FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 5 (2022), 975–991
https://doi.org/10.22190/FUMI220226067D
Original Scientific Paper

KILLING MAGNETIC FLUX SURFACES IN HEISENBERG THREE-GROUP *

Mohamed H. Dida and Fouzi Hathout

Faculty of Science, Department of Mathematics
Saida University, 20000 Saida Algeria

Abstract. In this paper, we determine, in the Heisenberg group, the parametric Killing magnetic flux surfaces and their corresponding Killing scalar flux functions. An examples of each are given with a graphic representation in Euclidean space.
Key words: Heisenberg group; Flux, Killing, Magnetic, flux surface, Scalar flux function.

1. Introduction

The flux usually describes an effect that appears to pass through a surface or substance. It is a concept in applied mathematics and vector calculus that have many applications in physics which we can cite fluid mechanics, thermodynamics, electromagnetism, radiation, energy and in particular particle flux (see [3] and [13]). In vector calculus, the flux is a scalar quantity, defined as the surface integral of the perpendicular component of a vector field over a surface.

Let $\mathbb{M}$ be an arbitrary surface in a Riemannian manifold $(\mathbb{N}, g)$, $\overrightarrow{n}$ is the normal vector and $\overrightarrow{V}$ is a smooth vector field on $\mathbb{N}$. The flux $\mathcal{V}$ corresponding to the vector $\overrightarrow{V}$, passes through the surface $\mathbb{M}$ is given by

$$\mathcal{V} = \int_{\mathbb{M}} g(\overrightarrow{V}, \overrightarrow{n})ds$$

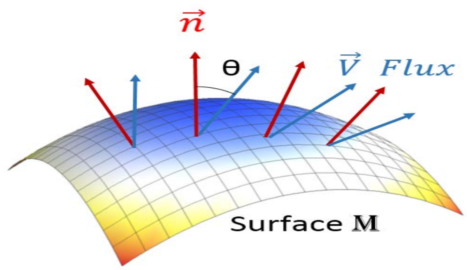
Received February 26, 2022, accepted: August 27, 2022
Communicated by Mića Stanković
Corresponding Author: Fouzi Hathout, Faculty of Science, Department of Mathematics, Saida university, 20000 Saida Algeria | E-mail: f.hathout@gmail.com, fouzi.hathout@univ-saida.dz
2010 *Mathematics Subject Classification.* Primary 53A35; Secondary 53C21, 37C10

*The paper was supported by Laboratory of fundamental and applied mathematics, University of Oran, 31000 Algeria.

For simplicity, we denote the vectors $\overrightarrow{V}, \overrightarrow{n}$ by V, n. The normal component of flux $\mathcal{V}$ is

$$g(V, n) = v \cos \theta$$

here θ is the angle between V and n. When the angle θ is $\frac{\pi}{2}$ $(i.e. V \perp n.)$, the surface $\mathbb{M}$ is called *flux surface.*



The flux surfaces $\mathbb{M}$ can also be characterized following the character of the flux vector V. If V is a magnetic vector fields (i.e. which is zero divergences according to Biot and Savart's law see [2], [7] and [10]) then $\mathbb{M}$ is called *flux surface according to the magnetic vector fields* V and we denoted it for simplify V-magnetic flux surface. Moreover, if V is a Killing i.e. magnetic vector fields satisfying the Killing equation

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0$$

then $\mathbb{M}$ is called *Killing V-magnetic flux surface,* where ∇ is a connection and X, Y are a vector fields on $\mathbb{M}$.

When $\mathbb{M}$ is V-magnetic flux surface, the vector V does not cross $\mathbb{M}$ anywhere (i.e. magnetic flux V traversing $\mathbb{M}$ is zero). This gives rise to the definition of a *scalar flux function* f which $\nabla f \perp n$ and f is constant on $\mathbb{M}$.

When V is a magnetic field with toroidal nested flux surfaces, two magnetic fluxes can be defined from two corresponding surfaces (see [1] and [13]). The poloidal flux is defined by

$$\mathcal{V}_1 = \int_{\mathbb{M}_p} g(V, n) dS$$

where $\mathbb{M}_p$ is a ring-shaped ribbon stretched between the magnetic axis and the flux surface. Likewise, the toroidal flux is defined by

$$\mathcal{V}_2 = \int_{\mathbb{M}_t} g(V, n) dS$$

where $\mathbb{M}_t$ is a poloidal section of the flux surface.

Considered as fourth state of matter, the plasma is the famous example of flux surface. Plasma is a hot ionized gas composed of approximately equal numbers of positively charged ions and negatively charged electrons, making it a good electrical conductor. The electrical conductivity creates currents flowing in a plasma that

interact with magnetic field to create the force needed for containment. Ordinary matter ionizes and forms plasma at temperatures above about 5000 K, and most of the visible matter in the universe is in a plasma state.
Plasma particles can be confined and shaped by magnetic field lines that combine to act like an invisible bottle. By fixing magnetic field lines toroidal around the interior of the tokamak, the ions and electrons in the plasma are forced to move slightly around these field lines, preventing them from escaping from the container. If it is assumed that the plasma is magnetized everywhere, the magnetic field does not disappear on this surface. The authors, in [8], have shown that the outermost, bounding surface must be a flux surface, it is natural to suppose the confinement region to be filled with a sequence of flux surfaces, each enclosing the next. In fact, flux surfaces provides a barrier to collisionless charged particles in the magnetic field. Most of the universe is in the form of a plasma with a magnetic field perforated. (see for more detail [1], [2], [8] and [11]).

Recently in [10], the Killing magnetic flux surfaces were determined in Euclidian space. In our study, we determine all Killing magnetic flux surfaces and their corresponding Killing scalar flux functions in the three-dimensional Heisenberg group.

