pl} r(u^1)\cos u^2 + n^3_{Pl} h(u^1) - a_0 = 0. \tag{2.15}$$

If $g_2(u^1) = n^1_{Pl} r(u^1) = 0$ then $r(u^1) \neq 0$ implies $n^1_{Pl} = 0$ and consequently $\vec{N}_{Pl}$ is parallel to the axis of rotation. Now lines of intersection are the parts $u^2 \in I_2$ of the u^2–lines that correspond to solutions $u^1_0 \in I_1$ of $g_1(u^1) = n^3_{Pl} h(u^1) - a_0 = 0$. If $g_2(u^1) \neq 0$ then we can solve (2.15) to obtain

$$\cos u^2 = \cos u^2(u^1) = -\frac{g_1(u^1)}{g_2(u^2)}, \tag{2.16}$$

and the intersection is given by $u^2(u^1) \in I_2$ from (2.16) for those values $u^1 \in I_2$ that satisfy

$$\left|\frac{g_1(u^1)}{g_2(u^1)}\right| \leqslant 1.$$

2.3. The Intersections of Surfaces of Revolution

Let RS and RS^* be surfaces of revolution given by the parametric representations (2.1) and

$$\vec{x}^*(u^{*i}) = (r^*(u^{*1})\cos u^{*2}, r^*(u^{*1})\sin u^{*2}, h^*(u^{*1}))$$

with domains $D = I_1 \times I_2$ and $D^* = I^*_1 \times I^*_2$. We also assume that $r(u^1) > 0$ on I_1 and $r^*(u^{*1}) > 0$ on I^*_1, and

$$|r'(u^1)| + |h'(u^1)| > 0 \text{ on } I_1 \text{ and } |r^{*'}(u^{*1})| + |h^{*'}(u^1)| > 0 \text{ on } I^*_1. \tag{2.17}$$

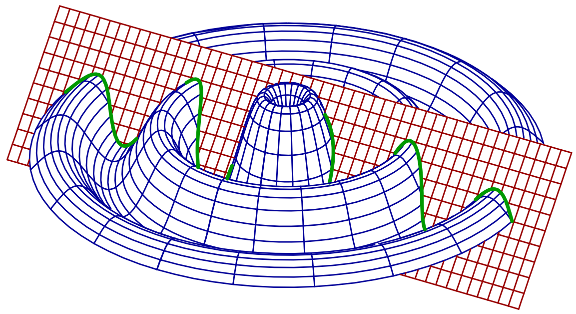

Fig. 2.2: The intersection of a plane and a surface of revolution

The lines of intersection of RS and RS^* are given by

$$\vec{x}(u^i) = \vec{x}^*(u^{*i}) \text{ for } (u^1, u^2), (u^{*1}, u^{*2}) \in D \cap D^*. \tag{2.18}$$

Squaring the equations for the first two components in (2.18), adding them and taking into account that $r(u^1), r^*(u^{*1}) > 0$, we obtain together with the third equation

$$r(u^1) = r^*(u^{*1}) \text{ and } h(u^1) = h^*(u^{*1}) \text{ for } u^1, u^{*1} \in J_1 = I_1 \cap I_1^*, \tag{2.19}$$

and then $\cos u^2 = \cos u^{*2}$ and $\sin u^2 = \sin u^{*2}$ from the first two equations. Since the map $u \mapsto (\cos u, \sin u)$ is one–to-one on $(0, 2\pi)$, we obtain $u^2 = u^{*2}$ for $u^2, u^{*2} \in J_2 = I_2 \cap I_2^*$. Now, by (2.17), at least one of the functions r, r^*, h and h^* has a local inverse. We assume that there exists an interval $J \subset J_1$ and a function $\varphi : r^*(J) \to \mathbb{R}$ with $\varphi(r^*(u^{*1})) = u^{*1}$ for all $u^{*1} \in J$. The other cases are treated similarly. Then we obtain from the first equation in (2.19) that $u^{*1} = \varphi(r(u^1)$ and substituting this in the second equation in (2.19), we get $h(u^1) = h^*(\varphi(r(u^1)))$, hence the corresponding parts of the lines of intersection are given by the equation

