FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 3 (2022), 651–666
https://doi.org/10.22190/FUMI220217046A
Original Scientific Paper

IMPULSIVE STURM-LIOUVILLE PROBLEMS ON TIME SCALES

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Abstract. In this paper, we consider an impulsive Sturm–Lioville problem on Sturmian time scales. We investigate the existence and uniqueness of the solution of this problem. We study some spectral properties and self-adjointness of the boundary-value problem. Later, we construct the Green function for this problem. Finally, an eigenfunction expansion is obtained.
Key words: Impulsive Sturm–Liouville problems, maximal and minimal operators, Green's function, self-adjoint operator, eigenfunction expansion.

1. Introduction

Impulsive differential equations have recently been subject to an increasing number of investigations since they occur in mathematical modeling of various areas of science, e.g., population dynamics, economics, optimal control, and chemotherapy. These equations have been studied by several authors (see [3, 9, 16, 17, 18, 21]).

In the '80s, Stefan Hilger introduced the theory of time scales. This theory aims to unify continuous-time and discrete-time equations. Due to its numerous application in science, many authors have obtained a lot of results about this subject (see [4, 7, 5, 6, 10, 11, 13, 20]).

On the other hand, there are some papers including impulsive dynamic equations. The results on such problems can be found, for example, in [1, 8, 12, 15].

Received February 17, 2022, accepted: August 09, 2022
Communicated by Dijana Mosić
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2010 *Mathematics Subject Classification.* Primary 34N05, 34L10, ; Secondary 47A10, 47B25.

Our goal in this paper is to study some spectral properties of the impulsive dynamic Sturm–Liouville problem on Sturmian time scales defined as

$$-\left[p(t)\,z^{\Delta}(t)\right]^{\nabla}+q(t)\,z(t)=\lambda z(t)\,,\ t\in J=[a,c)\cup(c,b], \tag{1.1}$$

where $\mathbb{T}$ is a Sturmian time scale, $J\subset\mathbb{T}$, $\lambda\in\mathbb{C}$, $q(.)$ is a real-valued continuous function, $p(.)$ is nabla differentiable function on J, $p(t)\neq 0$ for all $t\in J$, and $p^{\nabla}(.)$ is continuous.

Our paper is organized as follows. In the second section, an existence and uniqueness theorem for Eq. (1.1) with impulsive condition is proved. The minimal and maximal operators have been introduced, some results are presented and the self-adjointness of the boundary-value problem have been investigated. In the third section, Green's function is constructed. Finally, an eigenfunction expansion is given in the last section.

First, we recall some necessary fundamental concepts of time scales, and we refer to [10, 11, 7, 13] for more details.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ are defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}\,,\ t\in\mathbb{T}$$

and

$$\rho(t)=\sup\{s\in\mathbb{T}:s<t\}\,,\ t\in\mathbb{T}.$$

A point $t\in\mathbb{T}$ is left scattered if $\sigma(t)\neq t$ and left dense if $\sigma(t)=t$. A point $t\in\mathbb{T}$ is right scattered if $\rho(t)\neq t$ and right dense if $\rho(t)=t$. Moreover, $\sigma(\rho(t))=\rho(\sigma(t))=t,\ t\in\mathbb{T}$ (see [2]).

2. Impulsive Sturm–Liouville equation on Sturmian time scales

In this section, the existence and uniqueness of solutions of the impulsive dynamic Sturm–Liouville equation on Sturmian time scales are proved. Later, minimal symmetric operators are introduced in the regular cases. Furthermore, some results are given.

Let us consider the dynamic Sturm–Liouville equation

$$\Gamma(z):=-\left[p(t)\,z^{\Delta}(t)\right]^{\nabla}+q(t)\,z(t)=\lambda z(t)\,,\ t\in J, \tag{2.1}$$

with impulsive condition

$$Z(c+)=CZ(c-)\,,$$

where $\mathbb{T}$ is a Sturmian time scale, $J\subset\mathbb{T}$, $\lambda\in\mathbb{C}$, $q(.)$ is a real-valued continuous function, $p(.)$ is nabla differentiable function on J, $p(t)\neq 0$ for all $t\in J$, $p^{\nabla}(.)$ is continuous,

$$Z=\begin{pmatrix} z \\ pz^{\Delta}\end{pmatrix},\ C=\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix},$$

and d_1, d_2 are real numbers such that $\det C = d_1 d_2 > 0$. Note that $Z(c+)$ and $Z(c-)$ represent right and left limits with respect to the time scale, and in addition, the points $a, b, c \in J$ are left dense.

It has been denoted by $L^2_\nabla(J)$ that the Hilbert space with the inner product

$$(f,g) := \int_a^c f(t)\,\overline{g(t)}\nabla t + \delta \int_c^b f(t)\,\overline{g(t)}\nabla t,$$

where $f, g \in L^2_\nabla(J)$ and $\delta = \frac{1}{d_1 d_2}$.

Then we obtain the following theorem.

