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CONVERGENCE OF COMPLEX UNCERTAIN TRIPLE SEQUENCE VIA METRIC OPERATOR, p-DISTANCE AND COMPLETE CONVERGENCE

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Abstract. In this paper, we have introduced three new types of convergence concepts, namely convergence in p-distance, completely convergence and convergence in metric by considering triple sequences of complex uncertain variable. We have established the interrelationships among these notions and also with the existing ones. In this process, we have proven that the notions of convergence in metric and convergence in almost surely are equivalent in nature. Overall, this study presents a more complete scenario of interconnections between the notions of convergences initiated in different directions.
Keywords: uncertainty space, complex uncertain sequence, uncertain uncertain metric, p-distance, complete convergence

1. Introduction

Liu [15] first introduced the notion of uncertain sequence and its four types of convergences: namely convergence in measure via the set function uncertain measure, convergence in mean by means of expected value operator, convergence in distribution using the uncertainty distribution function, and convergence in almost surely by reducing the collection of all concerned uncertain events to such sub-collection whose uncertain measure is unity. Later, You [22] came up with another type of convergent uncertain sequence called uniformly almost surely convergent uncertain sequence by reducing the whole uncertainty space to the sub-collection

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constructed by excluding such events from the whole space whose uncertain measure approaches zero. He proved that every uncertain sequence which converges with respect to uniformly almost surely must converge in almost surely to the same limit. Chen et al. [3] extended the study of the above mentioned five types of convergence of real uncertain sequences to the sequence of complex uncertain variable due to Peng [17]. Thenafter, You and Yan [23] initiated two new types of convergence, namely convergence in p-distance and completely convergence of uncertain sequence. Ye and Zhu [21] redefined the concept of metric on uncertain variable given by Liu [15] and then studied convergence concept via the newly introduced metric. Few more studies on convergence of sequences of real/complex uncertain variable may be seen in [1, 2, 14].

In recent days researchers have been trying to expand the study of uncertain sequence spaces by considering double/triple sequences of complex uncertain variable. For instance, Saha et al. [18] studied almost convergence concept of complex uncertain sequences and the same has been introduced in double and triple sequences of complex uncertain variable by Das et al. [6, 12] and Nath et el. [16]. On the other hand, Tripathy and Nath [20] studied the notion of statistical convergence of complex uncertain sequences and again this concept was explored in double and triple sequences of complex uncertain variable by Das et al. [7, 8, 9]. Few characterizations of convergent complex uncertain sequences via matrix transformation were made by Das et al. [5, 10, 11] by introducing the notions of convergent complex uncertain series.

In this study, our aim is to introduce the concept of completely convergence, convergence in metric and convergence in p-distance in triple sequences of complex uncertain variable. We have examined the interconnections among these notions, also with the existing convergence concepts due to Das et al. (2021) and present the same in the form of a diagram. Before going to the main results section, we will give some ideas and results in the following preliminary section which are going to play an important role in the whole study.

Also, we need some basic and preliminary ideas about the existing definitions and results which will play a major role in this study.

2. Preliminaries

In this section we give preliminary ideas of uncertain measure, complex uncertain variable, expected value of uncertain variabl and uncertainty distribution which are used for calculation purpose in the whole research. Also, we state the definitions of convergent complex uncertain double sequences in mean, in measure, in distribution, in almost surely and with respect to uniformly almost surely.

Definition 2.1 [15] Let $\mathcal{L}$ be a σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:

Axiom 1 (Normality Axiom). $\mathcal{M}\{\Gamma\}=1$;

Axiom 2 (Duality Axiom). $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\}=1$, for any $\Lambda \in \mathcal{L}$;

Axiom 3 (Subadditivity Axiom). For every countable sequence $\{\Lambda_j\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space and each element Λ in $\mathcal{L}$ is called an event. In order to obtain an uncertain measure of compound events, a product uncertain measure is defined as follows:

Axiom 4 (Product Axiom). Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces, for k = 1,2,3..... The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{j=1}^{\infty} \Lambda_j\right\} = \bigwedge_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\},$$

where Λ_j are arbitrarily chosen events from Γ_j, for $j = 1, 2, 3, ...$ respectively.

Definition 2.2 [17] A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers. i.e., for any Borel set B of complex numbers, the set $\{\zeta \in B\}=\{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.

Definition 2.3 [15] The expected value operator of an uncertain variable ζ is defined by

$$E[\zeta] = \int_{0}^{+\infty} \mathcal{M}\{\zeta \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\zeta \leq r\}dr,$$

provided that at least one of the two integrals is finite.

Definition 2.4 [15] The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\},$$

for any real number x.

Definition 2.5 [21] The metric between uncertain variables ξ and η is defined as follows

$$D(\xi, \eta) = \inf\{x : \mathcal{M}\{|\xi - \eta| \leq x\} = 1\}.$$

Definition 2.6 [23] Let ζ and τ be complex uncertain variables. Then the p-distance between ζ and τ is defined by

$$D_p(\zeta, \tau) = (E[||\zeta - \tau||^p])^{\frac{1}{p+1}}, \quad p > 0.$$

Definition 2.7 [4] The complex uncertain triple sequence $\{\zeta_{lmn}\}$ is said to be convergent almost surely (a.s.) to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{l,m,n\to\infty} ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| = 0,$$

for every $\gamma \in \Lambda$.

Definition 2.8 [4] The complex uncertain triple sequence $\{\zeta_{lmn}\}$ is said to be convergent in measure to ζ if

$$\lim_{l,m,n\to\infty} \mathcal{M}\{||\zeta_{lmn} - \zeta|| \geq \varepsilon\} = 0,$$

for every given $\varepsilon > 0$.

Definition 2.9 [4] Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables with finite expected values. Then the complex uncertain double sequence $\{\zeta_{lmn}\}$ is said to be convergent in mean to ζ if

$$\lim_{l,m,n\to\infty} E[||\zeta_{lmn} - \zeta||] = 0.$$

Definition 2.10 [4] Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$, respectively. We say the complex uncertain triple sequence $\{\zeta_{lmn}\}$ converges in distribution to ζ if

$$\lim_{l,m,n\to\infty} \Phi_{lmn}(c) = \Phi(c).$$

for all c at which $\Phi(c)$ is continuous.