The paper is organized as follows. In section 2, we present an Overview on a geometry of Heisenberg 3-group $\mathbb{H}_3$ with its four Killing vectors representations. The section 3 is devoted to the determination of parameterisations of Killing flux surfaces with examples and its associate Killing magnetic scalar flux functions.

2. Geometry of Heisenberg group $\mathbb{H}_3$

The Heisenberg group $\mathbb{H}_3$ is a Lie group diffeomorphic to $\mathbb{R}^3$ with the standard representation in $SL(3,\mathbb{R})$ as

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x,y,z) \in \mathbb{R}^3 \right\}$$

endowed with the multiplication

$$(x_1,y_1,z_1)(x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2-x_1y_2).$$

All left-invariant Riemannian metrics on the $\mathbb{H}_3$ are isometric to the Riemannian metric $g_{\mathbb{H}_3}$ which is invariant with respect to the left-translations corresponding to that multiplication, given by

$$g_{\mathbb{H}_3} = \frac{1}{\lambda^2}dx^2 + dy^2 + (xdy+dz)^2. \tag{2.1}$$

where λ is a strictly positif real number.

We define an orthonormal basis $(e_i)_{i=\overline{1,3}}$, by

$$e_1 = \partial y - x\partial z, \quad e_2 = \lambda\partial x, \quad e_3 = \partial z, \tag{2.2}$$

and its dual basis $\left(\omega^i\right)_{i=\overline{1,3}}$, by

$$\omega^1 = dy; \quad \omega^2 = \frac{1}{\lambda}dx; \quad \omega^3 = xdy + dz.$$

The Lie bracket of the basis $(e_i)_{i=\overline{1,3}}$ are given by the following identities

$$[e_1, e_2] = \lambda e_3; \quad [e_1, e_3] = [e_2, e_3] = 0 \tag{2.3}$$

The Levi-Civita connection ∇ of the metric $g_{\mathbb{H}_3}$ with respect to the left-invariant basis $(e_i)_{i=\overline{1,3}}$ is

$$\left\{\begin{array}{lllll} \nabla_{e_1}e_1 = 0 & & \nabla_{e_2}e_1 = -\frac{\lambda}{2}e_3 & & \nabla_{e_3}e_1 = -\frac{\lambda}{2}e_2 \\ \nabla_{e_1}e_2 = \frac{\lambda}{2}e_3 & ; & \nabla_{e_2}e_2 = 0 & ; & \nabla_{e_3}e_2 = \frac{\lambda}{2}e_1 \\ \nabla_{e_1}e_3 = -\frac{\lambda}{2}e_2 & & \nabla_{e_2}e_3 = \frac{\lambda}{2}e_1 & & \nabla_{e_3}e_3 = 0 \end{array}\right. . \tag{2.4}$$

The Lie algebra of Killing vector fields of $(\mathbb{H}_3, g_{\mathbb{H}_3})$ is generated by the basis $\mathbb{K} = (K_i)_{i=\overline{1,4}}$, where the killing vectors $(K_i)_{i=\overline{1,4}}$ are presented in the following

$$\begin{aligned} \mathbf{K}_1 &= \partial z, \quad \mathbf{K}_2 = \partial y, \quad \mathbf{K}_3 = \partial x - y\partial z, \\ \mathbf{K}_4 &= \lambda^2 y\partial x - x\partial y - \frac{1}{2}(\lambda^2 y^2 - x^2)\partial z. \end{aligned} \tag{2.5}$$

We can rewrite the killing vectors in the base $(e_i)_{i=\overline{1,3}}$, using the Eq.(2.2), as

$$\begin{aligned} \mathbf{K}_1 &= e_3, \quad \mathbf{K}_2 = e_1 + xe_3, \quad \mathbf{K}_3 = \frac{1}{\lambda}e_2 - ye_3, \\ \mathbf{K}_4 &= -xe_1 + \lambda ye_2 + \frac{1}{2}(3x^2 - \lambda^2 y^2)e_3. \end{aligned} \tag{2.6}$$

(for more detail see [2], [5] and [6]).

3. Killing flux surfaces in $(\mathbb{H}_3, g_{\mathbb{H}_3})$

Definition 3.1. Let M be a smooth surface in a Riemannian manifold (N, g) and $\mathbf{n}$ be its normal vector field. The surface M is called a flux surface of a smooth vector field V on (N, g) if

$$g(V, \mathbf{n}) = 0$$

everywhere on M. Moreover, if V is a Killing magnetic field then M is called a Killing magnetic flux surface corresponding to V.

Lemma 3.1. *Let f be a scalar function in (N, g), then the Riemannian gradient of f is*

$$\nabla f = f_x\partial x + f_y\partial y + f_z\partial z = f_y e_1 + \frac{f_x}{\lambda}e_2 + (xf_y + f_z)\, e_3$$

In the sequel, to simplify, we denoted a flux surface M corresponding to the smooth vector field V by *V-flux surface* and if V is Killing magnetic vector M is denoted by *Killing V-flux surface*. We use the computer software "Wolfram Mathematica" to present the surface figures in Euclidean 3-space.