Theorem 2.1. *Eq.* (2.1) *has a unique solution z in $L^2_\nabla(J)$ such that*

$$Z(a,\lambda) = K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix},\ Z(c+) = CZ(c-), \tag{2.2}$$

where k_1, k_2 are given constants.

Proof. Let $z(t)$ be a solution of Eq. (2.1) and let $p(t)\,z^\Delta(t) = y(t)$. Then, we have

$$z^\Delta(t) = p^{-1}(t)\,y(t).$$

It follows from (2.1) that

$$y^\nabla(t) = (q(t) - \lambda)\,z(t).$$

Since

$$\begin{aligned} y^\Delta(t) &= y^\nabla(\sigma(t)) = (q(\sigma(t)) - \lambda)\,z(\sigma(t)) \\ &= (q(\sigma(t)) - \lambda)\left\{z(t) + [\sigma(t) - t]\,z^\Delta(t)\right\} \\ &= (q(\sigma(t)) - \lambda)\left\{z(t) + [\sigma(t) - t]\,p^{-1}(t)\,y(t)\right\}. \end{aligned}$$

we get the following first-order system:

$$Z^\Delta(t) = A(t,\lambda)\,Z(t,\lambda),$$

where

$$A(t,\lambda) = \begin{pmatrix} 0 & p^{-1}(t) \\ (q(\sigma(t)) - \lambda) & p^{-1}(t)\,[\sigma(t) - t]\,(q(\sigma(t)) - \lambda) \end{pmatrix},$$

and

$$Z(t) = \begin{pmatrix} z(t,\lambda) \\ y(t,\lambda) \end{pmatrix}.$$

It is clear that the matrix $A(t,\lambda)$ is regressive, i.e., $I_2 + \mu_\sigma(t)\,A(t,\lambda)$ ($\mu_\sigma(t) = \sigma(t) - t$) is invertible. By [[10], Theorem 5.8], we deduce that Eq. (2.1) has a unique solution $z(t,\lambda)$ with conditions (2.2). □

Consider the set

$$D_{\max}$$

$$= \left\{ z \in L^2_\nabla(J) : \begin{array}{c} z \text{ is } \Delta\text{-absolutely continuous and } pz^\Delta \text{ is } \nabla\text{-absolutely} \\ \text{continuous functions on all subintervals of } J, \\ \text{one-sided limits } z\,(c\pm) \text{ and } pz^\Delta\,(c\pm) \text{ exist and finite,} \\ Z\,(c+) = CZ\,(c-) \text{ and } \Gamma(z) \in L^2_\nabla(J). \end{array} \right\}$$

Then we define the *maximal operator* $L_{\max}$ on $D_{\max}$ by the equality $L_{\max}z = \Gamma(z)$. The operator $L_{\min}$, that is the restriction of the operator $L_{\max}$ to $D_{\min}$ is called the *minimal operator,* where

$$D_{\min} = \left\{ z \in D_{\max} : z\,(a) = p\,(a)\,z^\Delta\,(a) = z\,(b) = p\,(b)\,z^\Delta\,(b) = 0 \right\}. \tag{2.3}$$

For arbitrary two functions $z_1, z_2 \in D_{\max}$, we have the following Green's formula

$$(\Gamma(z_1), z_2) - (z_1, \Gamma(z_2))$$

$$= \int_a^c \left[\Gamma(z_1)(t)\overline{z_2(t)} - z_1(t)\overline{\Gamma(z_2)(t)} \right] \nabla t$$

$$+\delta \int_c^b \left[\Gamma(z_1)(t)\overline{z_2(t)} - z_1(t)\overline{\Gamma(z_2)(t)} \right] \nabla t$$

$$= \delta[z_1, z_2]\,(b) - \delta[z_1, z_2]\,(c+) + [z_1, z_2]\,(c-) - [z_1, z_2]\,(a)\,, \tag{2.4}$$

where

$$[z_1, z_2]\,(t) := W_\Delta\,(z_1, \overline{z_2})\,(t) = p\,(t) \left\{ z_1(t)\overline{z_2^\Delta(t)} - z_1^\Delta(t)\overline{z_2(t)} \right\}.$$

Theorem 2.2. *The operator $L_{\min}$ is Hermitian.*

Proof. From (2.3) and (2.4), we get the desired result. □

Theorem 2.3. *Let $\eta \in L^2_\nabla(J)$. Then, the equation*

$$\Gamma\,(z) = \eta \tag{2.5}$$

has a solution $z\,(t)$ satisfying the conditions

$$Z\,(a) = Z\,(b) = 0,\; Z\,(c+) = CZ\,(c-)\,, \tag{2.6}$$

if and only if the function η is orthogonal to all solutions of the equation $\Gamma\,(z) = 0$ satisfying the condition $Z\,(c+) = CZ\,(c-)$.