Definition 2.11 [4] The triple sequence $\{\zeta_{lmn}\}$ of complex uncertain variable is said to be convergent uniformly almost surely to ζ if there exists a sequence of events $\{E_k\}$ with $\mathcal{M}\{E_k\} \to 0$ for each x such that $\{\zeta_{lmn}\}$ converges uniformly to ζ in $\Gamma - E_k$, for any fixed $k \in \mathbb{N}$.

Definition 2.12 [15] Let $\Lambda_1, \Lambda_2, ...$ be a sequence of events with $\mathcal{M}\{\Lambda_i\} = 1$ for $i = 1, 2, ...$, respectively. Then

$$\mathcal{M}\left\{\bigcap_{i=1}^{\infty} \Lambda_i\right\} = \mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} = 1.$$

3. Convergence in Metric, p-distance and Complete Convergence

This section starts with providing the formal definition of three new types of convergences of complex uncertain triple sequences, namely completely convergence,

convergence in p-distance and convergence in metric.
Before defining convergent complex uncertain double sequence in metric, we need to give the idea of metric between two complex uncertain variables.

Definition 3.1 If α and β be any two complex uncertain variables then the metric between them may be given by

$$D(\alpha, \beta) = \inf\{x : \mathcal{M}\{||\alpha - \beta|| \leq x\} = 1\}.$$

Definition 3.2 A complex uncertain triple sequence $\{\zeta_{lmn}\}$ is said to convergent in metric and it converges to a finite limit ζ if

$$\lim_{l,m,n\to\infty} D(\zeta_{lmn}, \zeta) = 0.$$

Definition 3.3 A complex uncertain triple sequence $\{\zeta_{lmn}\}$ is said to be convergent in p-distance to ζ if the following condition is satisfied:

$$\lim_{l,m,n\to\infty} D_p(\zeta_{lmn}, \zeta) = 0.$$

Definition 3.4 A complex uncertain triple sequence $\{\zeta_{lmn}\}$ is said to be completely convergent to a finite limit ζ if for any preassigned $\varepsilon > 0$

$$\lim_{l,m,n\to\infty} \left(\sum_{x=l}^{\infty} \sum_{y=m}^{\infty} \sum_{z=n}^{\infty} \mathcal{M}\{||\zeta_{xyz} - \zeta|| \geq \varepsilon\} \right) = 0.$$

4. Interrelationship between Different Convergent Concepts

In this section, we establish the interrelationships between several existing convergence concepts with the newly introduced notions of convergence. At the end of the section, we depict a diagram which presents a more complete version of interrelationships between these convergence concepts.

Theorem 4.1 If a complex uncertain triple sequence is convergent in p-distance to ζ, the it converges in measure with limit being preserved.
Proof: Consider the triple sequence $\{\zeta_{lmn}\}$ of complex uncertain triple sequence which converges to ζ in p-distance.
Then, by definition of convergence in p-distance

$$\lim_{l,m,n\to\infty} D_p(\zeta_{lmn}, \zeta) = 0.$$

This implies that

$$\lim_{l,m,n\to\infty} \{E\left[||\zeta_{lmn} - \zeta||^p\right]\}^{\frac{1}{p+1}} = 0. \tag{3.1}$$

From equation 3.4, we have

$$\lim_{l,m,n\to\infty} E\left[||\zeta_{lmn}-\zeta||^p\right]=0.$$

Let $\varepsilon>0$ be given.
Then by Markov inequality,

$$\mathcal{M}\{||\zeta_{lmn}-\zeta||\geq\varepsilon\}\leq\frac{E[||\zeta_{lmn}-\zeta||^p]}{\varepsilon^p}.$$

Therefore, $\lim\limits_{l,m,n\to\infty}\mathcal{M}\{||\zeta_{lmn}-\zeta||\geq\varepsilon\}$
$\leq\lim\limits_{l,m,n\to\infty}\frac{E[||\zeta_{lmn}-\zeta||^p]}{\varepsilon^p}$
$=\frac{1}{\varepsilon^p}\lim\limits_{l,m,n\to\infty}E[||\zeta_{lmn}-\zeta||^p]$
$=\frac{1}{\varepsilon^p}.0=0$
From the above inequality, we can conclude that

$$\lim_{l,m,n\to\infty}\mathcal{M}\{||\zeta_{lmn}-\zeta||\geq\varepsilon\}=0.$$

Hence, the complex uncertain triple sequence is convergent in measure to ζ.

Remark 4.2 A convergent complex uncertain triple sequence in measure doesn't necessarily imply that it is convergent in p-distance.
The validity of the statement is verified in the following example.

Example 4.3 Let us consider the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ where $\Gamma=\{\gamma_1,\gamma_2,...\}$, $\mathcal{L}=P(\Gamma)$ ($P(\gamma)$ being the power set of Γ) and the set function $\mathcal{M}$ which is called uncertain measure is defined by $\mathcal{M}\{\gamma_p\}=\frac{1}{2p},\ \ p=1,2,...$
Define the complex uncertain triple sequence $\{\zeta_{lmn}\}$ ran by three indices by

$$\zeta_{lmn}(\gamma_k)=\begin{cases}2ki, & \text{if } lmn=k;\\ 0, & \text{otherwise};\end{cases}$$

and consider the complex uncertain variable $\zeta\equiv 0$.
Thus, for any $\varepsilon>0$, the complex uncertain triple sequence $\{\zeta_{nml}\}$ satisfies the following:

$$\mathcal{M}\{||\zeta_{lmn}-\zeta||\geq\varepsilon\}=\mathcal{M}\{\gamma:||\zeta_{lmn}(\gamma)-\zeta(\gamma)||\geq\varepsilon\}=\mathcal{M}\{\gamma_{lmn}\}=\frac{1}{2lmn}.$$

Therefore, $\lim\limits_{l,m,n\to\infty}\mathcal{M}\{||\zeta_{lmn}-\zeta||\geq\varepsilon\}=\lim\limits_{l,m,n\to\infty}\frac{1}{2lmn}=0.$
Thus $\{\zeta_{lmn}\}$ converges in measure to ζ.
Again, for each $l,m,n\geq 1$, the uncertainty distribution function of the real uncertain variable $||\zeta_{lmn}-\zeta||=||\zeta_{lmn}||$ is given by

$$\Phi_{lmn}(x)=\begin{cases}0 & \text{if } x<0;\\ 1-\frac{1}{2lmn} & \text{if } 0\leq x<2lmn;\\ 1, & \text{if } x\geq 2lmn.\end{cases}$$