3.1. K_1-Flux surfaces in $(\mathbb{H}_3, g_{\mathbb{H}_3})$

Let $\mathbb{M}$ be a surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ and $X(u,v) = (x(u,v), y(u,v), z(u,v))$ be its parametrization. The tangent vectors $X_u = \frac{\partial X}{\partial u}$ and $X_v = \frac{\partial X}{\partial v}$ are described by

$$\begin{aligned} X_u &= x_u\partial x + y_u\partial y + z_u\partial z = y_u e_1 + \frac{x_u}{\lambda} e_2 + (xy_u + z_u)\, e_3 \\ X_v &= x_v\partial x + y_v\partial y + z_v\partial z = y_v e_1 + \frac{x_v}{\lambda} e_2 + (xy_v + z_v)\, e_3 \end{aligned}$$

Its normal vector $\mathbf{n}$ in the base $(e_i)_{i=\overline{1,3}}$ is

$$\mathbf{n} = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{1}{\|X_u \times X_v\|} \begin{pmatrix} \frac{x_u}{\lambda}(xy_v + z_v) - \frac{x_v}{\lambda}(xy_u + z_u) \\ y_v(xy_u + z_u) - y_u(xy_v + z_v) \\ \frac{1}{\lambda}(y_u x_v - x_u y_v) \end{pmatrix} \tag{3.1}$$

Before starting the determination of Killing magnetic flux surfaces in $(\mathbb{H}_3, g_{\mathbb{H}_3})$, we will confront to the resolution of PDEs, therefore, we use the following notation and proposition.

Notation 3.1. *Let $x(u,v)$, $y(u,v)$ and $z(u,v)$ be a functions in $\mathbb{R}^2$. We denote by the real functions $\psi_{1,2,3}(u,v)$ =constant, the solution of ODEs*

$$\frac{du}{x_u} = -\frac{dv}{x_v}, \quad \frac{du}{y_u} = -\frac{dv}{y_v} \quad \text{and} \quad \frac{du}{z_u} = -\frac{dv}{z_v}$$

respectively.

Proposition 3.1. *Let f P and Q be a real functions with a respect to the parameters u, v. The solutions of the PDE*

$$Pf_u + Qf_v = 0$$

are:
1. If $P \equiv 0$ (resp. $Q = 0$) then $f(u,v) = f(u)$ (resp. $f(u,v) = f(u)$)
2. f $P, Q \neq 0$ then $f(u,v) = \varphi(\psi(u,v))$ where $\psi(u,v) = cst$ is solution of ODE

$$\frac{du}{P} = -\frac{dv}{Q}$$

and φ an arbitrary real function in u and v.

Proof. See the method to solve PDE in [12]. □

Theorem 3.1. *Let $\mathbb{M}$ be a surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ and $X(u,v) = (x(u,v), y(u,v), z(u,v))$ be its parametrization. Then $\mathbb{M}$ is a K_1-flux surface of the Killing vector field K_1 given in Eq.(2.6) if and only if*

$$x_v y_u - x_u y_v = 0$$

Proof. It's a direct consequence by using the inner product, in the orthonormal base $(e_i)_{i=\overline{1,3}}$ defined in the Definition 3.1, of the normal vector $\mathbf{n}$ given from the Eq.(3.1) and the Killing vector K_1. □

Proposition 3.2. *All K_1-flux surfaces in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ are parameterized by*

$$\begin{aligned} \mathbf{1}.\ X(u,v) &= (x(u,v), \varphi(\psi_1(u,v)), z(u,v)), \\ \mathbf{2}.\ X(u,v) &= (\varphi(\psi_2(u,v)), y(u,v), z(u,v)), \\ \mathbf{3}.\ X(u,v) &= (\varphi_1(u), \varphi_2(u), z(u,v)) \\ \mathbf{4}.\ X(u,v) &= (\varphi_1(v), \varphi_2(v), z(u,v)) \end{aligned}$$

where x, z y and $\varphi, \varphi_{1,2}$ are an arbitrary smooth functions in $\mathbb{R}^2$ and $\mathbb{R}$, respectively.

Proof. Using the Notation 3.1 and the Proposition 3.1, the parameterizations $X(u,v)$ is a general solution of the first order linear PDE given in the Theorem 3.1 for an arbitrary functions x and y for the assertions 1 and 2, respectively. For the assertions 3 and 4, it's direct consequence from assertions 1 of the Proposition 3.1. □

Example 3.1. **1.** Let $x(u,v) = uv$, from the Proposition 3.2, we have

$$\frac{du}{v} = -\frac{dv}{u}$$

its solution is

$$\psi_1(u,v) = u^2 + v^2 = c \text{ constant} \tag{3.2}$$

then the surface $\mathbb{M}_1$ parameterized by

$$X(u,v) = \left(uv, \varphi\left(u^2+v^2\right), z(u,v)\right)$$

is K_1-flux surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$, where φ and z are an arbitrary smooth functions in $\mathbb{R}$ and $\mathbb{R}^2$ respectively. (See Figure 3.1)

2. Let $y(u,v) = (c + a\cos v)\cos u$, similarly, using the Proposition 3.2, we have

$$\frac{-du}{(c+a\cos v)\sin u} = \frac{dv}{a\sin v\cos u}$$

its solution is

$$\psi_2(u,v) = (2a\sin v + cv)\sin u = c \text{ constant}$$

then the surface $\mathbb{M}_2$ parameterized by

$$X(u,v) = (\varphi((2a\sin v + cv)\sin u), (c + a\cos v)\cos u, z(u,v))$$

is K_1-flux surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$, where φ and z are an arbitrary smooth functions in $\mathbb{R}$ and $\mathbb{R}^2$, respectively.(See Figure 3.2)

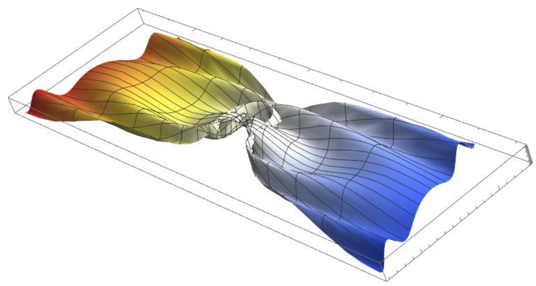


FIG. 3.1: K_1-flux surface $X(u;v) = (uv, \sin(u^2+v^2), u^3+v))$ in $(\mathbb{H}^3; g_{\mathbb{H}^3})$ presented in $(\mathbb{R}^3, g_{euc})$.