Proof. Let $z(t)$ be the solution of the equation $\Gamma(z) = \eta$ satisfying the conditions

$$Z(a) = 0,\ Z(c+) = CZ(c-)\,. \tag{2.7}$$

There exists one such solution. Let us denote by y_1 and y_2, a fundamental system of solutions of the equation $\Gamma(y) = 0$ satisfying the conditions

$$Y(c+) = CY(c-)\,,$$

$$\begin{aligned} Y_1(b) &= \begin{pmatrix} y_1(b) \\ p(b)\, y_1^{\Delta}(b) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ Y_2(b) &= \begin{pmatrix} y_2(b) \\ p(b)\, y_2^{\Delta}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{2.8}$$

From (2.4), we obtain

$$(\eta, y_i) = (\Gamma(z), y_i)$$

$$= \delta[z, y_i](b) - \delta[z, y_i](c+) + [z, y_i](c-) - [z, y_i](a) + (z, \Gamma(y_i))\,. \tag{2.9}$$

By (2.7), we deduce that $[z, y_i](a) = 0$ and $\delta[z, y_i](c+) - [z, y_i](c-) = 0$. Since $\Gamma(y_i) = 0$, we have

$$(\eta, y_i) = \delta[z, y_i](b) = \begin{cases} -\delta p(b)\, z^{\Delta}(b) & \text{for } i = 1, \\ \delta z(b) & \text{for } i = 2. \end{cases} \tag{2.10}$$

□

It follows from Theorem 2.3 that

$$\Omega \oplus F = L^2_{\nabla}(J), \tag{2.11}$$

where Ω is the set of all solutions of the equation $\Gamma(z) = 0$ satisfying the condition $Z(c+) = CZ(c-)$, and F is the range of the operator $L_{\min}$.

Theorem 2.4. *Let $\alpha_1, \alpha_2, \alpha_3,\ \alpha_4 \in \mathbb{C}$. Then there exists a function $z \in D_{\max}$ satisfying the following conditions*

$$\begin{aligned} Z(a) &= \begin{pmatrix} z(a) \\ p(a)\, z^{\Delta}(a) \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \\ Z(b) &= \begin{pmatrix} z(b) \\ p(b)\, z^{\Delta}(b) \end{pmatrix} = \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}. \end{aligned} \tag{2.12}$$

Proof. Let $\alpha = \alpha_2 = 0$ and let η be an arbitrary vector in $L^2_{\alpha}(J)$ satisfying the conditions

$$(\eta, y_i) = \begin{cases} -\alpha_4 & \text{for } i = 1, \\ \alpha_3 & \text{for } i = 2. \end{cases} \tag{2.13}$$

where y_1 and y_2 are a fundamental system of solutions of the equation $\Gamma(y) = 0$. There exists such a vector η. Indeed if we put $\eta = c_1 y_1 + c_2 y_2$, then (2.13) provide a system of equations in the constants c_i $(i = 1, 2)$ whose determinant is the same as the Gram determinant for the linearly independent functions y_1, y_2, and does not vanish.

Let x denote the solution of the equation $\Gamma(x) = \eta$ satisfying the conditions

$$X(a) = \begin{pmatrix} x(a) \\ p(a)\, x^{\Delta}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; X(c+) = CX(c-), \tag{2.14}$$

From (2.4), we conclude that

$$(\eta, y_i) = (\Gamma(x), y_i) =$$

$$\delta[x, y_i](b) - \delta[x, y_i](c+) + [x, y_i](c-) - [x, y_i](a) + (x, \Gamma(y_i)). \tag{2.15}$$

By virtue of $\Gamma(y_i) = 0$ and (2.14), we see that $[x, y_i](a) = 0$ and $\delta[x, y_i](c+) - [x, y_i](c-) = 0$.

It follows now from (2.8) and (2.10) that

$$[x, y_i](b) = \begin{cases} -p(b)\, x^{\Delta}(b) & \text{for } i = 1, \\ x(b) & \text{for } i = 2. \end{cases}$$

From (2.13) and (2.15), we get

$$X(b) = \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}.$$

Thus, we have constructed a function $x \in D_{\max}$ such that

$$X(a) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; X(b) = \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}.$$

Similarly, we can construct a function $x_1 \in D_{\max}$ such that

$$X_1(a) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \; X_1(b) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence the function $z = x + x_1 \in D_{\max}$ satisfies (2.12). □

Theorem 2.5. *$D_{\min}$ is dense in $L^2_{\nabla}(J)$.*

Proof. Let $\zeta \perp D_{\min}$ and let y be any particular solution of the equation $\Gamma(y) = \zeta$ satisfying the condition $Y(c+) = CY(c-)$. Therefore, for $z \in D_{\min}$, we conclude that

$$(y, L_{\min} z) = (L_{\max} y, z) = (\Gamma(y), z) = (\zeta, z) = 0.$$

From Theorem 2.3 and (2.11), we get $y \in \Omega$ and $\zeta = \Gamma(y) = 0$. □

Theorem 2.2 and Theorem 2.5 imply that $L_{\min}$ is a symmetric operator.