$$h(u^1) - h^*(\varphi(r(u^1))) = 0 \text{ for } u^1 \in J.$$

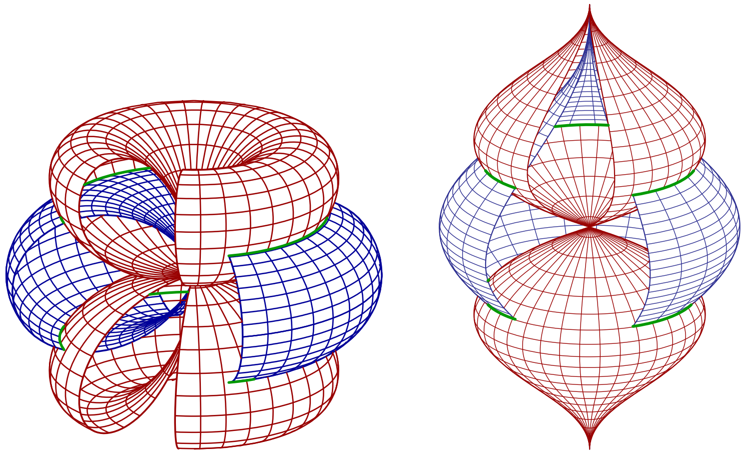

Fig. 2.3: Surfaces of revolution and their intersections

3. Surface of Revolution with Axis Along an Arbitrary Axis

Now we consider general surfaces of revolution generated by the rotation of a planar curve about an arbitrary axis. It turns out that the solutions of the problems that arise in the graphical representation can be reduced to the methods described in detail in Subsection 2.1.

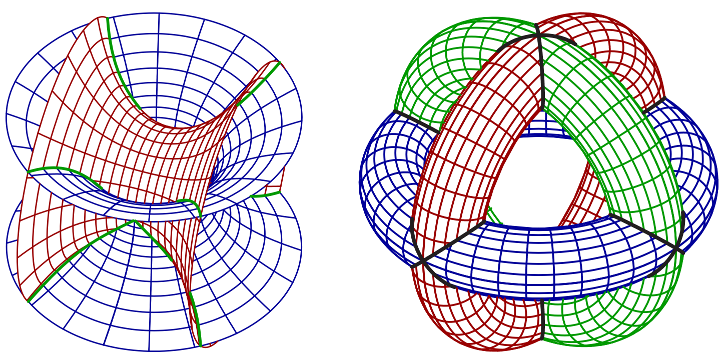

Fig. 3.1: Catenoids and tori and their lines of intersections

We observe that the geometry of a surface is independent of the choice of the coordinate system, in particular, the curves on a surface are determined in terms of the parameters of the surface. Thus we can use the methods in Section 2. to determine them. We make a transformation of the coordinate system, solve the visibility, contour and intersection problems in the new coordinate system exactly as in Section 2., and finally return to the original coordinate system.

Now we deal with the graphical representation of some intersections of general surfaces of revolution. First we consider some special cases.

