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SOME COMMON FIXED POINT THEOREMS IN COMPLETE WEAK PARTIAL METRIC SPACES INVOLVING AUXILIARY FUNCTIONS

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Abstract. In this paper, we establish some common fixed point theorems and a coincidence point theorem on complete weak partial metric spaces using auxiliary functions. We also give some examples in support of the results. The results proved in this paper extend and generalize several results from the existing literature.
Key words: fixed-point theorems, complete weak partial metric spaces, coincidence point theorem.

1. Introduction

The famous Banach contraction principle has been generalized in many directions, whether by generalizing the contractive condition or by extending the domain of the function. Matthews [22] introduced the concept of partial metric space as a part of the study of denotational semantics of dataflow networks [21, 22, 31, 33]. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation. In partial metric spaces the distance of a point in the self may not be zero. Introducing partial metric space, Matthews extended the Banach contraction principle [7] and proved the fixed point theorem in this space. The concept of partial Hausdorff metric was given by Aydi et al. [5] and they established a fixed point theorem for multivalued mappings in partial metric spaces. Excluding the idea of small self-distance Heckmann [16] generalized the partial metric space to weak partial metric space. Recall that Heckmann [16]

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has shown that, if p is a weak partial metric on X, then for all $x, y \in X$, we have the following weak small self-distance property:

$$p(x,y) \geq \frac{p(x,x)+p(y,y)}{2}.$$

Weak small self-distance property reflects that $WPMS$ are not far from small self-distance axiom. Clearly, every PMS is a $WPMS$, but not conversely. Some results have recently been obtained in [3], [6], [13], [14].

The study of common fixed points was initiated by Jungck [19] in 1986, and this notion has attracted many researchers to establish the existence of common fixed points by using various contractive conditions.

This direction of research produced a consistent literature on fixed point, common fixed point and coincidence point theorems in various ambient spaces. For more details see [1, 2, 8, 9, 11, 15, 17, 20, 23, 24, 25, 26, 27, 30, 32].

In 2017, Imdad et al. [18] established coupled and tripled fixed point results for (ψ, ϕ) contractions on complete weak partial metric spaces which generalize certain corresponding results of Ayadi et al. [4] and some others.

In 2019, Dhawan and Kaur [12] (Mathematics **2019**,7,193) introduced the notion of $\mathcal{F}$-generalized contractive type mappings by using C-class function and established some common fixed point theorems for weakly isotone increasing set valued mappings in the setting of ordered partial metric spaces and give an example in support of the result.

Recently, Popa and Patriciu [28] have proved a general fixed point theorem for a mapping satisfying an implicit relation in the framework of weak partial metric spaces, which is different of the results from [3] and [14].

Quite recently, Popa and Patriciu [29] have proved a general fixed point theorems for two pairs of absorbing mappings in the setting of weak partial space, using implicit relation and give an example in support of the result.

The aim of this paper is to investigate some unique common fixed point and a coincidence point theorems for two self mappings satisfying auxiliary functions in the framework of complete weak partial metric spaces. The results of findings extend and generalize several comparable results in the existing literature.

2. Definitions and Lemmas

Now, we give some basic definitions and auxiliary results on partial metric space (PMS) and weak partial metric space (WPMS).

Definition 2.1. ([21, 22]) Let X be a nonempty set and $p \colon X \times X \to [0, \infty)$ be such that for all $x, y, z \in X$ the following postulates are satisfied:

$(PM1)$ $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

$(PM2)$ $p(x,x) \leq p(x,y)$,

$(PM3)$ $p(x,y) = p(y,x)$,

$(PM4)$ $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Then p is called partial metric on X and the pair (X,p) is called partial metric space.

Remark 2.1. It is clear that, if $p(x,y) = 0$, then $x = y$. But, if $x = y$, $p(x,x)$ may not be zero.

Each partial metric space on a set X generates a T_0 topology τ_p on X which has a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Similarly, closed p-ball is defined as $B_p[x,\varepsilon] := \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$

A sequence $\{x_n\}$ in the partial metric space (X,p) converges with respect to τ_p to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$.

If p is a partial metric on X, then the functions $d_w, d_p \colon X \times X \to [0,\infty)$ given by

$$d_w(x,y) = p(x,y) - \min\{p(x,x), p(y,y)\} \tag{2.1}$$

and

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{2.2}$$

are ordinary metrics on X.

Remark 2.2. Let $\{x_n\}$ be a sequence in a PMS (X,p) and $x \in X$, then $\lim_{n\to\infty} d_w(x_n,x) = 0$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Proposition 2.1. *([3]) Let (X,p) be a PMS, then d_p and d_w are equivalent metrics on X.*

Definition 2.2. ([22]) Let (X,p) be a partial metric space. Then

(1) a sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite,

(2) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Definition 2.3. ([16]) A weak partial metric space on a nonempty set X is a function $p \colon X \times X \to [0,\infty)$ such that for all $x,y,z \in X$, the following is satisfied:

$(WPM1)$: $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

$(WPM2)$: $p(x,y) = p(y,x)$,

$(WPM3)$: $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

Then p is called weak partial metric on X and the pair (X,p) is called weak partial metric space (in short WPMS).

If $p(x,y) = 0$, then $x = y$.

It is obvious that, every partial metric space is a weak partial metric space, but the converse is not true. For example, if $X = [0,\infty)$ and $p(x,y) = \frac{x+y}{2}$, then (X,p) is a weak partial metric space and (X,p) is not a partial metric space. For another example, for $x,y \in \mathbb{R}$ the function $p(x,y) = \frac{e^x+e^y}{2}$ is a non partial metric but weak partial metric on $\mathbb{R}$.

Remark 2.3. ([3]) If (X,p) be a WPMS, but not a PMS, then the function d_p as in (2.2) may not be an ordinary metric on X. For example, let $X = [0,\infty)$ and let $p\colon X \times X \to [0,\infty)$ defined by $p(x,y) = \frac{x+y}{2}$. Then it is clear that $d_p(x,y) = 0$ for all $x,y \in X$, so d_p is not a metric on X. Note that, in this case $d_w(x,y) = \frac{1}{2}|x-y|$.

Proposition 2.2. ([3]) *Let $a,b,c \in [0,\infty)$, then we have*

$$\min\{a,c\} + \min\{b,c\} \le \min\{a,b\} + c.$$

Proposition 2.3. ([3]) *Let (X,p) be a WPMS, then $d_w\colon X \times X \to \mathbb{R}$ defined as in (2.1) is an ordinary metric on X.*

Definition 2.4. A point x in X is called a coincidence point of two self mappings f and $\mathcal{S}$ of X if $fx = \mathcal{S}x$ for each $x \in X$.

Example 2.1. Let $f(x) = \frac{x^3}{4}$ and $\mathcal{S}(x) = x^4$ for all $x \in [0, \frac{1}{4}]$. Then f and $\mathcal{S}$ have two coincidence point 0 and $\frac{1}{4}$. Clearly, they commute at 0 but not at $\frac{1}{4}$.

Remark 2.4. In a weak partial metric space, the convergent Cauchy sequence and the completeness are defined as in partial metric space.