Theorem 2.6. *The equality* $L_{\max} = L^*_{\min}$ *holds.*

Proof. Since $L_{\max} \subset L^*_{\min}$, it suffices to show that $L^*_{\min} \subset L_{\max}$. Let $\zeta \in D^*_{\min}$, where $D^*_{\min}$ is the domain of the operator $L^*_{\min}$. Let $L^*_{\min}\zeta = \nu$ and $\xi\,(x)$ be any particular solution of the equation $\Gamma\,(\xi) = \nu$ satisfying the condition $Z\,(c+) = CZ\,(c-)$. Then, for every $z \in D_{\min}$, we obtain

$$(\nu, z) = (\Gamma\,(\xi)\,, z) = (L_{\max}\xi, z) = (\xi, L_{\min}z)\,. \tag{2.16}$$

Therefore, we have

$$(\nu, z) = (L^*_{\min}\zeta, z) = (\zeta, L_{\min}z)\,. \tag{2.17}$$

Subtracting (2.17) from (2.16), we conclude that $(\xi - \zeta, L_{\min}z) = 0$, i.e., $\xi - \zeta \in F^{\perp}$.

It follows from (2.11) that $\xi - \zeta \in \Omega \subset D_{\max}$ and $\xi \in D_{\max}$. Then we have $\Gamma\,(\xi - \zeta) = 0$, i.e., $\Gamma(\zeta) = \Gamma(\xi) = \nu = L^*_{\min}\zeta$ and $L^*_{\min} \subset L_{\max}$. □

Theorem 2.7. *The following relation holds*

$$L^*_{\max} = L_{\min}.$$

Proof. It follows from Theorem 2.6 that $L^*_{\max} = L^{**}_{\min} \supset L_{\min}$. Hence

$$L^*_{\max} \subset L^*_{\min} = L_{\max}, \tag{2.18}$$

due to $L_{\min} \subset L_{\max}$. Let $\xi \in D^*_{\max}$, where $D^*_{\max}$ is the domain of the operator $L^*_{\max}$. From (2.18), we see that $\xi \in D_{\max}$ and $L^*_{\max}\xi = L_{\max}\xi$. This yields $(L^*_{\max}\xi, z) = (\xi, L_{\max}z)\,,$ and $(L_{\max}\xi, z) = (\xi, L_{\max}z)$ for all $z \in D_{\max}$. From (2.4), it may be concluded that

$$\delta[\xi, z]\,(b) - \delta[\xi, z]\,(c+) + [\xi, z]\,(c-) - [\xi, z]\,(a) = 0 \tag{2.19}$$

for all $z \in D_{\max}$. Hence Eq. (2.19) is possible only if

$$\xi\,(a) = p\,(a)\,\xi^{\Delta}\,(a) = \xi\,(b) = p\,(b)\,\xi^{\Delta}\,(b) = 0,$$

i.e., $\xi \in D_{\min}$. Hence and from (2.18) it follows that $L^*_{\max} \subset L_{\min}$. □

By Theorem 2.7, we see that $L_{\min}$ is a closed symmetric operator.

Consider the impulsive dynamic Sturm–Liouville equation

$$\Gamma(z) = \lambda z(t),\;\; t \in J, \tag{2.20}$$

with the boundary conditions

$$a_{11}z(a) + a_{12}p\,(a)\,z^{\Delta}(a) = 0, \tag{2.21}$$

$$z\left(c+\right)-d_{1}z\left(c-\right)=0, \qquad (2.22)$$
$$p\left(c+\right)z^{\Delta}\left(c+\right)-d_{2}p\left(c-\right)z^{\Delta}\left(c-\right)=0,$$

$$a_{21}z(b)+a_{22}p\left(b\right)z^{\Delta}(b)=0, \qquad (2.23)$$

where $a_{ij},\ (i,j=1,2),\ d_1$ and d_2 are real numbers such that $a_{11}^2+a_{12}^2\neq 0,\ a_{21}^2+a_{22}^2\neq 0$ and $d_1d_2>0$.

Theorem 2.8. *The impulsive dynamic Sturm–Liouville operator L defined by* (2.20)–(2.23) *is self-adjoint on the space $L_{\nabla}^2(J)$.*

Proof. From (2.4), (2.21), (2.23), and (2.22), we can get the desired result. □

From Theorem 2.8, we have the following corollary.

Corollary 2.1. *The impulsive dynamic Sturm–Liouville operator L defined by* (2.20)–(2.23) *has the following properties.*

(i) All eigenvalues are real and simple.

(ii) The eigenfunctions $f(t,\mu_1)$ and $g(t,\mu_2)$ corresponding to distinct eigenvalues μ_1 and μ_2 are orthogonal.

3. Construction of Green's function

Consider the following boundary value problem

$$-\left[p\left(t\right)z^{\Delta}\left(t\right)\right]^{\nabla}+q\left(t\right)z\left(t\right)=h\left(t\right),\ t\in J, \qquad (3.1)$$

together with the conditions (2.21)-(2.23). Denote by

$$\varphi\left(t,\lambda\right)=\begin{cases}\varphi_1\left(t,\lambda\right), & t\in[a,c)\\ \varphi_2\left(t,\lambda\right), & t\in(c,b]\end{cases}$$

and

$$\psi\left(t,\lambda\right)=\begin{cases}\psi_1\left(t,\lambda\right), & t\in[a,c)\\ \psi_2\left(t,\lambda\right), & t\in(c,b]\end{cases}$$

two basic solutions of Eq. (2.1) which satisfy the following initial conditions

$$\begin{aligned}\varphi_1(a)&=-a_{12},\ p\left(a\right)\varphi_1^{\Delta}(a)=-a_{11},\\ \psi_2(b)&=-a_{22},\ p\left(b\right)\psi_2^{\Delta}(b)=-a_{21},\end{aligned}$$

and both transmission conditions. It is clear that

$$D\left(\lambda\right)=-W_{\Delta}\left(\varphi,\psi\right)=-W_{1,\Delta}\left(\varphi_1,\psi_1\right)=-\delta W_{2,\Delta}\left(\varphi_2,\psi_2\right)\neq 0.$$