Then calculation for the expected value of $||\zeta_{lmn}-\zeta||$ gives

$$E[||\zeta_{lmn}-\zeta|| = E[||\zeta_{lmn}||] = \int\limits_{2lmn}^{\infty}(1-1)dx + \int\limits_{0}^{2lmn}\left(1-\left(1-\tfrac{1}{2lmn}\right)\right)dx = 1.$$

Then, $E[||\zeta_{lmn}-\zeta||^p]$
$= \int\limits_0^\infty \mathcal{M}\{||\zeta_{lmn}-\zeta||^p \geq x\}dx$
$\geq \int\limits_0^\infty \mathcal{M}\{||\zeta_{lmn}-\zeta|| \geq x\}dx$
$= E[||\zeta_{lmn}-\zeta||] = E[||\zeta_{lmn}||] = 1.$
Consequently,

$$\lim_{l,m,n\to\infty} D_p(\zeta_{lmn},\zeta) = \lim_{l,m,n\to\infty}\left\{E[||\zeta_{lmn}-\zeta||^p]^{\frac{1}{p+1}}\right\} \geq 1 \neq 0.$$

Hence, the triple sequence $\{\zeta_{lmn}\}$ doesn't converge to ζ in p-distance.

Datta and Tripathy [13] have already proved the interrelationship between convergence in measure and convergence in distribution in a double sequence. In this context, in the following we state the same for complex uncertain triple sequence, as it is a direct consequence.

Theorem 4.4 A complex uncertain triple sequence which is convergent in measure is also converges to the same limit in distribution.
Proof: Let $\{\zeta_{nml}\}$ be a convergent triple sequence of complex uncertain variable with $\zeta_{nml} = \xi_{nml} + i\eta_{nml}$ in measure. Then ξ_{nml}, η_{nml} both converges to ξ, η respectively in measure.
Then for any given $\varepsilon > 0$, we have

$$\mathcal{M}\left\{||\xi_{nml}-\xi|| \geq \alpha - a\right\} < \tfrac{\varepsilon}{2}$$

and

$$\mathcal{M}\left\{||\eta_{nml}-\eta|| \geq \beta - b\right\} < \tfrac{\varepsilon}{2}$$

Let $z = a + ib$ be a point of continuity of the complex uncertainty distribution Φ. Then for any $\varepsilon > 0$, there exist $\alpha > a, \beta > b$, such that

$$\mathcal{M}\{||\xi_{nml}-\xi|| \geq \alpha - a\} \leq \tfrac{\varepsilon}{2}$$

and

$$\mathcal{M}\{||\eta_{nml}-\eta|| \geq \beta - b\} \leq \tfrac{\varepsilon}{2}.$$

Therefore, for the real numbers $\alpha > a, \beta > b$,

$$\begin{aligned}\{\xi_{nml} \leq a, \eta_{nml} \leq b\} = &\{\xi_{nml} \leq a, \eta_{nml} \leq b, \xi \leq \alpha, \eta \leq \beta\}\\ &\cup \{\xi_{nml} \leq a, \eta_{nml} \leq b, \xi > \alpha, \eta > \beta\}\\ &\cup \{\xi_{nml} \leq a, \eta_{nml} \leq b, \xi > \alpha, \eta \leq \beta\}\end{aligned}$$

$$\subseteq \{\xi_{nml} \leq \alpha, \eta_{nml} \leq \beta\}$$
$$\cup \{|\xi_{nml} - \xi| \geq \alpha - a\} \cup \{|\eta_{nml} - \eta| \geq \beta - b\}.$$

Applying subadditivity axiom of uncertain measure, we get
$\Phi_{nml}(z) = \Phi_{nml}(a + ib)$
$\leq \Phi(\alpha + i\beta) + \mathcal{M}\{||\xi_{nml} - \xi|| \geq \alpha - a\} + \mathcal{M}\{||\eta_{nml} - \eta|| \geq \beta - b\}$
$< \Phi(\alpha + i\beta) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \Phi(\alpha + i\beta) + \varepsilon.$
Thus, $\Phi_{nml}(z) \leq \Phi(\alpha + i\beta) + \varepsilon$ and hence $\sup \Phi_{nml}(z) \leq \Phi(\alpha + i\beta)$, for all $\alpha > a, \beta > b$.
Considering $\alpha \to a, \beta \to b$, we get $\alpha + i\beta \to a + ib$ and so,

$$\lim_{n,m,l\to\infty} \sup \Phi_{nml}(z) \leq \Phi(z). \tag{3.2}$$

Again $\{\xi \leq x, \eta \leq y\}$
$= \{\xi_{nml} \leq a, \eta_{nml} \leq b, \xi \leq x, \eta \leq y\}$
$\cup \{\xi_{nml} \leq a, \eta_{nml} > b, \xi \leq x, \eta \leq y\}$
$\cup \{\xi_{nml} > a, \eta_{nml} \leq b, \xi \leq x, \eta \leq y\}$
$\cup \{\xi_{nml} > a, \eta_{nml} > b, \xi \leq x, \eta \leq y\}$
$\subseteq \{\xi_{nml} \leq a, \eta_{nml} \leq b\} \cup \{|\xi_{nml} - \xi| \geq a - x\} \cup \{|\eta_{nml} - \eta| \geq b - y\}.$
That gives like the previous step, we can write
$\Phi(x + iy) \leq \Phi_{nml}(a + ib) + \mathcal{M}\{||\xi_{nml} - \xi|| \geq a - x\}$
$+ \mathcal{M}\{||\eta_{nml} - \eta|| \geq b - y\}$
$< \Phi_{nml}(a + ib) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \Phi_{nml}(a + ib) + \varepsilon$, for all $x < a, y < b$.
Hence, $\Phi(x + iy) \leq \inf \Phi_{nml}(z)$ and taking $x \to a, y \to b$, we get

$$\Phi(z) \leq \lim_{n,m,l\to\infty} \inf \Phi_{nml}(z). \tag{3.3}$$

From the Equation (3.2) and the Equation (3.3), we conclude that $\Phi_{nml}(z) \to \Phi(z)$, as $n, m, l \to \infty$. Therefore, the complex uncertain triple sequence $\{\zeta_{nml}\}$ almost converges in distribution to $\zeta = \xi + i\eta$.