Lemma 2.1. ([3]) *Let (X,p) be a weak partial metric space (WPMS).*

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in (X,d_w).

(b) (X,p) is complete if and only if (X,d_w) is complete.

Lemma 2.2. ([28]) *Let (X,p) be a weak partial metric space and $\{x_n\}$ is a sequence in (X,p). If $\lim_{n\to\infty} x_n = x$ and $p(x,x) = 0$, then $\lim_{n\to\infty} p(x_n,y) = p(x,y)$, for all $y \in X$.*

Lemma 2.3. ([10]) *Let (X,p) be a partial metric space and let $\{x_n\}$ be a sequence in (X,p) such that $\lim_{n\to\infty} p(x_{n+1},x_n)=0$.*

If the sequence $\{x_{2n}\}$ is not a Cauchy sequence in (X,p), then there exists $\varepsilon>0$ and two subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of positive integers with $n(k)>m(k)>k$ such that the four sequences

$$p\big(x_{2m(k)},x_{2n(k)+1}\big),\ p\big(x_{2m(k)},x_{2n(k)}\big),\ p\big(x_{2m(k)-1},x_{2n(k)+1}\big),\ p\big(x_{2m(k)-1},x_{2n(k)}\big)$$

tend to $\varepsilon>0$ when $k\to\infty$.

Remark 2.5. Remark 2.2 is still true for weak partial metric spaces.

3. Main Results

In this section, we shall prove some unique common fixed point and a coincidence point theorems via auxiliary functions in the setting of complete weak partial metric spaces.

The following classes of the auxiliary functions are used in the main results of this paper.

(1) Let Ψ be the family of continuous and monotone non-decreasing functions $\psi\colon[0,\infty)\to[0,\infty)$ such that $\psi(t)=0$ if and only if $t=0$.

(2) Let Φ be the family of lower semi-continuous functions $\phi\colon[0,\infty)\to[0,\infty)$ such that $\phi(t)=0$ if and only if $t=0$.

Theorem 3.1. *Let F and G be two self-maps on a complete weak partial metric space (X,p) satisfying the condition:*

$$\psi\big(p(Fx,Gy)\big)\le\psi\big(\mathcal{M}(x,y)\big)-\phi\big(\mathcal{N}(x,y)\big), \tag{3.1}$$

for all $x,y\in X$, where

$$\mathcal{M}(x,y)=\max\Big\{p(x,y),p(x,Fx),p(y,Gy),\frac{1}{3}[p(x,y)+p(x,Gy)+p(y,Fx)]\Big\},$$

$$\mathcal{N}(x,y)=\max\Big\{p(x,y),\frac{1}{4}[p(x,Fx)+p(y,Gy)],\frac{1}{4}[p(x,Gy)+p(y,Fx)]\Big\},$$

$\psi\in\Psi$ and $\phi\in\Phi$. Then F and G have a unique common fixed point z in X with $p(z,z)=0$.

Proof. For each $x_0\in X$. Let $x_{2n+1}=Fx_{2n}$ and $x_{2n+2}=Gx_{2n+1}$ for $n=0,1,2,\ldots$. We prove that $\{x_n\}$ is a Cauchy sequence in (X,p). It follows from (3.1) for $x=x_{2n}$ and $y=x_{2n-1}$ that

$$\begin{aligned}\psi\big(p(x_{2n+1},x_{2n})\big)&=\psi\big(p(Fx_{2n},Gx_{2n-1})\big)\\&\le\psi\big(\mathcal{M}(x_{2n},x_{2n-1})\big)-\phi\big(\mathcal{N}(x_{2n},x_{2n-1})\big),\end{aligned}\tag{3.2}$$

where

$$
\begin{aligned}
\mathcal{M}(x_{2n}, x_{2n-1}) &= \max\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n}, Fx_{2n}), p(x_{2n-1}, Gx_{2n-1}), \\
&\qquad \frac{1}{3}[p(x_{2n}, x_{2n-1}) + p(x_{2n}, Gx_{2n-1}) + p(x_{2n-1}, Fx_{2n})]\Big\} \\
&= \max\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), \\
&\qquad \frac{1}{3}[p(x_{2n}, x_{2n-1}) + p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})]\Big\}, \qquad (3.3)
\end{aligned}
$$

using condition $(WPM3)$, we have

$$p(x_{2n-1}, x_{2n+1}) \leq p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}) - p(x_{2n}, x_{2n}),$$

or

$$
\begin{aligned}
&\frac{1}{3}[p(x_{2n}, x_{2n-1}) + p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})] \\
&\leq \frac{1}{3}[p(x_{2n}, x_{2n-1}) + p(x_{2n-1}, x_{2n}) \\
&\qquad + p(x_{2n}, x_{2n+1})] \\
&= \frac{1}{3}[p(x_{2n}, x_{2n-1}) + p(x_{2n}, x_{2n-1}) \\
&\qquad + p(x_{2n}, x_{2n+1})] \text{ (by (WPM2))} \\
&\leq \max\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n+1}, x_{2n})\Big\}, \qquad (3.4)
\end{aligned}
$$

by equations (3.3), (3.4) and using (WPM2), we have

$$\mathcal{M}(x_{2n}, x_{2n-1}) \leq \max\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n+1}, x_{2n})\Big\}, \qquad (3.5)$$

and

$$
\begin{aligned}
\mathcal{N}(x_{2n}, x_{2n-1}) &= \max\Big\{p(x_{2n}, x_{2n-1}), \frac{1}{4}[p(x_{2n}, Fx_{2n}) + p(x_{2n-1}, Gx_{2n-1})], \\
&\qquad \frac{1}{4}[p(x_{2n}, Gx_{2n-1}) + p(x_{2n-1}, Fx_{2n})]\Big\} \\
&= \max\Big\{p(x_{2n}, x_{2n-1}), \frac{1}{4}[p(x_{2n}, x_{2n+1}) + p(x_{2n-1}, x_{2n})], \\
&\qquad \frac{1}{4}[p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})]\Big\}, \qquad (3.6)
\end{aligned}
$$

from condition $(WPM3)$, we have

$$p(x_{2n-1}, x_{2n+1}) \leq p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}) - p(x_{2n}, x_{2n}),$$

or

$$\frac{1}{4}[p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})]$$

$$\begin{aligned} &\le \frac{1}{4}[p(x_{2n-1},x_{2n})+p(x_{2n},x_{2n+1})] \\ &\le \max\Big\{p(x_{2n},x_{2n-1}),p(x_{2n+1},x_{2n})\Big\}, \end{aligned} \tag{3.7}$$

by equations (3.6) and (3.7), we have

$$\mathcal{N}(x_{2n},x_{2n-1}) \le \max\Big\{p(x_{2n},x_{2n-1}),p(x_{2n+1},x_{2n})\Big\}, \tag{3.8}$$

From equations (3.2), (3.5) and (3.8), we have

$$\begin{aligned} \psi\big(p(x_{2n+1},x_{2n})\big) &\le \psi\Big(\max\Big\{p(x_{2n},x_{2n-1}),p(x_{2n+1},x_{2n})\Big\}\Big) \\ &\quad -\phi\Big(\max\Big\{p(x_{2n},x_{2n-1}),p(x_{2n+1},x_{2n})\Big\}\Big). \end{aligned} \tag{3.9}$$