Then we obtain the general solution of Eq. (2.1) in the form

$$z(t,\lambda)=\begin{cases}\alpha_1\varphi_1(t,\lambda)+\beta_1\psi_1(t,\lambda), & t\in[a,c)\\ \alpha_2\varphi_2(t,\lambda)+\beta_2\psi_2(t,\lambda), & t\in(c,b],\end{cases}$$

where $\alpha_1,\alpha_2,\beta_1$ and β_2 are arbitrary constants. Applying the method of variation of constants, we search the general solution of the non-homogenous Eq. (3.1) in the form

$$z(t,\lambda)=\begin{cases}\alpha_1(t,\lambda)\varphi_1(t,\lambda)+\beta_1(t,\lambda)\psi_1(t,\lambda), & t\in[a,c)\\ \alpha_2(t,\lambda)\varphi_2(t,\lambda)+\beta_2(t,\lambda)\psi_2(t,\lambda), & t\in(c,b],\end{cases}$$

where the functions $\alpha_1(t,\lambda)$, $\alpha_2(t,\lambda)$, $\beta_1(t,\lambda)$ and $\beta_2(t,\lambda)$ satisfy the following equations: for $t\in[a,c)$,

$$\begin{cases}\alpha_1^{\Delta}(t,\lambda)\varphi_1(t,\lambda)+\beta_1^{\Delta}(t,\lambda)\psi_1(t,\lambda)=0,\\ \qquad\alpha_1^{\Delta}(t,\lambda)\left[p(\rho(t))\varphi_1^{\Delta}(\rho(t),\lambda)\right]\\ +\beta_1^{\Delta}(t,\lambda)\left[p(\rho(t))\psi_1^{\Delta}(\rho(t),\lambda)\right]=h(t),\end{cases}\tag{3.2}$$

and

$$\begin{cases}\alpha_2^{\Delta}(t,\lambda)\varphi_2(t,\lambda)+\beta_2^{\Delta}(t,\lambda)\psi_2(t,\lambda)=0,\\ \qquad\alpha_2^{\Delta}(t,\lambda)\left[p(\rho(t))\varphi_2^{\Delta}(\rho(t),\lambda)\right]\\ +\beta_2^{\Delta}(t,\lambda)\left[p(\rho(t))\psi_2^{\Delta}(\rho(t),\lambda)\right]=h(t),\end{cases}\tag{3.3}$$

for $t\in(c,b]$.

On the other hand,

$$\begin{vmatrix}\varphi_1(t,\lambda) & \psi_1(t,\lambda)\\ p(\rho(t))\varphi_1^{\Delta}(\rho(t),\lambda) & p(\rho(t))\psi_1^{\Delta}(\rho(t),\lambda)\end{vmatrix}=W_{1,\Delta}(\varphi_1,\psi_1)(\rho(t))\neq 0,$$

and

$$\begin{vmatrix}\varphi_2(t,\lambda) & \psi_2(t,\lambda)\\ p(\rho(t))\varphi_2^{\Delta}(\rho(t),\lambda) & p(\rho(t))\psi_2^{\Delta}(\rho(t),\lambda)\end{vmatrix}=W_{2,\Delta}(\varphi_2,\psi_2)(\rho(t))\neq 0.$$

If we use the formula

$$z(t)=z(\rho(t))+(t-\rho(t))z^{\Delta}(\rho(t)),$$

we get

$$\begin{vmatrix} \varphi_1(t,\lambda) & \psi_1(t,\lambda) \\ p(\rho(t))\varphi_1^{\Delta}(\rho(t),\lambda) & p(\rho(t))\psi_1^{\Delta}(\rho(t),\lambda) \end{vmatrix}$$

$$= p(\rho(t))\left[\varphi_1(\rho(t),\lambda) + (t-\rho(t))\varphi_1^{\Delta}(\rho(t),\lambda)\right]\psi_1^{\Delta}(\rho(t),\lambda)$$

$$-p(\rho(t))\left[\psi_1(\rho(t),\lambda) + (t-\rho(t))\psi_1^{\Delta}(\rho(t),\lambda)\right]\varphi_1^{\Delta}(\rho(t),\lambda)$$

$$= p(\rho(t))\left[\varphi_1(\rho(t),\lambda)\psi_1^{\Delta}(\rho(t),\lambda) - \psi_1(\rho(t),\lambda)\varphi_1^{\Delta}(\rho(t),\lambda)\right]$$