3.1.1. Scalar flux functions

Definition 3.2. Let f be a function on (N, g). Then f is called a scalar flux function corresponding to the magnetic vector field V if its value is constant on the V-magnetic flux surface M, and

$$g(V, \nabla f) = 0$$

we denoted here f, to simplify, a V-magnetic scalar flux function. Moreover, if V is magnetic and Killing, f is denoted by Killing V-magnetic scalar flux function.

Now, we can present the following theorem.

Theorem 3.2. *Let $\mathbb{M}$ be a K_1-magnetic flux surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$. Then the function f given by*

$$f(x, y, z) = \psi(xz - y)$$

and constant on $\mathbb{M}$ is K_1-magnetic scalar flux function to $\mathbb{M}$, where ψ is an arbitrary smooth function in $\mathbb{R}$.

Proof. Using the Definition 3.2 and the Lemma 3.1, we have

$$g(K_1, \nabla f) = xf_y + f_z = 0$$

by solving the above first order PDE we get K_1-magnetic scalar flux function f. □

Example 3.2. Using the Example 3.1, we have the K_1-magnetic flux surface $\mathbb{M}$ parameterized by $X(u,v) = \left(uv, \varphi\left(u^2+v^2\right), z(u,v)\right)$, where φ and z are an arbitrary smooth functions. The K_1-magnetic scalar flux function f to $\mathbb{M}$, from the Theorem 3.2, is in

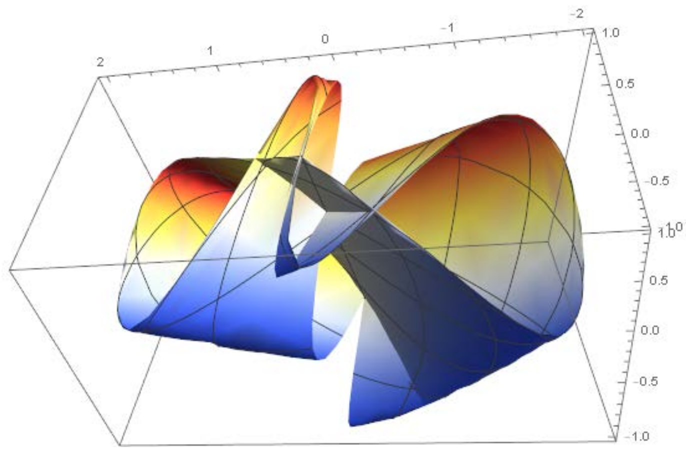


FIG. 3.2: K_1-Flux surface $X(u,v) = (2\sin v \sin u; \cos v \cos u; \sin uv)$ in $(\mathbb{H}^3, g_{\mathbb{H}^3})$ presented in $(\mathbb{R}^3, g_{euc})$.

the form $f(x,y,z) = \psi(xz-y)$ and it must be constant on $\mathbb{M}$, (i.e. $f(X(u,v)) \equiv C$ a constant). We have

$$f(X(u,v)) = \psi(uv\ z(u,v) - y\left(u^2+v^2\right))$$

by choosing the functions

$$\begin{cases} y = Id_{\mathbb{R}} \\ z(u,v) = \frac{u}{v} + \frac{v}{u} \end{cases}$$

we obtain

$$f(X(u,v)) = \psi(0) \text{ a constant}$$

then $f(x,y,z) = \psi(xz-y)$ is K_1-magnetic scalar flux function to the magnetic K_1-flux surface $\mathbb{M}$ parameterized by $X(u,v)=\left(uv, u^2+v^2, \frac{u}{v}+\frac{v}{u}\right)$ given in Figures 3.3.

3.2. K_2-Flux surfaces in $(\mathbb{H}_3, g_{\mathbb{H}_3})$

In this subsection, we characterise and present all K_2-flux surfaces corresponding to the Killing vector field K_2 given in the Eq.(2.6).

Theorem 3.3. *Let $\mathbb{M}$ be a surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ and $X(u,v) = (x(u,v), y(u,v), z(u,v))$ be its parametrization. Then $\mathbb{M}$ is a K_2-flux surface if and only if*

$$x_u z_v - x_v z_u = 0$$

Proof. The proof is similar as the proof of the Theorem 3.1 using the Killing vector K_2 given in Eq.(2.6). □

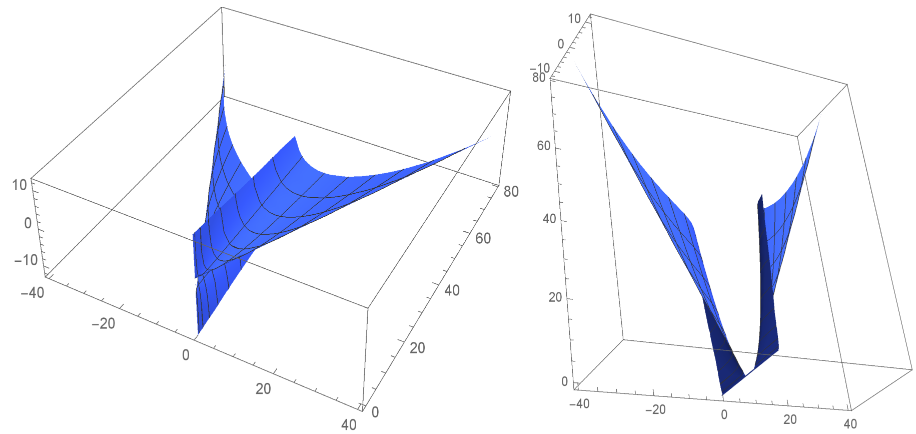


FIG. 3.3: K_1-Magnetic flux surface $\mathbb{M}$ parameterized by $X(u,v)=\left(uv, u^2+v^2, \frac{u}{v}+\frac{v}{u}\right)$