If $p(x_{2n+1},x_{2n}) > p(x_{2n},x_{2n-1})$, then from equation (3.9) and using the property of ψ, ϕ, we get

$$\begin{aligned} \psi\big(p(x_{2n+1},x_{2n})\big) &\le \psi\big(p(x_{2n+1},x_{2n})\big)-\phi\big(p(x_{2n+1},x_{2n})\big) \\ &< \psi\big(p(x_{2n+1},x_{2n})\big), \end{aligned} \tag{3.10}$$

which is a contradiction since $p(x_{2n+1},x_{2n}) > 0$. So, we have $p(x_{2n+1},x_{2n}) \le p(x_{2n},x_{2n-1})$, that is, $\{p(x_{2n+1},x_{2n})\}$ is a non-increasing sequence of positive real numbers. Thus there exists $\rho \ge 0$ such that

$$p(x_{2n+1},x_{2n}) = \rho. \tag{3.11}$$

Suppose that $\rho > 0$. Taking the lower limit in (3.9) as $n \to \infty$ and using (3.10) and the properties of ψ, ϕ, we have

$$\psi(\rho) \le \psi(\rho) - \liminf_{n\to\infty} \phi(\rho) \le \psi(\rho) - \phi(\rho) < \psi(\rho),$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} p(x_{2n+1},x_{2n}) = 0,$$

which implies

$$\lim_{n\to\infty} p(x_{n+1},x_n) = 0. \tag{3.12}$$

Now, we shall prove that $\{x_{2n}\}$ is a Cauchy sequence in (X,p). On the contrary, assume that $\{x_{2n}\}$ is not a Cauchy sequence in (X,p), then by Lemma 2.3, there exists $\varepsilon > 0$ and two subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ with $n(k) > m(k) > k$ such that the sequences

$$p(x_{2m(k)},x_{2n(k)+1}),\ p(x_{2m(k)},x_{2n(k)}),\ p(x_{2m(k)-1},x_{2n(k)+1}),\ p(x_{2m(k)-1},x_{2n(k)})$$

tend to $\varepsilon > 0$ when $k \to \infty$.

Now, using the given contractive condition (3.1) for $x = x_{2m(k)}$ and $y = x_{2n(k)+1}$, we have

$$\begin{aligned} \psi\big(p(x_{2m(k)}, x_{2n(k)+1})\big) &= \psi\big(p(Fx_{2m(k)-1}, Gx_{2n(k)}\big) \\ &\le \psi\big(\mathcal{M}(x_{2m(k)-1}, x_{2n(k)})\big) \\ &\quad -\phi\big(\mathcal{N}(x_{2m(k)-1}, x_{2n(k)})\big), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \mathcal{M}(x_{2m(k)-1}, x_{2n(k)}) &= \max\Big\{p(x_{2m(k)-1}, x_{2n(k)}), \\ &\quad p(x_{2m(k)-1}, Fx_{2m(k)-1}), p(x_{2n(k)}, Gx_{2n(k)}), \\ &\quad \frac{1}{3}[p(x_{2m(k)-1}, x_{2n(k)}) + p(x_{2m(k)-1}, Gx_{2n(k)}) \\ &\quad +p(x_{2n(k)}, Fx_{2m(k)-1})]\Big\} \\ &= \max\Big\{p(x_{2m(k)-1}, x_{2n(k)}), \\ &\quad p(x_{2m(k)-1}, x_{2m(k)}), p(x_{2n(k)}, x_{2n(k)+1}), \\ &\quad \frac{1}{3}[p(x_{2m(k)-1}, x_{2n(k)}) + p(x_{2m(k)-1}, x_{2n(k)+1}) \\ &\quad +p(x_{2n(k)}, x_{2m(k)})]\Big\}. \end{aligned} \tag{3.14}$$

Taking the limit as $k \to \infty$ and using (WPM2) and (3.12) in (3.14), we get

$$\mathcal{M}(x_{2m(k)-1}, x_{2n(k)}) \to \max\{\varepsilon, 0, 0, \varepsilon\} = \varepsilon, \tag{3.15}$$

and

$$\begin{aligned} \mathcal{N}(x_{2m(k)-1}, x_{2n(k)}) &= \max\Big\{p(x_{2m(k)-1}, x_{2n(k)}), \\ &\quad \frac{1}{4}[p(x_{2m(k)-1}, Fx_{2m(k)-1}) + p(x_{2n(k)}, Gx_{2n(k)})], \\ &\quad \frac{1}{4}[p(x_{2m(k)-1}, Gx_{2n(k)}) + p(x_{2n(k)}, Fx_{2m(k)-1})]\Big\} \\ &= \max\Big\{p(x_{2m(k)-1}, x_{2n(k)}), \\ &\quad \frac{1}{4}[p(x_{2m(k)-1}, x_{2m(k)}) + p(x_{2n(k)}, x_{2n(k)+1})], \\ &\quad \frac{1}{4}[p(x_{2m(k)-1}, x_{2n(k)+1}) + p(x_{2n(k)}, x_{2m(k)})]\Big\}. \end{aligned} \tag{3.16}$$

Taking the limit as $k \to \infty$ and using (WPM2) and (3.12) in (3.16), we obtain

$$\mathcal{N}(x_{2m(k)-1}, x_{2n(k)}) \to \max\{\varepsilon, 0, \frac{\varepsilon}{3}\} = \varepsilon. \tag{3.17}$$

Thus, by (3.13) for any $k \to \infty$, we have

$$\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. Hence, we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{3.18}$$

Since $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, p).

On the other hand, since

$$\begin{aligned} d_w(x_n, x_m) &\le p(x_n, x_m) - \min\Big\{p(x_n, x_n), p(x_m, x_m)\Big\} \\ &\le p(x_n, x_m). \end{aligned}$$

Therefore, taking the limit as $n, m \to \infty$ and using (3.18), we have

$$\lim_{n,m\to\infty} d_w(x_n, x_m) = 0. \tag{3.19}$$

This shows that $\{x_n\}$ is also a Cauchy sequence in the metric space (X, d_w). Since (X, p) is complete, then from Lemma 2.1, the sequence $\{x_n\}$ converges in the metric space (X, d_w), say to a point $z \in X$ and $\lim_{n\to\infty} d_w(x_n, z) = 0$. Again from Lemma 2.1, we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3.20}$$