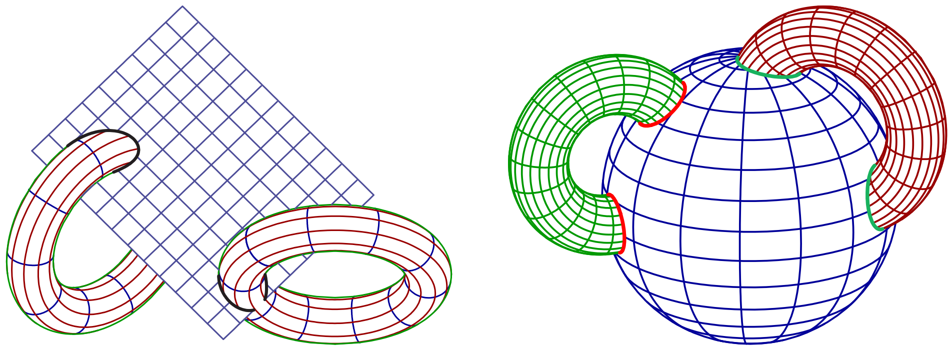

FIG. 3.2: Tori, plane and sphere and their lines of intersections

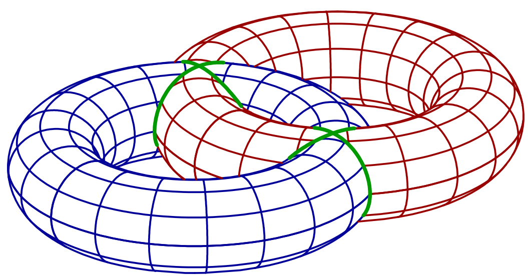

FIG. 3.3: Tori and their lines of intersection

3.1. The Intersection of Spheres

Let S_1 and S_2 be parts of spheres with centres in C_1 and C_2, radii $r_1 > 0$ and $r_2 > 0$, and domains D_1 and D_2 for their parameters. We write $\vec{c}_1$, $\vec{c}_2$ for the position vectors of C_1 and C_2, $\vec{d} = \vec{c}_2 - \vec{c}_1$, $d = \|\vec{d}\|$ and $IS = S_1 \cap S_2$.

First we consider the trivial cases.
If $d = 0$, then $C_1 = C_2$ and $IS = \emptyset$ for $r_1 \neq r_2$, and $IS = S_1 = S_2$ for $r_1 = r_2$.
If $d > r_1 + r_2$ then obviously $IS = \emptyset$.
Now we consider the case $0 < d < |r_1 - r_2|$. If $r_1 > r_2$ then $r_2 + d < r_1$ and the points $X \in S_2$ satisfy

$$\|\overrightarrow{OX} - \overrightarrow{OC_1}\| \leqslant \|\overrightarrow{OX} - \overrightarrow{OC_2}\| + \|\vec{d}\| = r_2 + d < r_1,$$

that is, they are in the interior of the closed ball $B_{r_1}(C_1)$ (closed ball of radius r centered at X_0 is $B_r(X_0) = \{X \in \mathbb{R}^3 : d(X, X_0) \leqslant r\}$), hence $IS = \emptyset$. If $r_2 > r_1$ then $r_1 + d < r_2$ and the points $X \in S_1$ satisfy

$$\|\overrightarrow{OX} - \overrightarrow{OC_2}\| \leqslant \|\overrightarrow{OX} - \overrightarrow{OC_1}\| + \|\vec{d}\| = r_1 + d < r_2,$$

that is they are in the interior of the closed ball $B_{r_2}(C_2)$, hence $IS = \emptyset$.

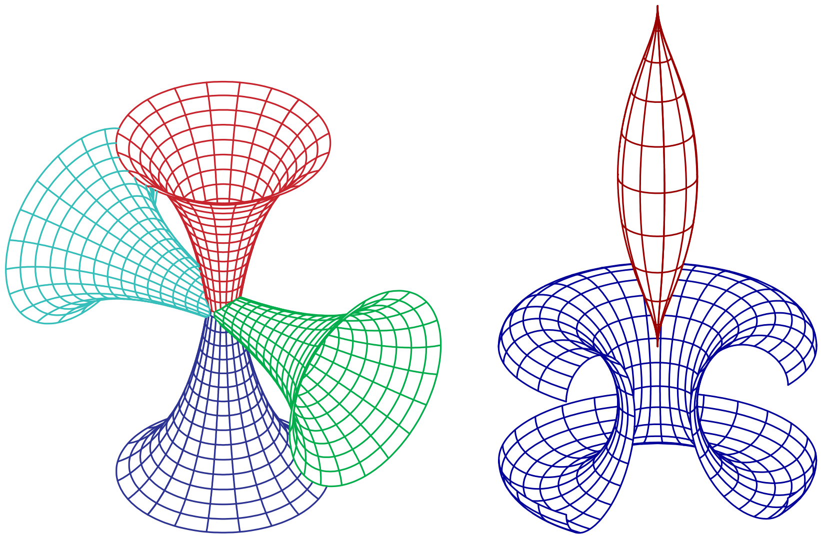

FIG. 3.4: Left: four pseudo–spheres. Right: general surfaces of revolution

For the nontrivial case, let $0 < |r_1 - r_2| \leqslant d \leqslant r_1 + r_2$. Then a point on S_1 with position vector $\vec{x}(u^i)$ in the intersection IS has to satisfy the equation