Theorem 4.5 A convergent complex uncertain triple sequence p-distance converges in distribution by preserving the limit of convergence.
Proof: This is obvious observing the theorem 4.1 and theorem 4.4.

Remark 4.6 The convergence of a complex uncertain triple sequence in distribution doesn't necessarily imply its convergence in p-distance. The claim has been demonstrated in the example given below.

Example 4.7 Let us consider uncertainty space $\Gamma = \{\gamma_1, \gamma_2\}$ with the σ-algebra $\mathcal{L} = P(\Gamma)$ and the uncertain measure $\mathcal{M}$ be defined by $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = \frac{1}{2}$.
Let the complex uncertain triple sequence $\{\zeta_{lmn}\}$ be defined by

$$\zeta_{lmn}(\gamma_k) = \begin{cases} i, & \text{if } k = 1; \\ -i, & \text{if } k = 2. \end{cases}$$

Suppose $\zeta \equiv -\zeta_{lmn}$.
Then, we obtain complex uncertainty distribution functions Φ_{lmn}, Φ of both the

variables as follows:

$$\Phi(z) = \Phi_{lmn}(z) = \Phi_{lmn}(a+ib) = \begin{cases} 0, & if \quad a < 0, b \in (-\infty, \infty); \\ 0, & if \quad a \geq 0, b < -1; \\ 0.2, & if \quad a \geq 0, -1 \leq b < 1; \\ 0.7, & if \quad a \geq 0, 1 \leq b < 2; \\ 1, & if \quad a \geq 0, 1, b \geq 2. \end{cases}$$

Since both the distribution functions are identical, conclusion can be made that the complex uncertain triple sequence $\{\zeta_{lmn}\}$ is convergent in distribution to ζ.
On the contrary, $||\zeta_{lmn}(\gamma) - \zeta(\gamma)||_p = ||2i||^p = 2^p, \quad$ for $\gamma = \gamma_1, \gamma_2$
and $E[||\zeta_{lmn} - \zeta||_p] = \int\limits_0^{2^p} 1 \; dx = 2^p$.
Thus,

$$\lim_{l,m,n\to\infty} D_p(\zeta_{lmn}, \zeta) = \lim_{l,m,n\to\infty} E[||\zeta_{lmn} - \zeta||^p]^{\frac{1}{p+1}} = 2^{\frac{p}{p+1}} \neq 0.$$

Hence, the complex uncertain triple sequence is not convergent in p-distance to ζ.

We now establish the interrelationships of completely convergent complex uncertain triple sequences with its convergence in uniformly almost surely, almost surely and measure.

Theorem 4.8 If a complex uncertain triple sequence $\{\zeta_{lmn}\}$ is completely convergent to ζ, then it converges with respect to uniformly almost surely therein.
Proof: Let $\{\zeta_{lmn}\}$ be the complex uncertain triple sequence which completely converges to ζ.
Then for any preassigned $\varepsilon > 0$,

$$\lim_{l,m,n\to\infty} \sum_{x=l}^{\infty} \sum_{y=m}^{\infty} \sum_{z=n}^{\infty} \mathcal{M}\{||\zeta_{xyz} - \zeta|| \geq \varepsilon\} = 0.$$

Now, applying the subadditivity axiom of uncertain measure we get,

$$\mathcal{M}\left\{\bigcup_{x=l}^{\infty} \bigcup_{y=m}^{\infty} \sum_{z=n}^{\infty} \{||\zeta_{xyz} - \zeta|| \geq \varepsilon\}\right\} \leq \sum_{x=l}^{\infty} \sum_{y=m}^{\infty} \sum_{z=n}^{\infty} \mathcal{M}\{||\zeta_{xyz} - \zeta|| \geq \varepsilon\}.$$

Taking limits of l, m and n to infinity, we get
$\lim\limits_{l,m,n\to\infty} \mathcal{M}\left\{\bigcup\limits_{x=l}^{\infty} \bigcup\limits_{y=m}^{\infty} \bigcup\limits_{z=n}^{\infty} \{||\zeta_{xyz} - \zeta|| \geq \varepsilon\}\right\} \leq \lim\limits_{l,m,n\to\infty} \sum\limits_{x=l}^{\infty} \sum\limits_{y=m}^{\infty} \sum\limits_{z=n}^{\infty} \mathcal{M}\{||\zeta_{xyz} - \zeta|| \geq \varepsilon\} = 0.$
Hence, the complex uncertain triple sequence is convergent in uniformly almost surely to ζ.

Theorem 4.9 A completely convergent complex uncertain triple sequence converges in almost surely therein.

Proof: Let the complex uncertain triple sequence $\{\zeta_{lmn}\}$ be converges completely to the limit ζ.

Then for any $\varepsilon > 0$,

$$\lim_{l,m,n\to\infty}\sum_{x=l}^{\infty}\sum_{y=m}^{\infty}\sum_{z=n}^{\infty}\mathcal{M}\{||\zeta_{xyz}-\zeta||\geq\varepsilon\}=0.$$

Now, $\mathcal{M}\left\{\bigcap_{l,m,n=1}^{\infty}\bigcup_{x=l}^{\infty}\bigcup_{y=m}^{\infty}\bigcup_{z=n}^{\infty}\{||\zeta_{xyz}-\zeta||\geq\varepsilon\}\right\}$

$\leq \mathcal{M}\left\{\bigcup_{x=l}^{\infty}\bigcup_{y=m}^{\infty}\bigcup_{z=n}^{\infty}\{||\zeta_{xyz}-\zeta||\geq\varepsilon\}\right\}$

$\leq \sum_{x=l}^{\infty}\sum_{y=m}^{\infty}\sum_{z=n}^{\infty}\mathcal{M}\{||\zeta_{xyz}-\zeta||\geq\varepsilon\}$