Proposition 3.3. *All K_1-flux surfaces in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ are parameterized by*

$$\begin{array}{rcl} \mathbf{1}.\ X(u,v) & = & (x(u,v), y(u,v), \varphi(\psi_1(u,v))), \\ \mathbf{2}.\ X(u,v) & = & (\varphi(\psi_3(u,v)), y(u,v), z(u,v)), \\ \mathbf{3}.\ X(u,v) & = & (\varphi_1(u), y(u,v), \varphi_2(u)) \\ \mathbf{4}.\ X(u,v) & = & (\varphi_1(v), y(u,v), \varphi_2(v)) \end{array}$$

where x, z y and $\varphi, \varphi_{1,2}$ are an arbitrary smooth functions in $\mathbb{R}^2$ and $\mathbb{R}$, respectively.

Proof. The proof is similar to the proof of the Proposition 3.2. □

Example 3.3. Let $x(u,v) = uv$, from the Proposition 3.2, the Notation 3.1 and a same computation as the Example 3.1, we have

$$\psi_1(u,v) = u^2 + v^2 = c \text{ constant}$$

then the surface $\mathbb{M}$ parameterized by

$$X(u,v)=\left(uv, y(u,v), \varphi\left(u^2+v^2\right)\right)$$

is K_2-flux surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$, where φ and u are an arbitrary smooth functions in $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

3.2.1. K_2-Magnetic scalar flux functions

A same as Subsubsection 3.1.1., we have the following theorem.

Theorem 3.4. *Let $\mathbb{M}$ be a K_2-magnetic flux surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$. Then the function f given by*

$$f(x,y,z) = \psi(xy - \left(1+x^2\right)z)$$

and constant on $\mathbb{M}$ is K_2-magnetic scalar flux function to $\mathbb{M}$, where ψ is an arbitrary real smooth function.

Proof. Using the Definition 3.2 and the Lemma 3.1, we have

$$g(K_1, \nabla f) = \left(1+x^2\right) f_y + x f_z = 0 = 0$$

we get K_2-magnetic scalar flux function f by solving the above linear first order PDE. □

Example 3.4. We want to determine the K_1-magnetic scalar flux function f to $\mathbb{M}$ given in Example 3.3. From the Theorem 3.4, the K_1-magnetic scalar flux function is in the form $f(x,y,z) = \psi(xy - \left(1+x^2\right)z)$ and it must be constant on $\mathbb{M}$, i.e. $f(X(u,v)) \equiv C$ a constant. We have

$$f(X(u,v)) = \psi(uv * y(u,v) - \left(1+u^2v^2\right)\varphi\left(u^2+v^2\right))$$

by choosing the functions

$$y = \frac{c}{uv} + \left(\frac{1}{uv} + uv\right)\varphi\left(u^2+v^2\right);\ c \in \mathbb{R}$$

we obtain

$$f(X(u,v)) = \psi(c) \text{ a constant}$$

then f is K_2-magnetic scalar flux function to the K_2-flux surface $\mathbb{M}$ parameterized by

$$X(u,v) = \left(uv, \frac{c}{uv} + \left(\frac{1}{uv} + uv\right)\varphi\left(u^2+v^2\right), \varphi\left(u^2+v^2\right)\right)$$

where φ and c are real function and constant, respectively. We present $\mathbb{M}$ for c=1 and $\varphi(x) = \sin x$, in Figures 3.4.

3.3. K_3-Flux surfaces in $(\mathbb{H}_3, g_{\mathbb{H}_3})$

Similarly, as in the above subsections, we have the following theorem for K_3-flux surfaces.

Theorem 3.5. *Let $\mathbb{M}$ be a surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ and $X(u,v) = (x(u,v), y(u,v), z(u,v))$ be its parametrization. Then $\mathbb{M}$ is a K_3-flux surface if and only if*

$$\begin{cases} x_u y_v - x_v y_u = 0 \text{ and} \\ y_v z_u - y_u z_v = 0 \end{cases}$$

Proof. The proof is similar as the Theorem 3.1 using the Killing vector K_3. □

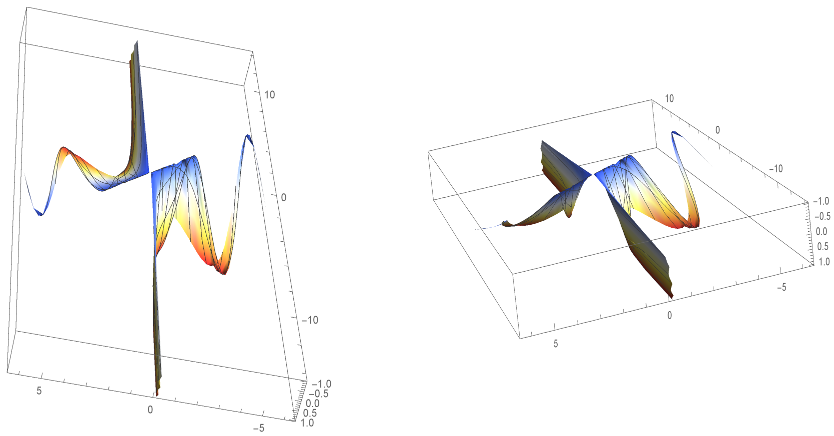


FIG. 3.4: K_2-flux surface $\mathbb{M}$ for K_2-magnetic scalar flux function f

Proposition 3.4. *The parametric surfaces in* $(\mathbb{H}_3, g_{\mathbb{H}_3})$ *with the parametrization* $X(u,v)$ *given by*