$$\|\vec{x}(u^i) - \vec{c}_2\|^2 = r_2^2.$$

This yields

$$\begin{aligned} r_2^2 &= \|(\vec{x}(u^i) - \vec{c}_1) - \vec{d}\|^2 = r_1^2 - 2(\vec{x}(u^i) - \vec{c}_1) \bullet \vec{d} + \|\vec{d}\|^2 \\ &= r_1^2 - \vec{d} \bullet \left(2(\vec{x}(u^i) - \vec{c}_1) - \vec{d}\right) = r_1^2 - 2\vec{d} \bullet \left(\vec{x}(u^i) - \frac{1}{2}(\vec{c}_1 + \vec{c}_2)\right) \end{aligned}$$

which is equivalent to

$$\vec{d} \bullet \left(\vec{x}(u^i) - \frac{1}{2}(\vec{c}_1 + \vec{c}_2) + \frac{(r_2^2 - r_1^2)\vec{d}}{2d^2}\right) = 0.$$

The points with position vectors $\vec{x}(u^i)$ that satisfy this equation are in a plane PL orthogonal to the vector $\vec{d}$ and through the point P_0 with position vector

$$\begin{aligned} \overrightarrow{OP_0} = \frac{1}{2d^2}\left(d^2(\vec{c}_2 + \vec{c}_1) + (r_1^2 - r_2^2)\vec{d}\right) &= \vec{c}_1 + \frac{1}{2d^2}\left(r_1^2 + d^2 - r_2^2\right) \bullet \vec{d} \\ &= \vec{c}_2 - \frac{1}{2d^2}\left(r_2^2 + d^2 - r_1^2\right) \bullet \vec{d}. \end{aligned}$$

We observe that $|r_1 - r_2| \leqslant d \leqslant r_1 + r_2$ implies $-d \leqslant r_1 - r_2 \leqslant d \leqslant r_1 + r_2$, hence $-r_2 \leqslant r_1 - d \leqslant r_2$, $r_2 \leqslant r_1 + d$, $-r_1 \leqslant r_2 - d \leqslant r_1$ and $r_1 \leqslant r_2 + d$, that is

$|r_1 - d| \leqslant r_2$, $r_2 \leqslant r_1 + d$, $|r_2 - d| \leqslant r_1$ and $r_1 \leqslant r_2 + d$. Thus we have

$$\begin{aligned} -2r_1 d = r_1^2 + d^2 - (r_1 + d)^2 \leqslant r_1^2 + d^2 - r_2^2 = (r_1 - d)^2 - r_2^2 + 2r_1 d \\ \leqslant r_2^2 - r_2^2 + 2r_1 d = 2r_1 d, \end{aligned}$$

that is, $|r_1^2 + d^2 - r_2^2| \leqslant 2r_1 d$. Similarly we obtain $|r_2^2 + d^2 - r_1^2| \leqslant 2r_2 d$, and consequently

$$\|\overrightarrow{OP_0} - \vec{c}_1\| = \frac{1}{2d}\left|r_1^2 + d^2 - r_2^2\right| \leqslant r_1 \text{ and } \|\overrightarrow{OP_0} - \vec{c}_2\| = \frac{1}{2d}\left|r_2^2 + d^2 - r_1^2\right| \leqslant r_2$$

that is, $P_0 \in B(C_1, r_1) \cap B(C_2, r_2)$. Thus $IS = PL \cap S_1 = PL \cap S_2$.

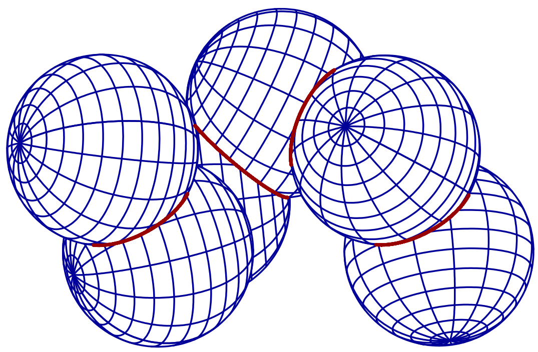

FIG. 3.5: General spheres and their intersections

3.2. The Intersection of a General Surface of Revolution and a Sphere

Let RS be a general surface of revolution with a local coordinate system with origin in C_1 and unit vectors $\vec{e}_L^{\,k}$ $(k = 1, 2, 3)$ along its coordinate axes, such that RS may be given by a parametric representation

$$(3.1) \qquad \vec{x}(u^i) = r(u^1)\cos u^2 \vec{e}_L^{\,1} + r(u^1)\sin u^2 \vec{e}_L^{\,2} + h(u^1)\vec{e}_L^{\,3} + \vec{c}_1 \text{ for } (u^1, u^2) \in D_1.$$