Then

$\lim_{l,m,n\to\infty}\mathcal{M}\left\{\bigcap_{l,m,n=1}^{\infty}\bigcup_{x=l}^{\infty}\bigcup_{y=m}^{\infty}\bigcup_{z=n}^{\infty}\{||\zeta_{xyz}-\zeta||\geq\varepsilon\}\right\}\leq\lim_{l,m,n\to\infty}\sum_{x=l}^{\infty}\sum_{y=m}^{\infty}\sum_{z=n}^{\infty}\mathcal{M}\{||\zeta_{xyz}-$ $\zeta||\geq\varepsilon\}=0.$

Hence,

$$\lim_{l,m,n\to\infty}\mathcal{M}\left\{\bigcap_{l,m,n=1}^{\infty}\bigcup_{x=l}^{\infty}\bigcup_{y=m}^{\infty}\bigcup_{z=n}^{\infty}\{||\zeta_{xyz}-\zeta||\geq\varepsilon\}\right\}=0.$$

Consequently, $\{\zeta_{lmn}\}$ converges to ζ in almost surely.

Remark 4.10 The converse of the above theorem is not true. This is illustrated in the example given below.

Example 4.11 Consider an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ with $\Gamma=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$. Let $\mathcal{M}$ be defined by

$$\mathcal{M}\{\Lambda\}=\begin{cases}0, & \text{if } \Lambda=\phi;\\ 1, & \text{if } \Lambda=\Gamma;\\ 0.6, & \text{if } \gamma_1\in\Lambda;\\ 0.4, & \text{if } \gamma_1\notin\Lambda.\end{cases}$$

Define the complex uncertain variables ζ_{lmn} and ζ by

$$\zeta_{lmn}(\gamma)=\begin{cases}i, & \text{if } \gamma=\gamma_1;\\ 2i, & \text{if } \gamma=\gamma_2;\\ 3i, & \text{if } \gamma=\gamma_3;\\ 4i, & \text{if } \gamma=\gamma_4;\\ 0, & \text{otherwise.}\end{cases}$$

for $l,m,n\in\mathbb{N}$ and $\zeta(\gamma)=0,\ \forall\ \gamma\in\Gamma$.

It is obvious that $\zeta_{lmn}\equiv\zeta$, except only for $\gamma=\gamma_1,\gamma_2,\gamma_3,\gamma_4$ and so the triple

sequence $\{\zeta_{lmn}\}$ is convergent to ζ with respect to almost surely.
On the contrary, for any small positive ε,

$$\mathcal{M}\{||\zeta_{lmn} - \zeta|| \geq \varepsilon\} = \mathcal{M}\{\gamma : ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\} = 1.$$

Therefore, $\sum\limits_{x=l}^{\infty}\sum\limits_{y=m}^{\infty}\sum\limits_{z=n}^{\infty}\mathcal{M}\{||\zeta_{xyz} - \zeta|| \geq \varepsilon\} = \infty$ and so

$$\lim_{l,m,n\to\infty}\sum_{x=l}^{\infty}\sum_{y=m}^{\infty}\sum_{z=n}^{\infty}\mathcal{M}\{||\zeta_{xyz} - \zeta|| \geq \varepsilon\} = \infty.$$

Before establishing the relation between convergence in measure and complete convergence of complex uncertain triple sequences, we would like to state the following theorem for complex uncertain triple sequence, as it is established by the Datta and Tripathy [13] for complex uncertain double sequence.

Theorem 4.12 Convergence with respect to uniformly almost surely implies convergence in measure for a complex uncertain triple sequence.
Proof: Let the triple sequence $\{\zeta_{nml}\}$ of complex uncertain variable converges with respect to uniformly almost surely to ζ. Then for any $\varepsilon > 0$, there exists $\delta > 0$ and an event B with uncertain measure approaching zero and the triple sequence $\{\zeta_{nml}\}$ converges uniformly to ζ on $\Gamma - B$. That means, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$||\zeta_{nml}(\gamma) - \zeta(\gamma)|| < \delta, \quad \text{for all } n, m, l \geq n_0 \text{ and all } \gamma \in \Gamma - B.$$

Accordingly we have,

$$\bigcup_{n=x}^{\infty}\bigcup_{m=y}^{\infty}\bigcup_{l=z}^{\infty}\{||\zeta_{nml}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\} \subseteq B.$$

Applying the subadditivity axiom of uncertain measure, we obtain

$$\mathcal{M}\left\{\bigcup_{n=x}^{\infty}\bigcup_{m=y}^{\infty}\bigcup_{l=z}^{\infty}||\zeta_{nml}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\right\} \leq \mathcal{M}\{B\} < \delta.$$

Conversely, suppose

$$\mathcal{M}\left\{\bigcup_{n=x}^{\infty}\bigcup_{m=y}^{\infty}\bigcup_{l=z}^{\infty}||\zeta_{nml}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\right\} = 0 \text{ holds true.}$$

Then for any $\nu > 0$ and $a \geq 1$, there exists a positive integer a_s such that

$$\mathcal{M}\left\{\bigcup_{x=a_s}^{\infty}\bigcup_{y=a_s}^{\infty}\bigcup_{z=a_s}^{\infty}||\zeta_{nml} - \zeta|| \geq \tfrac{1}{a}\right\} < \tfrac{\nu}{2^a}.$$

Consider $B = \bigcup\limits_{a=1}^{\infty} \bigcup\limits_{x=a_s}^{\infty} \bigcup\limits_{y=a_s}^{\infty} \bigcup\limits_{z=a_s}^{\infty} \left\{||\zeta_{nml} - \zeta|| \geq \frac{1}{a}\right\}.$
Then
$\mathcal{M}\{B\} \leq \sum\limits_{a=1}^{\infty} \mathcal{M}\left\{\bigcup\limits_{x=a_s}^{\infty} \bigcup\limits_{y=a_s}^{\infty} \bigcup\limits_{z=a_s}^{\infty} ||\zeta_{nml} - \zeta|| \geq \frac{1}{a}\right\}$
$\leq \sum\limits_{a=1}^{\infty} \frac{\nu}{2^a} = \nu.$
Moreover, $\sup_{\gamma \in \Gamma - B} ||\zeta_{nml} - \zeta|| < \frac{1}{a}$, where $a = 1, 2, 3, ...$ and $n, m, l > a_s$.
Since the triple sequence $\{\zeta_{nml}\}$ converges uniformly almost surely to ζ.
Then for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathcal{M}\left\{\bigcup_{n=x}^{\infty} \bigcup_{m=y}^{\infty} \bigcup_{l=z}^{\infty} ||\zeta_{nml} - \zeta|| \geq \delta\right\} < \varepsilon$$

and $\mathcal{M}\left\{||\zeta_{nml}(\gamma) - \zeta(\gamma)|| \geq \delta\right\}$
$\leq \mathcal{M}\left\{\bigcup\limits_{n=x}^{\infty} \bigcup\limits_{m=y}^{\infty} \bigcup\limits_{l=z}^{\infty} ||\zeta_{nml}(\gamma) - \zeta(\gamma)|| \geq \delta\right\} < \varepsilon.$
Hence, the triple sequence $\{\zeta_{nml}\}$ converges in measure to ζ.