$$= W_{1,\Delta}(\varphi_1,\psi_1)(\rho(t)).$$

From (3.2) and (3.3), we conclude that

$$\begin{aligned} \alpha_1^{\Delta}(t,\lambda) &= \alpha_1^{\nabla}(\sigma(t),\lambda) = -\tfrac{1}{D(\lambda)}h(t)\psi_1(t,\lambda), \\ \beta_1^{\Delta}(t,\lambda) &= \beta_1^{\nabla}(\sigma(t),\lambda) = \tfrac{1}{D(\lambda)}h(t)\varphi_1(t,\lambda), \end{aligned}$$

for $t \in [a,c)$, and

$$\begin{aligned} \alpha_2^{\Delta}(t,\lambda) &= \alpha_2^{\nabla}(\sigma(t),\lambda) = -\tfrac{1}{D(\lambda)}\delta h(t)\psi_2(t,\lambda), \\ \beta_2^{\Delta}(t,\lambda) &= \beta_2^{\nabla}(\sigma(t),\lambda) = \tfrac{1}{D(\lambda)}\delta h(t)\varphi_2(t,\lambda), \end{aligned}$$

for $t \in (c,b]$. Hence

$$\begin{aligned} \alpha_1(\sigma(t),\lambda) &= \tfrac{1}{D(\lambda)}\textstyle\int_t^c h(x)\psi_1(x,\lambda)\nabla x + \alpha_1,\ t \in [a,c), \\ \beta_1(\sigma(t),\lambda) &= \tfrac{1}{D(\lambda)}\textstyle\int_a^t h(x)\varphi_1(x,\lambda)\nabla x + \beta_1,\ t \in [a,c), \\ \alpha_2(\sigma(t),\lambda) &= \tfrac{1}{D(\lambda)}\delta\textstyle\int_t^b h(x)\psi_2(x,\lambda)\nabla x + \alpha_2,\ t \in (c,b], \\ \beta_2(\sigma(t),\lambda) &= \tfrac{1}{D(\lambda)}\delta\textstyle\int_c^t h(x)\varphi_2(x,\lambda)\nabla x + \beta_2,\ t \in (c,b], \end{aligned}$$

and

$$\begin{aligned} \alpha_1(t,\lambda) &= \tfrac{1}{D(\lambda)}\textstyle\int_{\rho(t)}^c h(x)\psi_1(x,\lambda)\nabla x + \alpha_1,\ t \in [a,c), \\ \beta_1(t,\lambda) &= \tfrac{1}{D(\lambda)}\textstyle\int_a^{\rho(t)} h(x)\varphi_1(x,\lambda)\nabla x + \beta_1,\ t \in [a,c), \\ \alpha_2(t,\lambda) &= \tfrac{1}{D(\lambda)}\delta\textstyle\int_{\rho(t)}^b h(x)\psi_2(x,\lambda)\nabla x + \alpha_2,\ t \in (c,b], \\ \beta_2(t,\lambda) &= \tfrac{1}{D(\lambda)}\delta\textstyle\int_c^{\rho(t)} h(x)\varphi_2(x,\lambda)\nabla x + \beta_2,\ t \in (c,b]. \end{aligned}$$

Thus, the general solution of the non-homogenous Eq. (3.1) is given by the formula

$$z(t,\lambda)=\begin{cases}\frac{\varphi_1(t,\lambda)}{D(\lambda)}\int_{\rho(t)}^{c}h(x)\,\psi_1(x,\lambda)\,\nabla x+\alpha_1\varphi_1(t,\lambda)\\ +\frac{\psi_1(t,\lambda)}{D(\lambda)}\int_{a}^{\rho(t)}h(x)\,\varphi_1(x,\lambda)\,\nabla x+\beta_1\psi_1(t,\lambda), & t\in[a,c),\\ \frac{\varphi_2(t,\lambda)}{D(\lambda)}\delta\int_{\rho(t)}^{b}h(x)\,\psi_2(x,\lambda)\,\nabla x+\alpha_2\varphi_2(t,\lambda)\\ +\frac{\psi_2(t,\lambda)}{D(\lambda)}\delta\int_{c}^{\rho(t)}h(x)\,\varphi_2(x,\lambda)\,\nabla x+\beta_2\psi_2(t,\lambda), & t\in(c,b].\end{cases}\tag{3.4}$$

From (3.4), we obtain

$$z^{\Delta}(t,\lambda)=\begin{cases}\frac{\varphi_1^{\Delta}(t,\lambda)}{D(\lambda)}\int_{\rho(t)}^{c}h(x)\,\psi_1(x,\lambda)\,\nabla x+\alpha_1\varphi_1^{\Delta}(t,\lambda)\\ +\frac{\psi_1^{\Delta}(t,\lambda)}{D(\lambda)}\int_{a}^{\rho(t)}h(x)\,\varphi_1(x,\lambda)\,\nabla x+\beta_1\psi_1^{\Delta}(t,\lambda), & t\in[a,c),\\ \frac{\varphi_2^{\Delta}(t,\lambda)}{D(\lambda)}\delta\int_{\rho(t)}^{b}h(x)\,\psi_2(x,\lambda)\,\nabla x+\alpha_2\varphi_2^{\Delta}(t,\lambda)\\ +\frac{\psi_2^{\Delta}(t,\lambda)}{D(\lambda)}\delta\int_{c}^{\rho(t)}h(x)\,\varphi_2(x,\lambda)\,\nabla x+\beta_2\psi_2^{\Delta}(t,\lambda), & t\in(c,b].\end{cases}\tag{3.5}$$