Furthermore, let S_2 be a part of a sphere given by $D_2 \subset (-\pi/2, \pi/2) \times (0, 2\pi)$, with its centre in C_2 and radius $r_2 > 0$. Again we write $\vec{d} = \vec{c}_2 - \vec{c}_1$. Then a point with the position vector $\vec{x}(u^i)$ on RS in the intersection $IS = RS \cap S_2$ has to satisfy the equation

$$\|(\vec{x}(u^i) - \vec{c}_1) - \vec{d}\|^2 = r_2^2.$$

If $\vec{d}$ has the components $d_{L,k}$ $(k = 1, 2, 3)$ with respect to the local coordinate system of RS, that is, if

$$\vec{d} = d_{L,1}\vec{e}_L^{\,1} + d_{L,2}\vec{e}_L^{\,2} + d_{L,3}\vec{e}_L^{\,3} \text{ where } d_{L,k} = \vec{d} \bullet \vec{e}_L^{\,k} \ (k = 1, 2, 3),$$

then we must have

$$(3.2)\quad \begin{aligned} r_2^2 &= r^2(u^1) + h^2(u^1) - 2\vec{d} \bullet (\vec{x}(u^i) - \vec{c}_1) + \|\vec{d}\|^2 \\ &= r^2(u^1) + h^2(u^1) + \|\vec{d}\|^2 - \\ &\quad - 2\left(d_{L,1} r(u^1)\cos u^2 + d_{L,2} r(u^1)\sin u^2 + d_{L,3} h(u^1)\right). \end{aligned}$$

First we consider the case $d_{L,1} = d_{L,2} = 0$ when C_2 is on the axis of rotation of RS. Then we must find the zeros of

$$f(u^1) = r^2(u^1) + h^2(u^1) + \|\vec{d}\|^2 - r_2^2 - 2d_{L,3}h(u^1).$$

Now the intersection IS is given by the parts of the u^2–lines on RS that correspond to the zeros of f with parameters belonging to both D_1 and D_2.

Now we assume that $\vec{d}_{PL} = d_{L,1}\vec{e}_L^{\,1} + d_{L,2}\vec{e}_L^{\,2} \neq \vec{0}$. Let ϕ denote the polar angle of $\vec{d}_{PL}$ in the plane spanned by the vectors $\vec{e}_L^{\,1}$ and $\vec{e}_L^{\,2}$. Then we first consider the case where $\vec{d}_{PL} = \|\vec{d}_{PL}\|\vec{e}_L^{\,1}$. Now equation (3.2) reduces to

$$r^2(u^1) + h^2(u^1) + \|\vec{d}\|^2 - r_2^2 - 2d_{L,3}h(u^1) - 2\|\vec{d}_{PL}\| r(u^1)\cos u^2 = 0$$

or

$$\cos u^2 = a(u^1) = \frac{r^2(u^1) + h^2(u^2) + \|\vec{d}\|^2 - r_2^2 - 2d_{L,3}h(u^1)}{2\|\vec{d}_{PL}\| r(u^1)}.$$

If $|a(u^1)| \leqslant 1$ then we can solve for u^2 and obtain

$$u_1^2 = u_1^2(u^1) = \arccos(a(u^1)) \text{ and } u_2^2 = u_2^2(u^1) = 2\pi - \arccos(a(u^1)).$$

In the general case, we have to add the angle ϕ to the values u_1^2 and u_2^2. Now a point P is in the intersection IS if and only if its parameters satisfy $(u^1, u^2) \in D_1$ with respect to RS and $(v^1, v^2) \in D_2$ with respect to S_2.