Theorem 4.13 If a complex uncertain triple sequence converges completely to certain limit then it converges in measure to the same value.
Proof: This is obvious from theorem 4.8 and theorem 4.12.

Remark 4.14 A convergent complex uncertain triple sequence in measure may not be complete convergent. This is verified in the following example.

Example 4.15 Consider the uncertainty space and complex uncertain triple sequence taken in the example 4.3.
It has been observed that the complex uncertain triple sequence $\{\zeta_{lmn}\}$ taken there is convergent in measure to $\zeta \equiv 0$.
For all $l, m, n \geq 2$, we have
$\lim\limits_{l,m,n\to\infty} \sum\limits_{x=l}^{\infty} \sum\limits_{y=m}^{\infty} \sum\limits_{z=n}^{\infty} \mathcal{M}\{||\zeta_{xyz} - \zeta|| \geq \varepsilon\}$
$= \lim\limits_{l,m,n\to\infty} \sum\limits_{x=l}^{\infty} \sum\limits_{y=m}^{\infty} \sum\limits_{z=n}^{\infty} \mathcal{M}\{\gamma : ||\zeta_{xyz}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}$
$= \lim\limits_{l,m,n\to\infty} \sum\limits_{x=l}^{\infty} \sum\limits_{y=m}^{\infty} \sum\limits_{z=n}^{\infty} \mathcal{M}\{\gamma_p\}$
$= \lim\limits_{l,m,n\to\infty} \sum\limits_{x=l}^{\infty} \sum\limits_{y=m}^{\infty} \sum\limits_{z=n}^{\infty} \frac{1}{2p}$
$= \infty.$
Hence, the complex uncertain triple sequence doesn't converge completely to ζ.

Our next aim is to examine the relation between a completely convergent complex uncertain triple sequence and a convergent triple sequence in distribution.

The following theorem is an extension to triple sequence from complex uncertain double sequence due to Datta and Tripathy [13].

Theorem 4.16 A complex uncertain triple sequence converging completely means that it converges in distribution.
Proof: The proof is straightforward from the Theorem 4.13 and the Theorem 4.4.

Remark 4.17 Converse of the above theorem is not true in general, which is demonstrated in the following example.

Example 4.18 Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ and the complex uncertain triple sequence as in example 4.7.
Here, $||\zeta_{lmn} - \zeta|| = ||2i|| = 2, \quad$ for $\gamma = \gamma_1, \gamma_2$.
For a given $2 > \varepsilon > 0$, we have

$$\mathcal{M}\{||\zeta_{lmn} - \zeta|| \geq \varepsilon\} = \mathcal{M}\{\gamma_1, \gamma_2\} = \mathcal{M}\{\Gamma\} = 1.$$

Therefore, $\lim\limits_{l,m,n\to\infty} \sum\limits_{x=l}^{\infty} \sum\limits_{y=m}^{\infty} \sum\limits_{z=n}^{\infty} \mathcal{M}\{||\zeta_{xyz} - \zeta|| \geq \varepsilon\} \neq 0$.
Hence the complex uncertain triple sequence is not completely convergent to ζ.
We now move further to the part concerning the notion of convergence in metric.

Theorem 4.19 The complex uncertain triple sequence $\{\zeta_{lmn}\}$ is convergent in metric to some finite limit ζ if and only if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ (depending on ε) such that

$$\mathcal{M}\{||\zeta_{lmn} - \zeta|| < \varepsilon\} = 1, \;\; \forall\, l, m, n \geq N.$$

Proof: Let us assume that the complex uncertain triple sequence $\{\zeta_{lmn}\}$ is convergent in metric to ζ.
Then, $\lim\limits_{l,m,n\to 0} D(\zeta_{lmn}, \zeta) = 0$.
Then, for any $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$ such that

$$D(\zeta_{lmn}, \zeta) + \frac{\varepsilon}{2} < \varepsilon, \qquad \forall\, l, m, n \geq N(\varepsilon). \tag{3.4}$$

From the definition 3.1 of uncertain metric,

$$\mathcal{M}\{||\zeta_{lmn} - \zeta|| < D(\zeta_{lmn}, \zeta) + \tfrac{\varepsilon}{2}\} = 1$$

and using the above equation 3.4 we get as a consequence

$$\mathcal{M}\{||\zeta_{lmn} - \zeta|| < \varepsilon\} = 1, \quad \forall\, l, m, n \geq N(\varepsilon).$$

Conversely, let us suppose for any given $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\mathcal{M}\{||\zeta_{lmn} - \zeta|| < \varepsilon\} = 1, \;\; \forall\, l, m, n \geq N.$$

The above equality holds true for all possible positive values of ε, means that the smaller ε we consider the only change it will make on N (the value of which will be larger, depending on ε). Thus we can conclude that the infimum of such ε will be zero and supremum of N will approach infinity.
Therefore, $\inf\{\varepsilon : \mathcal{M}\{\gamma : ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| \leq \varepsilon\} = 1\} = 0$
and in this case N tends to infinity.
As a consequence of this

$$\lim_{l,m,n\to\infty} D(\zeta_{lmn}, \zeta) = \lim_{l,m,n\to\infty} \inf\{x : \mathcal{M}\{\gamma : ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| \leq x\} = 1\} = 0.$$

Hence, the complex uncertain triple sequence $\{\zeta_{lmn}\}$ converges to ζ in metric.