Hence from (3.18) and (3.20), we get

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{3.21}$$

Now, we shall show that z is a common fixed point of F and G. Notice that due to (3.21), we have $p(z, z) = 0$. By (3.1) with $x = x_{2n}$ and $y = z$ and using (3.21), we have

$$\begin{aligned} \psi\big(p(x_{2n+1}, Gz)\big) &= \psi\big(p(Fx_{2n}, Gz)\big) \\ &\le \psi\big(\mathcal{M}(x_{2n}, z)\big) - \phi\big(\mathcal{N}(x_{2n}, z)\big), \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} \mathcal{M}(x_{2n}, z) &= \max\Big\{p(x_{2n}, z), p(x_{2n}, Fx_{2n}), p(z, Gz), \\ &\qquad \frac{1}{3}[p(x_{2n}, z) + p(x_{2n}, Gz) + p(z, Fx_{2n})]\Big\} \\ &= \max\Big\{p(x_{2n}, z), p(x_{2n}, x_{2n+1}), p(z, Gz), \\ &\qquad \frac{1}{3}[p(x_{2n}, z) + p(x_{2n}, Gz) + p(z, x_{2n+1})]\Big\}, \end{aligned}$$

passing the limit as $n \to \infty$ and using (3.21) in the above inequality, we obtain

$$\mathcal{M}(x_{2n}, z) \to \max\Big\{0, 0, p(z, Gz), \frac{p(z, Gz)}{3}\Big\} = p(z, Gz), \tag{3.23}$$

and

$$\begin{aligned}\mathcal{N}(x_{2n}, z) &= \max\Big\{p(x_{2n}, z), \frac{1}{4}[p(x_{2n}, Fx_{2n}) + p(z, Gz)], \\ &\qquad \frac{1}{4}[p(x_{2n}, Gz) + p(z, Fx_{2n})]\Big\} \\ &= \max\Big\{p(x_{2n}, z), \frac{1}{4}[p(x_{2n}, x_{2n+1}) + p(z, Gz)], \\ &\qquad \frac{1}{4}[p(x_{2n}, Gz) + p(z, x_{2n+1})]\Big\},\end{aligned}$$

passing the limit as $n \to \infty$ and using (3.21) in the above inequality, we obtain

$$\mathcal{N}(x_{2n}, z) \to \max\Big\{0, \frac{p(z, Gz)}{4}, \frac{p(z, Gz)}{4}\Big\} = \frac{p(z, Gz)}{4}. \tag{3.24}$$

Now, from (3.22), (3.23) and (3.24), we have

$$\psi\big(p(x_{2n+1}, Gz)\big) \le \psi\big(p(z, Gz)\big) - \phi\big(\frac{p(z, Gz)}{4}\big). \tag{3.25}$$

passing the limit as $n \to \infty$ in the above inequality and using the property of ψ, ϕ, we obtain

$$\psi\big(p(z, Gz)\big) \le \psi\big(p(z, Gz)\big) - \phi\big(\frac{p(z, Gz)}{4}\big) < \psi\big(p(z, Gz)\big),$$

which is a contradiction. Thus $Gz = z$, that is, z is a fixed point of G. Similarly, we can prove that z is also a fixed point of F. Hence, z is a common fixed point of F and G. Finally to prove uniqueness, suppose z' is another common fixed point of F and G such that $Fz' = z' = Gz'$ with $z \neq z'$. From (3.1) and (3.20), we have

$$\begin{aligned}\psi\big(p(z, z')\big) &= \psi\big(p(Fz, Gz')\big) \\ &\le \psi\big(\mathcal{M}(z, z')\big) - \phi\big(\mathcal{N}(z, z')\big),\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}\mathcal{M}(z, z') &= \max\Big\{p(z, z'), p(z, Fz), p(z', Gz'), \\ &\qquad \frac{1}{3}[p(z, z') + p(z, Gz') + p(z', Fz)]\Big\} \\ &= \max\Big\{p(z, z'), p(z, z), p(z', z'), \\ &\qquad \frac{1}{3}[p(z, z') + p(z, z') + p(z', z)]\Big\} \\ &= \max\Big\{p(z, z'), 0, 0, p(z, z')\Big\} \text{ (by (WPM2))} \\ &= p(z, z'),\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\mathcal{N}(z,z') &= \max\Big\{p(z,z'), \frac{1}{4}[p(z,Fz)+p(z',Gz')],\\ &\qquad \frac{1}{4}[p(z,Gz')+p(z',Fz)]\Big\}\\ &= \max\Big\{p(z,z'), \frac{1}{4}[p(z,z)+p(z',z')],\\ &\qquad \frac{1}{4}[p(z,z')+p(z',z)]\Big\}\\ &= \max\Big\{p(z,z'), 0, \frac{p(z,z')}{2}\Big\} \text{ (by (WPM2))}\\ &= p(z,z'). \qquad (3.28)\end{aligned}$$

Now, from equations (3.26)-(3.28) and using the property of ψ, ϕ, we obtain

$$\psi\big(p(z,z')\big) \le \psi\big(p(z,z')\big) - \phi\big(p(z,z')\big) < \psi\big(p(z,z'\big),$$

which is a contraction. Thus, $z = z'$. This shows that the common fixed point of F and G is unique. This completes the proof. □

If we take $F = G = T$ in Theorem 3.1, then we have the following result as corollaries.

Corollary 3.1. *Let T be a self-map on a complete weak partial metric space (X, p) satisfying the condition:*

$$\psi\big(p(Tx,Ty)\big) \le \psi\big(\mathcal{M}(x,y)\big) - \phi\big(\mathcal{N}(x,y)\big), \qquad (3.29)$$

for all $x, y \in X$, where

$$\mathcal{M}(x,y) = \max\Big\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{3}[p(x,y)+p(x,Ty)+p(y,Tx)]\Big\},$$

$$\mathcal{N}(x,y) = \max\Big\{p(x,y), \frac{1}{4}[p(x,Tx)+p(y,Ty)], \frac{1}{4}[p(x,Ty)+p(y,Tx)]\Big\},$$

$\psi \in \Psi$ and $\phi \in \Phi$. Then T has a unique fixed point u in X with $p(u,u) = 0$.

Corollary 3.2. *Let T be a self-map on a complete weak partial metric space (X, p) satisfying the condition:*

$$\psi\big(p(Tx,Ty)\big) \le \psi\big(\mathcal{M}(x,y)\big) - \phi\big(\mathcal{M}(x,y)\big),$$

for all $x, y \in X$, where $\mathcal{M}(x,y)$ as in Corollary 3.1, $\psi \in \Psi$ and $\phi \in \Phi$. Then T has a unique fixed point u in X with $p(u,u) = 0$.

Corollary 3.3. *Let T be a self-map on a complete weak partial metric space (X, p) satisfying the condition:*

$$\psi\big(p(Tx, Ty)\big) \leq \psi\big(\mathcal{N}(x, y)\big) - \phi\big(\mathcal{N}(x, y)\big),$$

for all $x, y \in X$, where $\mathcal{N}(x, y)$ as in Corollary 3.1, $\psi \in \Psi$ and $\phi \in \Phi$. Then T has a unique fixed point u in X with $p(u, u) = 0$.

Taking ψ to an identity mapping and $\phi(t) = (1-k)t$ for all $t \geq 0$, where $k \in (0, 1)$ in Corollary 3.2, then we obtain the following result.

Corollary 3.4. *Let T be a self-map on a complete weak partial metric space (X, p) satisfying the condition:*

$$p(Tx, Ty) \leq k \max\Big\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{3}[p(x, y) + p(x, Ty) + p(y, Tx)]\Big\},$$

for all $x, y \in X$, where $k \in (0, 1)$ is a constant. Then T has a unique fixed point u in X with $p(u, u) = 0$.

Taking ψ to an identity mapping and $\phi(t) = (1-q)t$ for all $t \geq 0$, where $q \in (0, 1)$ in Corollary 3.3, then we obtain the following result.