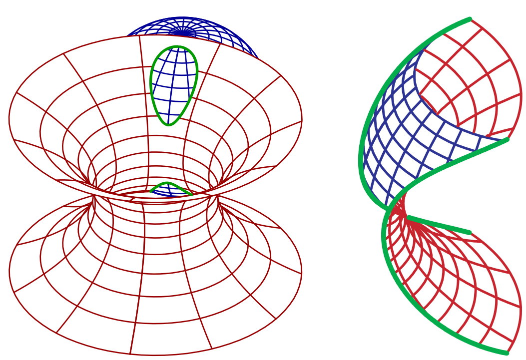

Fig. 3.6: Intersection of a catenoid and a sphere

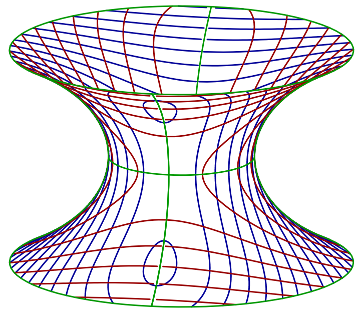

FIG. 3.7: Lines of intersection of a catenoid with spheres

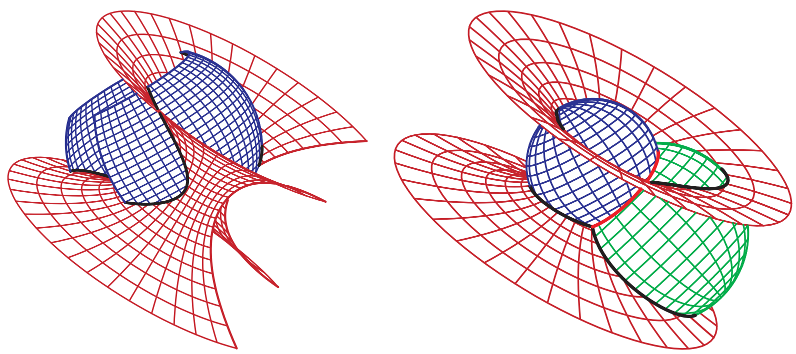

FIG. 3.8: Intersections of a catenoid and spheres

3.3. The Intersection of a General Surface of Revolution and a Surface Given by an Equation

Let RS be a general surface of revolution given by a parametric representation (3.1) with respect to its local coordinate system and S be a surface that can be given by an equation

$$F(x^1_{L,S}, x^2_{L,S}, x^3_{L,S}) = 0, \tag{3.3}$$

where $x^k_{L,S}$ $(k = 1, 2, 3)$ denote the coordinates of the points of S in the local coordinate system of S.

If C_S with position vector $\vec{c}_2$ is the origin of the local coordinate system of S and $\vec{e}^{\,k}_{L,S}$ $(k = 1, 2, 3)$ denote the unit vectors along the coordinate axes of the local coordinate system of S then a point $P = P(u^i)$ of RS with its position vector satisfying (3.1) in the intersection of RS and S has to satisfy equation (3.3) with

$$x^k_{L,S} = (\vec{x}(u^i) - \vec{c}_2) \bullet \vec{e}^{\,k}_{L,S} \quad \text{for } k = 1, 2, 3.$$

This involves finding the zeros of a real–valued function of two variables and drawing a curve given by an equation. The algorithms and methods needed for this task and their implementations are described in detail in [4, 5].

Two simple examples are the intersections of a general surface of revolution with a cone with vertex in C_2 and an angle 2β ($\beta \in (0, \pi/2)$ at its vertex, and with a circular cylinder of radius $r > 0$. Then equation (3.3) reduces to

$$\left(x^1_{L,S}\right)^2 + \left(x^2_{L,S}\right)^2 - \tan^2\beta \left(x^3_{L,S}\right)^2 = 0$$

and

$$\left(x^1_{L,S}\right)^2 + \left(x^2_{L,S}\right)^2 - r^2$$

for a cone and a cylinder, respectively.

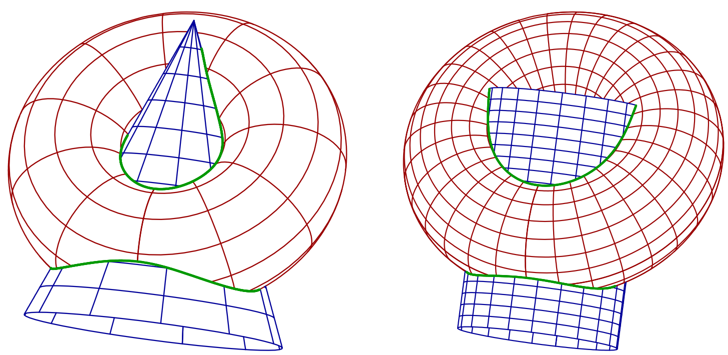

Fig. 3.9: Intersections of a torus, cone and cylinder

3.4. The Intersection of General Surfaces of Revolution

Finally we consider the intersection IS of general surfaces of revolution $RS^{(1)}$ and $RS^{(2)}$ given by the parametric representations