Theorem 4.20 A convergent complex uncertain triple sequence in metric converges in measure therein.
Proof: Let $\{\zeta_{lmn}\}$ be a convergent complex uncertain triple sequence in metric to ζ.
Then from the above theorem 4.20, for any given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}\{||\zeta_{lmn} - \zeta|| < \varepsilon\} = 1.$$

Then, applying the duality axiom of uncertain measure to the above we get

$$\mathcal{M}\{||\zeta_{lmn} - \zeta|| \geq \varepsilon\} = 0, \quad \forall\, l, m, n \geq n_0$$

and so $\lim\limits_{l,m,n\to\infty} \mathcal{M}\{||\zeta_{lmn} - \zeta|| \geq \varepsilon\} = 0, \quad \forall\, l, m, n \geq n_0.$
Hence, the complex uncertain triple sequence is convergent in measure to ζ.

Remark 4.21 The converse case of the above theorem is not true. This is demonstrated in the following example.

Example 4.22 Let us take the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\Gamma = \{\gamma_1.\gamma_2, ...\}, \mathcal{L} = P(\Gamma)$ and the uncertain measure be defined by

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup_{\gamma_m \in \Lambda} \frac{1}{m}, & \text{if } \sup_{\gamma_m \in \Lambda} \frac{1}{m} < 0.5; \\ 1 - \sup_{\gamma_m \in \Lambda^c} \frac{1}{m}, & \text{if } \sup_{\gamma_m \in \Lambda^c} \frac{1}{m} < 0.5; \\ 0.5 & \text{otherwise.} \end{cases}$$

Consider the complex uncertain variable ζ_{lmn} defined as follows:

$$\zeta_{lmn}(\gamma) = \begin{cases} (l + m + n + 1)i, & \text{if } \gamma = \gamma_{l+m+n}; \\ 0, & \text{otherwise.} \end{cases}$$

for all $l, m, n \in \mathbb{N}$ and $\zeta(\gamma) = 0, \ \forall\, \gamma \in \Gamma$.
Here, the uncertainty distribution function Φ_{lmn} of the uncertain variable $||\zeta_{lmn}|| =$

$||\zeta_{lmn} - \zeta||$ is given by

$$\Phi_{lmn}(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - \frac{1}{l+m+n}, & \text{if } 0 \leq x < (l+m+n); \\ 1, & \text{if } x \geq (l+m+n). \end{cases}$$

Now, for any preassigned $\varepsilon, \delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $l, m, n \geq n_0$
$\mathcal{M}\{||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| \geq \delta\}$
$= \mathcal{M}\{\gamma : ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| \geq \delta\}$
$= \mathcal{M}\{\gamma = \gamma_{l+m+n}\} = \frac{1}{l+m+n} < \varepsilon$.
Thus the complex uncertain triple sequence $\{\zeta_{lmn}\}$ converges in measure to ζ.
Now, the metric between complex uncertain variables ζ_{lmn} and ζ is given by
$D(\zeta_{lmn}, \zeta)$
$= \inf\{x : \mathcal{M}\{||\zeta_{lmn} - \zeta|| \leq x\} = 1\}$
$= \inf\{x : \mathcal{M}\{||\zeta_{lmn}|| \leq x\} = 1\}$
$= \inf\{x : \Phi_{lmn}(x) = 1\}$
$= l + m + n$, which yields in the following

$$\lim_{l,m,n\to\infty} D(\zeta_{lmn}, \zeta) = \lim_{l,m,n\to\infty} (l + m + n) = \infty.$$

Therefore, the complex uncertain triple sequence doesn't converge to ζ in metric.

Theorem 4.23 If a complex uncertain triple sequence converges to some limit in metric, then it is convergent in mean therein.
Proof: Let the triple sequence $\{\zeta_{lmn}\}$ of complex uncertain variable converges to a limit ζ in metric.
Then

$$\lim_{l,m,n\to\infty} D(\zeta_{lmn}, \zeta) = 0 \tag{3.5}$$

i.e., for a given $\varepsilon > 0$ there exists a natural number $n_0 \in \mathbb{N}$ such that

$$D(\zeta_{lmn}, \zeta) < \varepsilon, \quad \forall\, l, m, n \geq n_0.$$

Now, $E[||\zeta_{lmn} - \zeta||] = \int\limits_0^{+\infty} (1 - \Phi_{lmn}(x))dx = \int\limits_0^{D+\varepsilon} (1 - \Phi_{lmn}(x))dx \leq 1.(D + \varepsilon) = D + \varepsilon$
and thus $E[||\zeta_{lmn} - \zeta||] \leq D(\zeta_{lmn}, \zeta)$.
Taking limits of l, m and n to infinity, we get

$$\lim_{l,m,n\to\to\infty} E[||\zeta_{lmn} - \zeta||] \leq \lim_{l,m,n\to\infty} D(\zeta_{lmn}, \zeta) = 0, \qquad \text{by equation 3.5}$$

Consequently, the complex uncertain triple sequence $\{\zeta_{lmn}\}$ converges to ζ in mean.

Example 4.24 We consider a complex uncertain variable ζ_{lmn} such that uncertainty distribution Φ_{lmn} of the real uncertain variable $||\zeta_{lmn}||$ is given by

$$\Phi_{lmn}(x) = \begin{cases} 0, & \text{if } x \leq 0; \\ 1 - \frac{1}{l^2 m^2 n^2}, & \text{if } 0 < x < l + m + n; \\ 1, & \text{otherwise}, \end{cases}$$

uniformly for all l, m, n.
Let ζ be the complex uncertain variable defined by $\zeta(\gamma) = 0$, for all $\gamma \in \Gamma$, where Γ is the whole uncertainty space.
Then, Φ_{lmn} becomes the uncertainty distribution of $||\zeta_{lmn} - \zeta||$.
Therefore, like in the above theorem, calculation to the expected value of $||\zeta_{lmn} - \zeta||$ via uncertainty distribution function gives us

$$E[||\zeta_{lmn} - \zeta||] = \tfrac{1}{l^2m^2n^2}.(l + m + n)$$

and so $\lim\limits_{l,m,n\to\infty} E[||\zeta_{lmn} - \zeta||] = \lim\limits_{l,m,n\to\infty} \tfrac{1}{l^2m^2n^2}.(l + m + n) = 0.$
Hence, $\{\zeta_{lmn}\}$ is convergent in mean.
On the other hand, the metric between the complex uncertain variable ζ_{lmn} and ζ is given by
$D(\zeta_{lmn}, \zeta)$
$= \inf\{x : \mathcal{M}\{||\zeta_{lmn} - \zeta|| \le x\} = 1\}$
$= \inf\{x : \mathcal{M}\{||\zeta_{lmn}|| \le x\} = 1\}$
$= \inf\{x : \Phi_{lmn}(x) = 1\}$
$= l + m + n$, and so

$$\lim_{l,m,n\to\infty} D(\zeta_{lmn}, \zeta) = \lim_{l,m,n\to\infty} (l + m + n) = \infty.$$

Therefore, the complex uncertain triple sequence doesn't converge to ζ in metric.