Corollary 3.5. *Let T be a self-map on a complete weak partial metric space (X, p) satisfying the condition:*

$$p(Tx, Ty) \leq q \max\Big\{p(x, y), \frac{1}{4}[p(x, Tx) + p(y, Ty)], \frac{1}{4}[p(x, Ty) + p(y, Tx)]\Big\},$$

for all $x, y \in X$, where $q \in (0, 1)$ is a constant. Then T has a unique fixed point u in X with $p(u, u) = 0$.

Theorem 3.2. *Let G_1 and G_2 be two continuous self-maps on a complete weak partial metric space (X, p) satisfying the condition:*

$$\psi\big(p(G_1^m x, G_2^n y)\big) \leq \psi\big(\mathcal{M}(x, y)\big) - \phi\big(\mathcal{N}(x, y)\big), \tag{3.30}$$

for all $x, y \in X$, where m and n are some integers,

$$\mathcal{M}(x, y) = \max\Big\{p(x, y), p(x, G_1^m x), p(y, G_2^n y), \frac{1}{3}[p(x, y) + p(x, G_2^n y) + p(y, G_1^m x)]\Big\},$$

$$\mathcal{N}(x, y) = \max\Big\{p(x, y), \frac{1}{4}[p(x, G_1^m x) + p(y, G_2^n y)], \frac{1}{4}[p(x, G_2^n y) + p(y, G_1^m x)]\Big\},$$

$\psi \in \Psi$ and $\phi \in \Phi$. Then G_1 and G_2 have a unique common fixed point v in X with $p(v, v) = 0$.

Proof. Since G_1^m and G_2^n satisfy the conditions of the Theorem 3.1. So G_1^m and G_2^n have a unique common fixed point. Let v be the common fixed point. Then we have

$$\begin{aligned} G_1^m v = v &\Rightarrow G_1(G_1^m v) = G_1 v \\ &\Rightarrow G_1^m(G_1 v) = G_1 v. \end{aligned}$$

If $G_1 v = v_0$, then $G_1^m v_0 = v_0$. So $G_1 v$ is a fixed point of G_1^m. Similarly, $G_2^n(G_2 v) = G_2 v$, that is, $G_2 v$ is a fixed point of G_2^n.

Now, using equations (3.30) and (3.21), we have

$$\begin{aligned} \psi\big(p(v, G_1 v)\big) &= \psi\big(p(G_1^m v, G_1^m(G_1 v))\big) \\ &\leq \psi\big(\mathcal{M}(v, G_1 v)\big) - \phi\big(\mathcal{N}(v, G_1 v)\big), \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} \mathcal{M}(v, G_1 v) &= \max\Big\{p(v, G_1 v), p(v, G_1^m v), p(G_1 v, G_1^m(G_1 v)), \\ &\qquad \frac{1}{3}[p(v, G_1 v) + p(v, G_1^m(G_1 v)) + p(G_1 v, G_1^m v)]\Big\} \\ &= \max\Big\{p(v, G_1 v), p(v, v), p(G_1 v, G_1 v), \\ &\qquad \frac{1}{3}[p(v, G_1 v) + p(v, G_1 v) + p(G_1 v, v)]\Big\} \\ &= \max\Big\{p(v, G_1 v), 0, 0, p(v, G_1 v)\Big\} \text{ (using (WPM2))} \\ &= p(v, G_1 v), \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} \mathcal{N}(v, G_1 v) &= \max\Big\{p(v, G_1 v), \frac{1}{4}[p(v, G_1^m v) + p(G_1 v, G_1^m(G_1 v))], \\ &\qquad \frac{1}{4}[p(v, G_1^m(G_1 v)) + p(G_1 v, G_1^m v)]\Big\} \\ &= \max\Big\{p(v, G_1 v), \frac{1}{4}[p(v, v) + p(G_1 v, G_1 v)], \\ &\qquad \frac{1}{4}[p(v, G_1 v) + p(G_1 v, v)]\Big\} \\ &= \max\Big\{p(v, G_1 v), 0, \frac{p(v, G_1 v)}{2}\Big\} \text{ (using (WPM2))} \\ &= p(v, G_1 v). \end{aligned} \tag{3.33}$$

From equations (3.31)-(3.33) and using the property of ψ, ϕ, we obtain

$$\begin{aligned} \psi\big(p(v, G_1 v)\big) &\leq \psi\big(p(v, G_1 v)\big) - \phi\big(p(v, G_1 v)\big) \\ &< \psi\big(p(v, G_1 v)\big), \end{aligned}$$

which is a contradiction. Hence $v = G_1 v$ for all $v \in X$. Similarly, we can show that $v = G_2 v$. This shows that v is a common fixed point of G_1 and G_2. For the uniqueness of v, let $v' \neq v$ be another common fixed point of G_1 and G_2. Then clearly v' is also a common fixed point of G_1^m and G_2^n which implies $v = v'$. Thus G_1 and G_2 have a unique common fixed point in X. This completes the proof. □

Theorem 3.3. *Let $\{F_\alpha\}$ be a family of continuous self-maps on a complete weak partial metric space (X, p) satisfying the condition:*

$$\psi\big(p(F_\alpha x, F_\beta y)\big) \leq \psi\big(\mathcal{M}(x,y)\big) - \phi\big(\mathcal{N}(x,y)\big), \tag{3.34}$$

for all $x, y \in X$, where

$$\mathcal{M}(x,y) = \max\Big\{p(x,y), p(x,F_\alpha x), p(y,F_\beta y), \frac{1}{3}[p(x,y) + p(x,F_\beta y) + p(y,F_\alpha x)]\Big\},$$

$$\mathcal{N}(x,y) = \max\Big\{p(x,y), \frac{1}{4}[p(x,F_\alpha x) + p(y,F_\beta y)], \frac{1}{4}[p(x,F_\beta y) + p(y,F_\alpha x)]\Big\},$$

$\psi \in \Psi$, $\phi \in \Phi$ and $\alpha, \beta \in \mathcal{F}$ with $\alpha \neq \beta$, where $\mathcal{F}$ denote the family of all continuous functions $\alpha\colon [0,\infty) \to [0,\infty)$. Then there exists a unique $\mu \in X$ satisfying $F_\alpha(\mu) = \mu$ for all $\alpha \in \mathcal{F}$ with $p(\mu,\mu) = 0$.

Proof. Follows from Theorem 3.1 and by definition of continuity. □

Theorem 3.4. *Let $\mathcal{T}$ and f be two self-maps on a complete weak partial metric space (X, p) satisfying the condition:*

$$\psi\big(p(\mathcal{T}x, \mathcal{T}y)\big) \leq \psi\big(\mathcal{M}_1(x,y)\big) - \phi\big(\mathcal{M}_2(x,y)\big), \tag{3.35}$$

for all $x, y \in X$, where

$$\mathcal{M}_1(x,y) = \max\Big\{p(fx,fy), p(fx,\mathcal{T}x), p(fy,\mathcal{T}y), \frac{1}{3}[p(fx,fy) + p(fx,\mathcal{T}y) + p(fy,\mathcal{T}x)]\Big\},$$

$$\mathcal{M}_2(x,y) = \max\Big\{p(fx,fy), \frac{1}{4}[p(fx,\mathcal{T}x) + p(fy,\mathcal{T}y)], \frac{1}{4}[p(fx,\mathcal{T}y) + p(fy,\mathcal{T}x)]\Big\},$$

$\psi \in \Psi$ and $\phi \in \Phi$. If the range of f contains the range of $\mathcal{T}$ and $f(X)$ is a complete subspace of X, then $\mathcal{T}$ and f have a coincidence point $v \in X$, that is, $fv = \mathcal{T}v$ with $p(fv, fv) = 0$.