$$\vec{x}^{(1)}(u^i) = r_1(u^1)\cos u^2 \vec{e}^{\,1}_{L,1} + r_1(u^1)\sin u^2 \vec{e}^{\,2}_{L,1} + h_1(u^1)\vec{e}^{\,3}_{L,1}$$

and

$$\vec{x}^{(2)}(v^i) = r_2(v^1)\cos v^2 \vec{e}^{\,1}_{L,2} + r_2(v^1)\sin v^2 \vec{e}^{\,2}_{L,2} + h_2(v^1)\vec{e}^{\,3}_{L,2}.$$

Since the functions r_1, r_2, h_1 and h_2 satisfy the conditions in (2.17), at each u^i or v^i at least one of them has a local inverse. Here we treat the case that h_2 has a local inverse ψ in some interval $I^{(2)}_1$. The other cases are similar. Writing $\vec{d} = \vec{c}_2 - \vec{c}_1$ we see that a point of intersection must satisfy

$$\begin{aligned} r_2(v^1)\cos v^2 &= \left(\vec{x}^{(1)}(u^i) - \vec{d}\right) \bullet \vec{e}^{\,1}_{L,2}, \\ r_2(v^1)\sin v^2 &= \left(\vec{x}^{(1)}(u^i) - \vec{d}\right) \bullet \vec{e}^{\,2}_{L,2} \quad \text{and and and} \\ h_2(v^1) &= \left(\vec{x}^{(1)}(u^i) - \vec{d}\right) \bullet \vec{e}^{\,3}_{L,2}. \end{aligned}$$

Since

$$v^1 = v^1(u^i) = \psi\left((\vec{x}^{(1)}(u^i) - \vec{d}) \bullet \vec{e}^{\,3}_{L,2}\right) \text{ for } v^1 \in I_1^{(2)}, \tag{3.4}$$

we have to find the zeros of the function Φ with

$$\Phi(u^1,u^2) = r_2^2(v^1(u^i)) - \left(\left(\left(\vec{x}^{(1)}(u^i) - \vec{d}\right) \bullet \vec{e}^{\,1}_{L,2}\right)^2 + \left(\left(\vec{x}^{(1)}(u^i) - \vec{d}\right) \bullet \vec{e}^{\,2}_{L,2}\right)^2\right).$$

Then we have to compute the values $v_0^1 = v^1(u_0^1, u_0^1)$ from (3.4) that correspond to the zeros u_0^1 and u_0^2 of thefunction Φ and finally find the corresponding values $v_0^2 = v^2(u_0^1, u_0^2)$ from

$$\cos v_0^2 = \frac{(\vec{x}^1(u_0^i) - \vec{d}) \bullet \vec{e}^{\,1}_{L,2}}{r_2(v_0^1)} \text{ and } \sin v_0^2 = \frac{(\vec{x}^1(u_0^i) - \vec{d}) \bullet \vec{e}^{\,2}_{L,2}}{r_2(v_0^1)}.$$