Theorem 4.25 A complex uncertain triple sequence is convergent in metric if it converges to the same limit with respect to almost surely.
Proof: Let $\{\zeta_{lmn}\}$ be convergent to ζ in almost surely. Then, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that for any $\varepsilon > 0$, there exists a positive integer $N(\varepsilon) > 0$ such that

$$||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| < \varepsilon, \text{ for all } l, m, n \ge N(\varepsilon) \text{ and } \gamma \in \Lambda.$$

Thus we have, $\Lambda \subseteq \{\gamma : ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| < \varepsilon\}$, for all $l, m, n \ge N(\varepsilon)$.
Therefore, $\mathcal{M}\{\Lambda\} \le \mathcal{M}\{\gamma : ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| < \varepsilon\}$,
which means

$$\mathcal{M}\{\gamma : ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| < \varepsilon\} \ge 1, \text{ since } \mathcal{M}\{\Lambda\} = 1.$$

Again, we know that uncertain measure lies between 0 and 1 and so

$$\mathcal{M}\{\gamma : ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| < \varepsilon\} = 1, \text{ for all } l, m, n \ge N(\varepsilon).$$

This implies $D(\zeta_{lmn}, \zeta) < \varepsilon$ and so $\{\zeta_{mn}\}$ converges in metric to ζ.
For the converse part, let the complex uncertain triple sequence $\{\zeta_{lmn}\}$ converges in metric to ζ.
Then, for any given positive integer p, there exists a positive integer $N(p)$ such that

$$\mathcal{M}\{\gamma : ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| < \tfrac{1}{p}\} = 1, \text{ for all } l, m, n > N.$$

Consider the subcollection $\Lambda_p^{lmn} = \{\gamma : ||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| < \frac{1}{p}\}$.
Therefore, $\mathcal{M}\{\Lambda_p^{lmn}\} = 1$.
We construct another uncertain event Λ as follows:

$$\Lambda = \bigcap_{l,m,n=1}^{\infty} \bigcap_{l=N(p)}^{\infty} \bigcap_{m=N(p)}^{\infty} \bigcap_{n=N(p)}^{\infty} \Lambda_p^{lmn}$$

Then, by using the theorem 2.12 we can say that $\mathcal{M}\{\Lambda\} = 1$.
Thus, $||\zeta_{lmn}(\gamma) - \zeta(\gamma)|| < \frac{1}{p}$, for all $l, m, n \geq N(p)$ and $\gamma \in \Lambda$ and hence the triple sequence $\{\zeta_{lmn}\}$ converges to ζ in almost surely.

Corollary 4.26 For a complex uncertain triple sequence the following implication holds.
(i) Completely convergence $\Rightarrow$ Convergence in metric
(ii) Convergence in uniformly almost surely $\Rightarrow$ Convergence in metric
(iii) Convergence in metric $\Rightarrow$ Convergence in distribution
(iv) Completely convergence $\Rightarrow$ Convergence in metric
Proof: (i) This is straightforward from theorem 4.9 and the above theorem 4.25.

(ii) Datta and Tripathy [13] proved that if a complex uncertain double sequence converges with respect to uniformly almost surely, then it is convergent in almost surely. It is very easy to verify that this holds true for complex uncertain triple also. Then applying the above theorem 4.25, the result is obvious.

(iii) By combining results given in theorem 4.20 and theorem 4.4, proof is straightforward.

(iv)] In the above theorem 4.8, we have already shown that a completely convergent complex uncertain double sequence converges in uniformly almost surely to the same limit. Then applying the above condition (ii), the result can be obtained in a go.

Remark 4.27 Combining the results established in this article and merging with the previous work of Datta and Tripathy [13], we can depict the interrelationships between several convergence concepts as follows.
At first we label the convergence concepts to simplify the diagram of interconnections (one way arrow means one way implication and the reverse implication doesn't hold):
(1) convergence in mean
(2) convergence measure
(3) convergence in distribution
(4) convergence almost surely
(5) convergence in uniformly almost surely

(6) completely convergence
(7) convergence in p-distance
(8) convergence in metric

The figure of the interrelationships is given below in the next page.

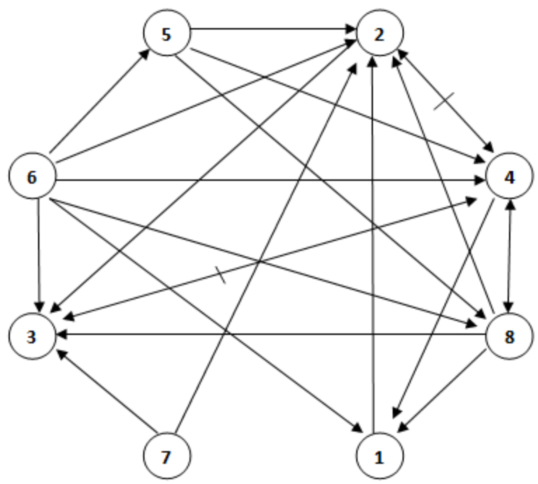

Fig. 3.1: Figure of interrelationships

4. Conclusion

In this article, we have introduced and investigated three new types of convergent triple sequences of complex uncertain variables. A comparative study has been made by producing a complete interconnection between several convergence concepts. Studies on complex uncertain sequences has been initiated in the last decade. Now it is drawing attention of researchers in various direction via generalization of results and this article will motivate for further investigation and application.

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