Proof. Let $x_0 \in X$ and choose a point x_1 in X such that $\mathcal{T}x_0 = fx_1, \ldots, \mathcal{T}x_n = fx_{n+1}$. Then from (3.35) for $x = x_{n-1}$ and $y = x_n$ we have successively

$$\begin{aligned} \psi\big(p(fx_n, fx_{n+1})\big) &= \psi\big(p(\mathcal{T}x_{n-1}, \mathcal{T}x_n)\big) \\ &\leq \psi\big(\mathcal{M}_1(x_{n-1},x_n)\big) - \phi\big(\mathcal{M}_2(x_{n-1},x_n)\big), \end{aligned} \tag{3.36}$$

where

$$\begin{aligned}\mathcal{M}_1(x_{n-1},x_n) &= \max\Big\{p(fx_{n-1},fx_n),p(fx_{n-1},\mathcal{T}x_{n-1}),p(fx_n,\mathcal{T}x_n),\\ &\qquad \frac{1}{3}[p(fx_{n-1},fx_n)+p(fx_{n-1},\mathcal{T}x_n)+p(fx_n,\mathcal{T}x_{n-1})]\Big\}\\ &= \max\Big\{p(fx_{n-1},fx_n),p(fx_{n-1},fx_n),p(fx_n,fx_{n+1}),\\ &\qquad \frac{1}{3}[p(fx_{n-1},fx_n)+p(fx_{n-1},fx_{n+1})+p(fx_n,fx_n)]\Big\}, \qquad (3.37)\end{aligned}$$

using condition $(WPM3)$, we have

$$p(fx_{n-1},fx_{n+1})\le p(fx_{n-1},fx_n)+p(fx_n,fx_{n+1})-p(fx_n,fx_n),$$

or

$$\begin{aligned}&\frac{1}{3}[p(fx_{n-1},fx_n)+p(fx_{n-1},fx_{n+1})+p(fx_n,fx_n)]\\ &\le \frac{1}{3}[p(fx_{n-1},fx_n)+p(fx_{n-1},fx_n)+p(fx_n,fx_{n+1})]\\ &\le \max\Big\{p(fx_{n-1},fx_n),p(fx_n,fx_{n+1})\Big\}, \qquad (3.38)\end{aligned}$$

by equations (3.37) and (3.38), we have

$$\mathcal{M}_1(x_{n-1},x_n)=\max\Big\{p(fx_{n-1},fx_n),p(fx_n,fx_{n+1})\Big\}. \qquad (3.39)$$

Similarly, we can show that

$$\mathcal{M}_2(x_{n-1},x_n)=\max\Big\{p(fx_{n-1},fx_n),p(fx_n,fx_{n+1})\Big\}. \qquad (3.40)$$

From equations (3.36), (3.39) and (3.40), we have

$$\begin{aligned}\psi\big(p(fx_n,fx_{n+1})\big) &\le \psi\Big(\max\Big\{p(p(fx_{n-1},fx_n),p(fx_n,fx_{n+1}))\Big\}\Big)\\ &\quad -\phi\Big(\max\Big\{p(p(fx_{n-1},fx_n),p(fx_n,fx_{n+1}))\Big\}\Big). \qquad (3.41)\end{aligned}$$

If $p(fx_n,fx_{n+1})>p(fx_{n-1},fx_n)$, then from equation (3.41) and using the property of ψ, ϕ, we get

$$\begin{aligned}\psi\big(p(fx_n,fx_{n+1})\big) &\le \psi\big(p(fx_n,fx_{n+1})\big)-\phi\big(p(fx_n,fx_{n+1})\big)\\ &< \psi\big(p(fx_n,fx_{n+1})\big), \qquad (3.42)\end{aligned}$$

which is a contradiction since $p(fx_n,fx_{n+1})>0$. So, we have $p(fx_n,fx_{n+1})\le p(fx_{n-1},fx_n)$, that is, $p(fx_n,fx_{n+1})$ is a non-increasing sequence of positive real numbers. Thus there exists $L\ge 0$ such that

$$\lim_{n\to\infty}p(fx_n,fx_{n+1})=L. \qquad (3.43)$$

Suppose that $L > 0$. Taking the lower in (3.41) as $n \to \infty$ and using (3.43) and the properties of ψ, ϕ, we have

$$\psi(L) \le \psi(L) - \liminf_{n\to\infty} \phi(L) \le \psi(L) - \phi(L) < \psi(L),$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} p(fx_n, fx_{n+1}) = 0. \tag{3.44}$$

On the other hand, since

$$\begin{aligned} d_w(fx_{n+1}, fx_n) &\le p(fx_{n+1}, fx_n) - \min\Big\{p(fx_n, fx_n), p(fx_{n+1}, fx_{n+1})\Big\} \\ &\le p(fx_{n+1}, fx_n). \end{aligned}$$

Therefore, taking the limit as $n \to \infty$ and using (3.44), we have

$$\lim_{n\to\infty} d_w(fx_{n+1}, fx_n) = 0. \tag{3.45}$$

Therefore we have for $m > n$

$$d_w(fx_m, fx_n) \le d_w(fx_m, fx_{m-1}) + \ldots + d_w(fx_{n+1}, fx_n).$$

Passing to the limit as $n, m \to \infty$ and using (3.45), we obtain

$$\lim_{n,m\to\infty} d_w(fx_n, fx_m) = 0. \tag{3.46}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_w). Since (X, p) is complete then from Lemma 2.1, the sequence $\{x_n\}$ converges in the metric space (X, d_w), say $x_n \to v \Rightarrow fx_n \to fv$ as $n \to \infty$, since $f(X)$ is a complete subspace of X, that is, $\lim_{n\to\infty} d_w(fx_n, fv) = 0$. Again from Lemma 2.1, we have

$$p(fv, fv) = \lim_{n\to\infty} p(fx_n, fv) = \lim_{n,m\to\infty} p(fx_n, fx_m). \tag{3.47}$$

Moreover, since $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_w), we have $\lim_{n,m\to\infty} d_w(fx_m, fx_n) = 0$. On the other hand since

$$p(fx_n, fx_n) + p(fx_{n+1}, fx_{n+1}) \le 2p(fx_n, fx_{n+1}), \tag{3.48}$$

from (3.44), we obtain

$$\lim_{n\to\infty} p(fx_n, fx_n) = 0. \tag{3.49}$$

Therefore, by the definition of d_w we obtain

$$p(fx_n, fx_m) = d_w(fx_n, fx_m) + \min\big\{p(fx_n, fx_n), p(fx_m, fx_m)\big\},$$

and so

$$\lim_{n,m\to\infty} p(fx_n, fx_m) = 0. \tag{3.50}$$

Thus from (3.47), we obtain

$$p(fv, fv) = \lim_{n\to\infty} p(fx_n, fv) = \lim_{n,m\to\infty} p(fx_n, fx_m) = 0. \tag{3.51}$$