$$\begin{array}{lcl} \mathbf{1}.\ X(u,v) & = & \left(x(u,v), \varphi_1(\psi_1(u,v)), \varphi_2(\overline{\psi}_2(u,v))\right), \\ \mathbf{2}.\ X(u,v) & = & \left(\varphi_1(\psi_2(u,v)), y(u,v), \varphi_2(\psi_2(u,v))\right), \\ \mathbf{3}.\ X(u,v) & = & \left(\varphi_2(\overline{\psi}_2(u,v)), \varphi_1(\psi_3(u,v)), z(u,v)\right) \end{array}$$

are K_3-flux surfaces, where x, y, z and $\varphi_{\underline{1,2}}$ are an arbitrary smooth functions in $\mathbb{R}^2$ and $\mathbb{R}$, respectively and the real function $\overline{\psi}_2\,(u,v)$ =constant the solution of ODE

$$\frac{du}{\varphi_1(\psi_{1,3}(u,v))_u} = -\frac{dv}{\varphi_1(\psi_{1,3}(u,v))_v}$$

for the assertions 1,3 respectively, and we assume that $\frac{\partial(x,y,z)}{\partial(u,v)} \neq 0$.

Proof. The PDEs of the Theorem 3.5 hold when the system

$$S : \begin{cases} y_v z_u - y_u z_v = 0 \\ x_u y_v - x_v y_u = 0 \end{cases}$$

vanish and using the same method in the Theorem 3.1 with the Notation 3.1 and the Proposition 3.1, we get all K_3-flux surfaces presented in the assertions $\overline{1,3}$. □

Example 3.5. We construct an example using the assertion 3 of the Proposition 3.4. Let $x(u,v) = u^2 + v^2$, using the Propositions 3.1, 3.4 and the Notation 3.1, we have

$$\frac{du}{2u} = -\frac{dv}{2v}$$

its solution is

$$\psi_1(u,v) = uv = c \text{ constant}$$

and $y(u,v) = \varphi_1(uv)$. By chousing $\varphi_1(x) = x^2$, again, we have

$$\frac{du}{\left((uv)^2\right)_u} = -\frac{dv}{\left((uv)^2\right)_v}$$

and

$$\overline{\psi}_2(u,v) = u^2 + v^2 = 0$$

then the surface $\mathbb{M}$ parameterized by

$$X(u,v) = \left(u^2 + v^2, (uv)^2, \varphi\left(u^2 + v^2\right)\right)$$

is K_3-flux surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$, where φ is an arbitrary smooth function. We present, in $(\mathbb{R}^3, g_{euc})$, the K_3-flux surface $X(u,v) = \left(u^2 + v^2, (uv)^2, \ln\left(u^2 + v^2\right)\right)$ in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ in Figures 3.5.

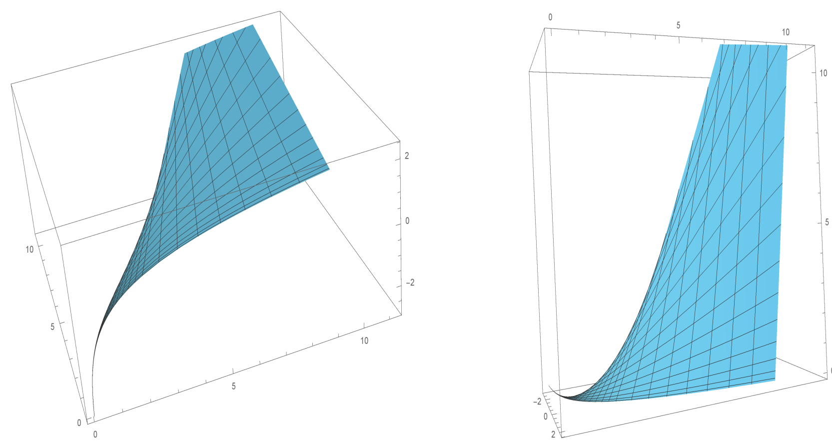


FIG. 3.5: K_3-Magnetic flux surface $X(u,v) = \left(u^2 + v^2, (uv)^2, \ln\left(u^2 + v^2\right)\right)$

3.3.1. K_3-Magnetic Scalar flux functions

Similarly as Subsubsections 3.1.1. and 3.2.1., we have,

Theorem 3.6. *Let $\mathbb{M}$ be a K_3-magnetic flux surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$. Then the function f given by*

$$f(x,y,z) = \psi\left(\lambda^2 x^2 + \ln y^2, \lambda^2 yx + z\right)$$

and constant on $\mathbb{M}$ is K_3-magnetic scalar flux function to $\mathbb{M}$, where ψ is an arbitrary smooth function in $\mathbb{R}^2$.

Proof. The proof is similar to the Theorems 3.2 and 3.4 proofs. □

Example 3.6. Let $y(u,v) = u+v$, using the assertion 2 of the Proposition 3.4 and similar computation as the the Example 3.5, we ge the K_3-magnetic flux surface $\mathbb{M}$ in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ parameterized by

$$X(u,v) = (\varphi_1(u+v), u+v, \varphi_2(u+v))$$

where $\varphi_{1,2}$ are an arbitrary smooth real functions. By choosing

$$\varphi_1 \equiv Id_{\mathbb{R}};\quad \varphi_2(x) = 2\ln x \text{ and } \psi(x,y) = x - y + c \ (c \in \mathbb{R})$$

the function

$$\begin{aligned} f(x,y,z) &= \psi\left(\lambda^2 x^2 + \ln y^2, \lambda^2 yx + z\right) \\ &= \lambda^2 x^2 + \ln y^2 - \lambda^2 yx - z + c \end{aligned}$$

is constant on K_4-magnetic flux surface $\mathbb{M}$ parameterized by

$$X(u,v) = (u+v, u+v, 2\ln(u+v))$$

in $(\mathbb{H}_3, g_{\mathbb{H}_3})$, from the Theorem 3.6, f is K_3-magnetic scalar flux function to $\mathbb{M}$.