From (2.21), we get

$$\beta_1=0.\tag{3.6}$$

By the condition (2.23), we obtain

$$\alpha_2=0.\tag{3.7}$$

From (3.6), (3.7), and (2.22), we get the following system:

$$\begin{cases}\beta_2\varphi_2(c+,\lambda)-d_1\alpha_1\psi_1(c-,\lambda)\\ =\frac{\varphi_2(c+,\lambda)}{D(\lambda)}\delta\int_{c}^{b}h(x)\,\psi_2(x,\lambda)\,\nabla x\\ -d_1\frac{\psi_1(c-,\lambda)}{D(\lambda)}\int_{a}^{c}h(x)\,\varphi_1(x,\lambda)\,\nabla x,\\ \beta_2p(c+)\,\varphi_2^{\Delta}(c+,\lambda)-\alpha_1d_2p(c-)\,\psi_1^{\Delta}(c-,\lambda)\\ =\frac{p(c+)\varphi_2^{\Delta}(c+,\lambda)}{D(\lambda)}\delta\int_{c}^{b}h(x)\,\psi_2(x,\lambda)\,\nabla x\\ -d_2\frac{p(c-)\psi_1^{\Delta}(c-,\lambda)}{D(\lambda)}\int_{a}^{c}h(x)\,\varphi_1(x,\lambda)\,\nabla x.\end{cases}$$

Since

$$\begin{vmatrix} \varphi_2(c+,\lambda) & -d_1\psi_1(c-,\lambda) \\ p(c+)\varphi_2^{\Delta}(c+,\lambda) & -d_2 p(c-)\psi_1^{\Delta}(c-,\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \varphi_2(c+,\lambda) & -\psi_2(c+,\lambda) \\ p(c+)\varphi_2^{\Delta}(c+,\lambda) & -p(c+)\psi_2^{\Delta}(c+,\lambda) \end{vmatrix}$$

$$= W_{2,\Delta}(\varphi_2,\psi_2) = -\tfrac{1}{\delta}D(\lambda),$$

we obtain

$$\alpha_1 = \frac{1}{D(\lambda)}\delta\int_c^b h(x)\psi_2(x,\lambda)\nabla x,$$

and

$$\beta_2 = \frac{1}{D(\lambda)}\int_a^c h(x)\varphi_1(x,\lambda)\nabla x.$$

Hence

$$z(t,\lambda) = \left(G(t,x,\lambda),\overline{h(t)}\right),\ t\in J,$$

where

$$G(t,x,\lambda) = \begin{cases} \dfrac{1}{D(\lambda)}\psi(t,\lambda)\varphi(x,\lambda), & a\le x\le t\le b,\ t\ne c,\ x\ne c \\ \dfrac{1}{D(\lambda)}\varphi(t,\lambda)\psi(x,\lambda), & a\le t\le x\le b,\ t\ne c,\ x\ne c. \end{cases} \tag{3.8}$$

4. Eigenfunction expansion

In this section, we shall give an eigenfunction expansion. In this context, we need the following definition and theorem.

Definition 4.1. Let $M(t,x)$ be a function in $\mathbb{C}^2$ where $a<t$ and $x<b$. If

$$\int_a^b\int_a^b |M(t,x)|^2\nabla t\nabla x < +\infty,$$

then $M(t,x)$ is called the ∇-Hilbert–Schmidt kernel.

Theorem 4.1. [19] *Let A be an operator defined as*

$$A\{x_i\} = \{y_i\}, \tag{4.1}$$

where

$$y_i = \sum_{k=1}^{\infty} a_{ik}x_k,\ i,k\in\mathbb{N} := \{1,2,3,,,\}. \tag{4.2}$$

If

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 < +\infty, \tag{4.3}$$

then A is the compact operator in l^2,where l^2 is the standard sequence space.

There is no loss of generality in assuming that $\lambda = 0$ is not an eigenvalue. Thus we have

$$G(t,x) := G(t,x,0)$$

$$= \begin{cases} -\dfrac{1}{W_\Delta(\varphi,\psi)}\psi(t)\varphi(x), & a \le x \le t \le b,\ t \ne c,\ x \ne c \\ -\dfrac{1}{W_\Delta(\varphi,\psi)}\varphi(t)\psi(x), & a \le t \le x \le b,\ t \ne c,\ x \ne c. \end{cases} \tag{4.4}$$

Theorem 4.2. *$G(t,x)$ defined by the formula (4.4) is a ∇-Hilbert–Schmidt kernel.*

Proof. It follows from (4.4) that

$$\int_a^b \nabla_\alpha t \int_a^t |G(t,x)|^2 \nabla x < +\infty,$$

and

$$\int_a^b \nabla_\alpha t \int_t^b |G(t,x)|^2 \nabla x < +\infty,$$

since the inner integral exists and the product $\psi(t)\varphi(x)$ belongs to $L^2_\nabla(J) \times L^2_\nabla(J)$. Then, we see that

$$\int_a^b \int_a^b |G(t,x)|^2 \nabla t \nabla x < +\infty. \tag{4.5}$$

□

Theorem 4.3. *Let $\mathcal{L}$ be an operator defined as*

$$(\mathcal{L}h)(t) = \left(G(t,x), \overline{h(x)}\right).$$

Then the operator $\mathcal{L}$ is self-adjoint and compact.