Now, we shall show that v is a coincidence point of $\mathcal{T}$ and f. Notice that due to (3.51), we have $p(fv, fv) = 0$. By (3.35), we have

$$\psi\big(p(\mathcal{T}v, \mathcal{T}x_n)\big) \le \psi\big(\mathcal{M}_1(v, x_n)\big) - \phi\big(\mathcal{M}_2(v, x_n)\big),$$

or

$$\psi\big(p(\mathcal{T}v, fx_{n+1})\big) \le \psi\big(\mathcal{M}_1(v, x_n)\big) - \phi\big(\mathcal{M}_2(v, x_n)\big), \tag{3.52}$$

where

$$\begin{aligned} \mathcal{M}_1(v, x_n) &= \max\Big\{p(fv, fx_n), p(fv, \mathcal{T}v), p(fx_n, \mathcal{T}x_n), \\ &\qquad \frac{1}{3}[p(fv, fx_n) + p(fv, \mathcal{T}x_n) + p(fx_n, \mathcal{T}v)]\Big\} \\ &= \max\Big\{p(fv, fx_n), p(fv, \mathcal{T}v), p(fx_n, fx_{n+1}), \\ &\qquad \frac{1}{3}[p(fv, fx_n) + p(fv, fx_{n+1}) + p(fx_n, \mathcal{T}v)]\Big\}, \end{aligned} \tag{3.53}$$

passing to the limit as $n \to \infty$ and using $p(fv, fv) = 0$ in (3.53), we obtain

$$\mathcal{M}_1(v, x_n) \to \max\Big\{0, p(fv, \mathcal{T}v), 0, \frac{1}{3}p(fv, \mathcal{T}v)\Big\} = p(fv, \mathcal{T}v), \tag{3.54}$$

and

$$\begin{aligned} \mathcal{M}_2(v, x_n) &= \max\Big\{p(fv, fx_n), \frac{1}{4}[p(fv, \mathcal{T}v) + p(fx_n, \mathcal{T}x_n)], \\ &\qquad \frac{1}{4}[p(fv, \mathcal{T}x_n) + p(fx_n, \mathcal{T}v)]\Big\} \\ &= \max\Big\{p(fv, fx_n), \frac{1}{4}[p(fv, \mathcal{T}v) + p(fx_n, fx_{n+1})], \\ &\qquad \frac{1}{4}[p(fv, fx_{n+1}) + p(fx_n, \mathcal{T}v)]\Big\}, \end{aligned} \tag{3.55}$$

passing to the limit as $n \to \infty$ and using $p(fv, fv) = 0$ in (3.55), we obtain

$$\mathcal{M}_2(v, x_n) \to \max\Big\{0, \frac{1}{4}p(fv, \mathcal{T}v), \frac{1}{4}p(fv, \mathcal{T}v)\Big\} = \frac{1}{4}p(fv, \mathcal{T}v). \tag{3.56}$$

From equations (3.52), (3.54) and (3.56), we have

$$\psi\big(p(\mathcal{T}v, fx_{n+1})\big) \le \psi\big(p(fv, \mathcal{T}v)\big) - \phi\big(\frac{1}{4}p(fv, \mathcal{T}v)\big). \tag{3.57}$$

Passing to the limit as $n \to \infty$ in (3.57) and using the property of ψ, ϕ, we obtain

$$\begin{aligned} \psi\big(p(fv, \mathcal{T}v)\big) &\leq \psi\big(p(fv, \mathcal{T}v)\big) - \phi\big(\frac{1}{4}p(fv, \mathcal{T}v)\big), \text{(by (WPM2))} \\ &< \psi\big(p(fv, \mathcal{T}v)\big), \end{aligned}$$

which is a contradiction. Hence, $\mathcal{T}v = fv$. This shows that v is a coincidence point of $\mathcal{T}$ and f. This completes the proof. □

Remark 3.1. If we take $g = I$, the identity map, $\mathcal{M}_1(x, y) = \mathcal{M}(x, y)$ and $\mathcal{M}_2(x, y) = \mathcal{N}(x, y)$ in Theorem 3.4, then we obtain Corollary 3.1 of this paper.

Now, we give some examples in support of our result.

Example 3.1. Let $X = [0, 1]$ and $p(x, y) = \frac{x+y}{2}$, then $d_w(x, y) = \frac{1}{2}|x - y|$. Therefore, since (X, d_w) is complete, then by Lemma 2.1, (X, p) is a complete weak partial metric space (WPMS). Consider the mapping $T: X \to X$, defined by

$$T(x) = \begin{cases} x - 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

and $\psi, \phi: [0, \infty) \to [0, \infty)$ defined by $\psi(t) = t$ and $\phi(t) = \frac{2t}{3}$ for all $t \geq 0$.

We claim that condition (3.29) of Corollary 3.1 is satisfied. For this, we consider the following cases.

Case 1. If $x = y = 0$, then

$$p(T(x), T(y)) = 0 \text{ and } \psi\Big(p(T(x), T(y))\Big) = 0,$$

and

$$\mathcal{M}(x, y) = 0, \mathcal{N}(x, y) = 0 \Rightarrow \psi\Big(\mathcal{M}(x, y)\Big) = 0 = \phi\Big(\mathcal{N}(x, y)\Big).$$

Hence

$$\psi\Big(p(T(x), T(y))\Big) = 0 \leq \psi\Big(\mathcal{M}(x, y)\Big) - \phi\Big(\mathcal{N}(x, y)\Big).$$

Case 2. If $x = y > 0$, then

$$p(T(x), T(y)) = p(x - 1, x - 1) = x - 1 \text{ and } \psi\Big(p(T(x), T(y))\Big) = x - 1.$$

Now

$$p(x, y) = \frac{x + y}{2}, \ p(x, T(x)) = \frac{2x - 1}{2}, \ p(y, T(y)) = \frac{2y - 1}{2},$$
$$p(x, T(y)) = \frac{x + y - 1}{2}, \ p(y, T(x)) = \frac{y + x - 1}{2},$$

and

$$\mathcal{M}(x, y) = \frac{x + y}{2} \Rightarrow \psi\Big(\mathcal{M}(x, y)\Big) = \frac{x + y}{2},$$
$$\mathcal{N}(x, y) = \frac{x + y}{2} \Rightarrow \phi\Big(\mathcal{N}(x, y)\Big) = \frac{x + y}{3}.$$

Hence from condition (3.29), we have

$$\begin{aligned}\psi\Big(p(T(x),T(y))\Big) &= x-1 \le \frac{x+y}{6} = \frac{x+y}{2} - \frac{x+y}{3}\\ &= \psi\Big(\mathcal{M}(x,y)\Big) - \phi\Big(\mathcal{N}(x,y)\Big).\end{aligned}$$

Thus, we have

$$\psi\Big(p(T(x),T(y))\Big) \le \psi\Big(\mathcal{M}(x,y)\Big) - \phi\Big(\mathcal{N}(x,y)\Big).$$