3.4. K_4-Flux surfaces in $(\mathbb{H}_3, g_{\mathbb{H}_3})$

Finally, as seen in the above subsections, we consider K_4-flux surfaces in $(\mathbb{H}_3, g_{\mathbb{H}_3})$.

Theorem 3.7. *Let $\mathbb{M}$ be a surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ and $X(u,v) = (x(u,v), y(u,v), z(u,v))$ be its parametrization. Then $\mathbb{M}$ is a K_4-flux surface if and only if*

$$\left(\frac{1}{2}y^2\lambda^2 - \frac{5}{2}x^2\right)(x_u y_v - x_v y_u) - x\,(x_u z_v - x_v z_u) - y\lambda^2\,(y_u z_v - y_v z_u) = 0$$

Proof. Using the Killing vector K_4, we get the proof with same proof as the Theorem 3.1. □

Proposition 3.5. *The parametric surfaces in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ with the parametrization $X(u,v)$ given as*

$$\begin{aligned} 1.\ X(u,v) &= (x(u,v), \varphi(\psi_1(u,v)), \varphi(\psi_1(u,v))), \\ 2.\ X(u,v) &= (\varphi(\psi_2(u,v)), y(u,v), \varphi(\psi_2(u,v))), \\ 3.\ X(u,v) &= (\varphi(\psi_3(u,v)), \varphi(\psi_3(u,v)), z(u,v)) \\ 4.\ X(u,v) &= \left(\varphi(\psi_3(u,v)), \tfrac{\pm\sqrt{5}}{\lambda}\varphi(\psi_3(u,v)), z(u,v)\right) \\ 5.\ X(u,v) &= \left(x(u,v), \tfrac{\pm\sqrt{5}}{\lambda}x(u,v), \varphi(\psi_1(u,v))\right) \end{aligned}$$

are K_4-flux surfaces, where x, y, z and φ are an arbitrary real smooth functions in $\mathbb{R}^2$ and $\mathbb{R}$ respectively, and we assume that $\frac{\partial(x,y,z)}{\partial(u,v)} \neq 0$.

Proof. Similarly as the Theorem 3.5, it's not easy to solve the PDE of the Theorem 3.7. We can find some solutions when the systems

$$S_1 : \begin{cases} x_u y_v - x_v y_u = 0 \\ x_u z_v - x_v z_u = 0 \\ y_u z_v - y_v z_u = 0 \end{cases} \text{ or } S_2 : \begin{cases} y \pm \frac{\sqrt{5}}{\lambda} x = 0 \\ x_u z_v - x_v z_u = 0 \\ y_u z_v - y_v z_u = 0 \end{cases}$$

vanish. Using the Notation 3.1 and the Proposition 3.1, the solutions of the system S_1 is

1. $(x(u,v), \varphi(\psi_1(u,v)), \varphi(\psi_1(u,v)))$,
2. $(\varphi(\psi_2(u,v)), y(u,v), \varphi(\psi_2(u,v)))$,
3. $(\varphi(\psi_3(u,v)), \varphi(\psi_3(u,v)), z(u,v))$

where φ is an arbitrary real function and $\frac{\partial(x,y,z)}{\partial(u,v)} \neq 0$. For the system S_2, for an arbitrary function z, the solutions of the equations $(S_2)_{2,3}$ are

$$x(u,v) = \varphi_1(\psi_3(u,v)) \text{ and } y(u,v) = \varphi_2(\psi_3(u,v))$$

substituting the last equations in the equation $(S_2)_1$, we get

$$\varphi_2 = \pm \frac{\sqrt{5}}{\lambda} \varphi_1$$

hence, the solution of (S_2) is $\left(\varphi_1(\psi_3(u,v)), \frac{\pm\sqrt{5}}{\lambda}\varphi_1(\psi_3(u,v)), z(u,v)\right)$, which prove the assertion.
For an arbitrary function x, we have $y = \pm\frac{\sqrt{5}}{\lambda} x$ by the equation $(S_2)_1$ and the equations $(S_2)_{2.3}$ become the same and

$$z(u,v) = \varphi(\psi_1(u,v))$$

and similar result when we take an arbitrary function y. Then the solution of (S_2) is

$$\left(x(u,v), \frac{\pm\sqrt{5}}{\lambda} x(u,v), \varphi(\psi_1(u,v))\right)$$

where $\psi_{1,2}$ are given in the Notation 3.1 and φ is an arbitrary real function.
□

Example 3.7. 1. Let $x(u,v) = \sin uv$, from the assertion 1 of Proposition 3.5 and the Notation 3.1, we have

$$\frac{du}{v} = -\frac{dv}{u}$$

its solution is

$$\psi_1(u,v) = u^2 + v^2 = c \text{ constant}$$

the surface $\mathbb{M}$ parameterized by

$$X(u,v) = \left(\sin uv, \varphi(u^2+v^2), \varphi(u^2+v^2)\right)$$

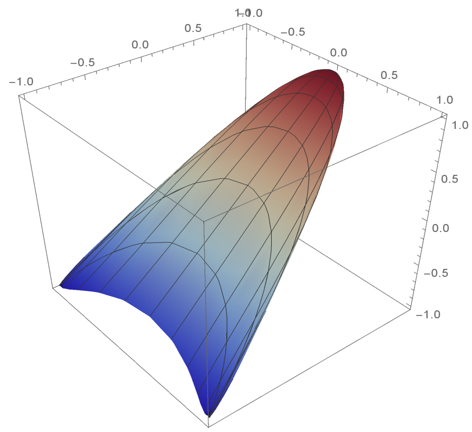


FIG. 3.6: Planar K_4-flux surface $X(u,v) = \left(\sin uv, \cos(u^2+v^2), \cos(u^2+v^2)\right)$

is K_4-flux surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$, where $\varphi_{1,2}$ are an arbitrary smooth real functions.
We present in $(\mathbb{R}^3, g_{euc})$, the K_4-flux surface $X(u,v) = \left(\sin uv, \cos\left(u^2+v^2\right), \cos(u^2+v^2)\right)$ in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ in Figure 3.6.