Proof. Let $f, g \in L^2_\nabla(J)$. Since $G(t,x) = G(x,t)$ we obtain

$$(\mathcal{L}f, g) = \int_a^b (\mathcal{L}f)(t)\overline{g(t)}\nabla t$$

$$= \int_a^b \left[\int_a^b G(t,x) f(x) \nabla x\right] \overline{g(t)} \nabla t$$

$$= \int_a^b f(x) \left[\int_a^b G(x,t)\overline{g(t)}\nabla t\right] \nabla x = (f, \mathcal{L}g).$$

Let

$$\phi_i = \phi_i\,(t) = \begin{cases} \phi_i^{(1)}(t), & t \in [a,c) \\ \phi_i^{(2)}(t), & t \in (c,b] \end{cases} \quad (i \in \mathbb{N})$$

be a complete, orthonormal basis of $L^2_\nabla(J)$. From Theorem 4.2, we can define

$$\begin{aligned} x_i &= (f,\phi_i) = \textstyle\int_a^c f^{(1)}\,(t)\,\overline{\phi_i^{(1)}\,(t)}\nabla t + \delta \int_c^b f^{(2)}\,(t)\,\overline{\phi_i^{(2)}\,(t)}\nabla t, \\ y_i &= (g,\phi_i) = \textstyle\int_a^c g^{(1)}\,(t)\,\overline{\phi_i^{(1)}\,(t)}\nabla t + \delta \int_c^b g^{(2)}\,(t)\,\overline{\phi_i^{(2)}\,(t)}\nabla t, \\ a_{ik} &= \textstyle\int_a^c \int_a^c G\,(x,t)\,\overline{\phi_i^{(1)}\,(x)\phi_k^{(1)}\,(t)}\nabla x \nabla t \\ &\quad + \delta^2 \textstyle\int_c^b \int_c^b G\,(x,t)\,\overline{\phi_i^{(2)}\,(x)\phi_k^{(2)}\,(t)}\nabla x \nabla t \ (i,k \in \mathbb{N}). \end{aligned}$$

It is clear that $L^2_\nabla(J)$ is mapped isometrically the space l^2. Consequently, our integral operator transforms into the operator defined by the formula (4.1) in the space l^2 by this mapping, and the condition (4.5) is translated into the condition (4.3). Hence this operator and with it, the original operator is compact. □

Theorem 4.4. *The eigenvalues of the operator L form an infinite sequence* $\{\lambda_i\}_{i=1}^{\infty}$ *of real numbers which can be ordered so that*

$$|\lambda_1| < |\lambda_2| < ... < |\lambda_i| < ..., \ |\lambda_i| \to \infty \ \text{as } i \to \infty.$$

The set of all normalized eigenfunctions of L forms an orthonormal basis for $L^2_\nabla(J)$ *and for* $z \in L^2_\nabla(J)$, $\mathcal{L}z = h$, $Lh = z$, $L\phi_i = \lambda_i\phi_i$ $(i \in \mathbb{N})$ *the eigenfunction expansion formula*

$$Lh = \sum_{i=1}^{\infty} \lambda_i (h,\phi_i)\phi_i$$

is valid.

Proof. From Theorem 4.3 and the Hilbert–Schmidt theorem (see [14]), we see that $\mathcal{L}$ has an infinite sequence of non-zero real eigenvalues $\{\eta_i\}_{i=1}^{\infty}$ with $\eta_i \to 0$ $(i \to \infty)$. Then, we obtain

$$|\lambda_i| = \frac{1}{|\eta_i|} \to \infty \text{ as } i \to \infty.$$

Furthermore, let $\{\phi_i\}_{i=1}^{\infty}$ denote an orthonormal set of eigenfunctions corresponding to $\{\eta_i\}_{i=1}^{\infty}$. Hence, we get

$$\begin{aligned} z &= Lh = \textstyle\sum_{i=1}^{\infty} (z,\phi_i)\,\phi_i = \sum_{i=1}^{\infty} (Lh,\phi_i)\,\phi_i \\ &= \textstyle\sum_{i=1}^{\infty} (h, L\phi_i)\,\phi_i = \sum_{i=1}^{\infty} \lambda_i\,(z,\phi_i)\,\phi_i. \end{aligned}$$

□

Statements and Declarations. This work does not have any conflict of interest.

Availability of data and materials. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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