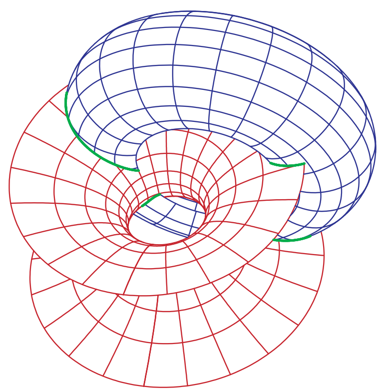

FIG. 3.10: Intersections of a catenoid and a torus

REFERENCES

1. J.G. ALCÁZAR: *On the Affine Image of a Rational Surface of Revolution*. Mathematics **(8)(11)** (2020), 2061.
2. J.G. ALCÁZAR and R. GOLDMAN: *Finding the Axis of Revolution of an Algebraic Surface of Revolution*. IEEE Transactions on Visualization and Computer Graphics **(22)(9)** (2016), 2082–2093.
3. J.C. ALEXANDER: *Closed geodesics on certain surfaces of revolution*. Journal of Geometry and Symmetry in Physics **(8)** (2006), 1–16.
4. M. FAILING: *Entwicklung numerischer Algorithmen zur computergrafischen Darstellung spezieller Probleme der Differentialgeometrie und Kristallographie*. Ph.D. Thesis, Giessen (1996), Shaker Verlag, Aachen.
5. M. FAILING and E. MALKOWSKY: *Ein effizienter Nullstellenalgorithmus zur computergrafischen Darstellung spezieller Kurven und Flächen*. Mitt. Math. Sem. Giessen **229** (1996), 11–25.

6. R. Hama and J. Kasemsuwan: *The Theory of Geodesics on Some Surface of Revolution*. Current Applied Science and Technology **(17)(1)** (2017), 42–47.
7. H.S. Heo, S. Je Hong, J.K. Seong, M.S. Kim and G. Elber: *The Intersection of Two Ringed Surfaces and Some Related Problems*. Graphical Models **(63)(4)** (2001), 228–244.
8. J. Jia, K. Tang and K.W. Kwok: *A Novel Algorithm for Computing Intersections of Two Surfaces of Revolution* in *Geometric Modeling: Techniques, Applications, Systems and Tools*. Kluwer Academic Publishers, 2004.
9. J. Jinyuan, G. Baciu and K.W. Kwok: *Quadric Decomposition for Computing the Intersections of Surfaces of Revolution*. Graph. Models **(66)(5)** (2004), 303-330.
10. X. Liu, M. Huang, S. Li and C. Ma: *Surfaces of Revolution (SORs) Reconstruction Using a Self-Adaptive Generatrix Line Extraction Method from Point Clouds*. Remote Sensing **(11)(9)** (2019), 1125.
11. P.C. López-Custodio, J.M. Rico, J.J. Cervantes-Sánchez and G.I. Pérez-Soto: *Reconfigurable Mechanisms From the Intersection of Surfaces*. Journal of Mechanisms and Robotics **(8)(2)** (2016), 021029.
12. E. Malkowsky: *Visualisation and animation in mathematics and physics*. Proceedings of the Institute of Mathematics of NAS of Ukraine **(50)(3)** (2004), 1415–1422.
13. E. Malkowsky and W. Nickel: *Computergrafik in der Differentialgeometrie*. Vieweg–Verlag, Braunschweig, 1993.
14. E. Malkowsky, F. Özger and V. Veličković: *Some Spaces Related to Cesaro Sequence Spaces and an Application to Crystallography*. MATCH Commun. Math. Comput. Chem., **vol 70, no. 3** (2013), 867-884.
15. E. Malkowsky, F. Özger and V. Veličković: *Some mixed paranorm spaces*. Filomat, **31:4** (2017), 1079-1098.
16. E. Malkowsky, F. Özger and V. Veličković: *Matrix transformations on mixed paranorm spaces*. Filomat, **31:10** (2017), 2957-2966.
17. E. Malkowsky, V. Rakovcević and V. Veličković: *Bounded linear and compact operators between the Hahn space and spaces of strongly summable and bounded sequences*. Bull. Sci.Math. Nat. Sci. Math. **(45)** (2020), 25–41.
18. E. Malkowsky and V. Veličković: *Topologies of some new sequence spaces, their duals, and the graphical representations of neighborhoods*. Topology and its Applications, **vol. 158 no. 12** (2011), 1369–1380.
19. E. Malkowsky and V. Veličković: *Some New Sequence Spaces, Their Duals and a Connection with Wulff's Crystal*. MATCH Commun. Math. Comput. Chem., **vol. 67 no. 3** (2012), 589-607.
20. E. Malkowsky and V. Veličković: *The duals of certain matrix domains of factorable triangles and some related visualisations*. Filomat, **27:5** (2013), 821-829.
21. R. Obradović: *Intersection Between Two Surfaces of Revolution*. Proceedings of *The* 10^{th} *International Conference on Geometry and Graphics*, Kyiv, Ukraine, **(1)** (2002), 108–112.
22. J. Vršek and . Lávička: *Determining surfaces of revolution from their implicit equations*. Journal of Computational and Applied Mathematics **(290)** (2015), 125–135.