Case 3. If $x > y = 0$, then

$$p(T(x),T(y)) = p(x-1,0) = \frac{x-1}{2} \text{ and } \psi\Big(p(T(x),T(y))\Big) = \frac{x-1}{2}.$$

Now

$$p(x,y) = \frac{x+y}{2},\ p(x,T(x)) = \frac{2x-1}{2},\ p(y,T(y)) = 0,$$
$$p(x,T(y)) = \frac{x}{2},\ p(y,T(x)) = \frac{x-1}{2},$$

and

$$\begin{aligned}\mathcal{M}(x,y) = \frac{2x-1}{2} &\Rightarrow \psi\Big(\mathcal{M}(x,y)\Big) = \frac{2x-1}{2},\\ \mathcal{N}(x,y) = \frac{2x-1}{8} &\Rightarrow \phi\Big(\mathcal{N}(x,y)\Big) = \frac{2x-1}{12}.\end{aligned}$$

Hence from condition (3.29), we have

$$\begin{aligned}\psi\Big(p(T(x),T(y))\Big) &= \frac{x-1}{2} \le \frac{5}{12}(2x-1) = \frac{1}{2}(2x-1) - \frac{1}{12}(2x-1)\\ &= \psi\Big(\mathcal{M}(x,y)\Big) - \phi\Big(\mathcal{N}(x,y)\Big).\end{aligned}$$

Thus, we have

$$\psi\Big(p(T(x),T(y))\Big) \le \psi\Big(\mathcal{M}(x,y)\Big) - \phi\Big(\mathcal{N}(x,y)\Big).$$

This shows that all the conditions of Corollary 3.1 are satisfied and so T has a unique fixed point in X, that is, 0 is the unique fixed point of T.

Example 3.2. Let $X = \{0,1,2,\dots,10\}$ and $p(x,y) = \frac{x+y}{2}$, then $d_w(x,y) = \frac{1}{2}|x-y|$. Therefore, since (X,d_w) is complete, then by Lemma 2.1, (X,p) is a complete weak partial metric space (WPMS). Consider the mapping $T\colon X \to X$, defined by

$$T(x) = \begin{cases} x-1, & \text{if } x \ne 0,\\ 0, & \text{if } x = 0.\end{cases}$$

(1) We claim that the inequality of Corollary 3.4 is satisfied with $k \in [0,1)$. For this, we consider the following cases.

Case 1. If $x = y = 0$, then

$$p(T(x),T(y)) = 0,$$

and

$$\max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{3}[p(x,y)+p(x,Ty)+p(y,Tx)]\right\} = 0.$$

Hence, we conclude that

$$p(T(x),T(y)) \\ \leq k \max\Big\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{3}[p(x,y)+p(x,Ty)+p(y,Tx)]\Big\},$$

where $k \in [0,1)$.

Case 2. If $x = y > 0$, then

$$p(T(x),T(y)) = p(x-1,x-1) = x-1,$$

and

$$\max\Big\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{3}[p(x,y)+p(x,Ty)+p(y,Tx)]\Big\} = \frac{x+y}{2}.$$

Hence, we have

$$p(T(x),T(y)) = x-1 \leq k\,x = k\,p(x,y),$$

where $k \in [0,1)$.

Case 3. If $x > y = 0$, then

$$p(T(x),T(y)) = p(x-1,0) = \frac{x-1}{2},$$

and

$$\max\Big\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{3}[p(x,y)+p(x,Ty)+p(y,Tx)]\Big\} = \frac{2x-1}{2}.$$

Hence, we have

$$p(T(x),T(y)) = \frac{x-1}{2} \leq k\,\frac{x}{2} = k\,p(x,y),$$

where $k \in [0,1)$.

Case 4. If $x > y > 0$, then

$$p(T(x),T(y)) = p(x-1,y-1) = \frac{x+y-2}{2},$$

and

$$\max\Big\{p(x,y),p(x,Tx),p(y,Ty),\frac{1}{3}[p(x,y)+p(x,Ty)+p(y,Tx)]\Big\} = \frac{x+y}{2}.$$

Hence, we have

$$p(T(x),T(y)) = \frac{x+y-2}{2} \leq k\,\frac{x+y}{2} = k\,p(x,y),$$

where $k \in [0,1)$.

This shows that all conditions of Corollary 3.4 are satisfied for $k \in [0,1)$ and so T has a unique fixed point in X. Indeed, $0 \in X$ is the unique fixed point in this case.

(2) Now, we claim that the inequality of Corollary 3.5 is satisfied. For this, we consider the following cases.

Case 1. If $x = y = 0$, then

$$p(T(x), T(y)) = 0,$$

and

$$\max\left\{p(x,y), \frac{1}{4}[p(x,Tx)+p(y,Ty)], \frac{1}{4}[p(x,Ty)+p(y,Tx)]\right\} = 0.$$

Hence, we conclude that

$$p(T(x), T(y)) \leq q \max\left\{p(x,y), \frac{1}{4}[p(x,Tx)+p(y,Ty)], \frac{1}{4}[p(x,Ty)+p(y,Tx)]\right\},$$

where $q \in [0, 1)$.

Case 2. If $x = y > 0$, then

$$p(T(x), T(y)) = p(x-1, x-1) = x-1,$$

and

$$\max\left\{p(x,y), \frac{1}{4}[p(x,Tx)+p(y,Ty)], \frac{1}{4}[p(x,Ty)+p(y,Tx)]\right\} = \frac{x+y}{2}.$$

Hence, we have

$$p(T(x), T(y)) = x-1 \leq q\, x = q\, p(x,y),$$

where $q \in [0, 1)$.

Case 3. If $x > y = 0$, then

$$p(T(x), T(y)) = p(x-1, 0) = \frac{x-1}{2},$$

and

$$\max\left\{p(x,y), \frac{1}{4}[p(x,Tx)+p(y,Ty)], \frac{1}{4}[p(x,Ty)+p(y,Tx)]\right\} = \frac{x+y}{2}.$$

Hence, we have

$$p(T(x), T(y)) = \frac{x-1}{2} \leq q\, \frac{x}{2} = q\, p(x,y),$$

where $q \in [0, 1)$.

Case 4. If $x > y > 0$, then

$$p(T(x), T(y)) = p(x-1, y-1) = \frac{x+y-2}{2},$$

and

$$\max\left\{p(x,y), \frac{1}{4}[p(x,Tx)+p(y,Ty)], \frac{1}{4}[p(x,Ty)+p(y,Tx)]\right\} = \frac{x+y}{2}.$$

Hence, we have

$$p(T(x), T(y)) = \frac{x+y-2}{2} \leq q\, \frac{x+y}{2} = q\, p(x,y),$$

where $q \in [0, 1)$.

This shows that all conditions of Corollary 3.5 are satisfied for $q \in [0, 1)$ and so T has a unique fixed point in X. Indeed, $0 \in X$ is the unique fixed point.

4. Conclusion

In this paper, we establish some unique common fixed point and a coincidence point theorems using auxiliary functions in the setting of complete weak partial metric spaces and give some consequences of the main results. We also give some examples to support our results. The results presented in this article extend and generalize several results from the existing literature.

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