2. Let $x(u,v) = u^2 + v$, similarly we have

$$\frac{du}{2u} = -dv$$

its solution is

$$\psi_1(u,v) = u - 2uv = c \text{ constant}$$

the surface $\mathbb{M}$ parameterized by

$$X(u,v) = \left(u^2 + v, \pm\frac{\sqrt{5}}{\lambda}\left(u^2+v\right), \varphi(u - 2uv)\right)$$

is Killing K_4-flux surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$, where φ is an arbitrary smooth real function. (see Figure 3.7)

3.4.1. K_4-Magnetic scalar flux functions

Following the Subsubsections 3.1.1., 3.2.1. and 3.3.1., we have,

Theorem 3.8. *Let $\mathbb{M}$ be a K_4-magnetic flux surface in $(\mathbb{H}_3, g_{\mathbb{H}_3})$. Then the function f given by*

$$f\left(x,y,z\right) = \psi\left(\left(\frac{3}{4}x^2 - 1\right)x^2 + \left(1 - \frac{\lambda^2}{2}x^2\right)y^2, x^3 - \lambda^2 y^2 x + 2yz\right)$$

and constant on $\mathbb{M}$ is K_4-magnetic scalar flux function to $\mathbb{M}$, where ψ an arbitrary smooth function.

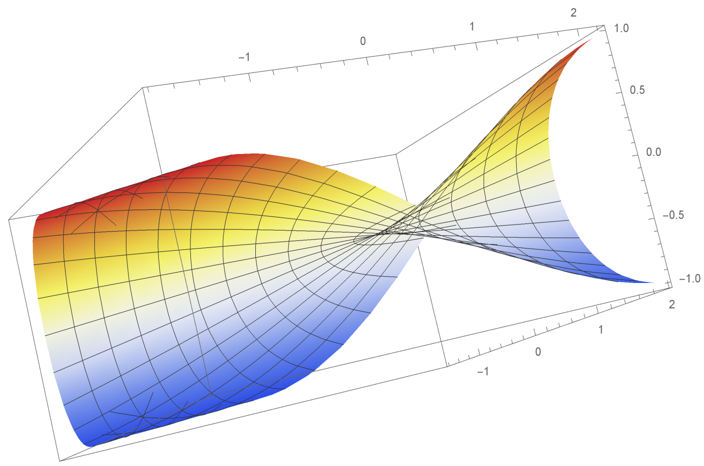


FIG. 3.7: Planar K_4-flux surface $X(u;v) = \left(u^2+v, \frac{\sqrt{5}}{2}\left(u^2+v\right), \sin(u-2uv)\right)$ in $(\mathbb{H}^3; g_{\mathbb{H}^3})$ presented in $(\mathbb{R}^3, g_{euc})$.

Proof. We get the proof a same as in the Theorems 3.2, 3.4 and 3.6. □

Example 3.8. Let $x(u,v) = u^2+v$, from the Example 3.7_2, we ge the K_4-magnetic flux surface $\mathbb{M}$ in $(\mathbb{H}_3, g_{\mathbb{H}_3})$ parameterized by

$$X(u,v) = \left(u^2+v, \frac{\sqrt{5}}{\lambda}\left(u^2+v\right), \varphi(u-2uv)\right)$$

where φ is an arbitrary smooth real function. By choosing

$$\lambda = \sqrt{5} \text{ and } \psi(a,b) = \psi(a)$$

the function

$$f(x,y,z) = \psi\left(\left(\frac{3}{4}x^2-1\right)x^2 + \left(1-\frac{3}{4}x^2\right)y^2, x^3 - \tfrac{3}{2}y^2x + 2yz\right)$$

is K_4-magnetic scalar flux function to K_4-flux surface $\mathbb{M}$ parameterized by

$$X(u,v) = \left(u^2+v, \left(u^2+v\right), \varphi(u-2uv)\right)$$

in $(\mathbb{H}_3, g_{\mathbb{H}_3})$, here $f(x,y,z) = \psi(0)$ a constant on $\mathbb{M}$.

REFERENCES

1. A. H. BOOZER: *Physics of magnetically confined plasmas*, Rev. Mod. Phys. **76**(2004), 1071-1141.
2. M. BARROS and A. ROMERO: *Magnetic vortices*, EPL **77**(2007), 34002.

3. R. B. Bird, W. E. Stewart and E. N. Lightfoot: *Transport Phenomena*, Wiley. ISBN 0-471-07392-X, (1960).
4. W. Batat and A. Zaeim: *On symmetries of the Heisenberg group*, arXiv:1710.04539v1.
5. L. Capogna, D. Danielli, S.D. Pauls and J.T. Tyson: *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem.* (Progress in Mathematics). Birkhäuser Verlag, Basel (2007).
6. K. Derkaoui and F. Hathout: Explicit formulas for Killing magnetic curves in three-dimensional Heisenberg group, Int. J. of Geo. Meth. in Modern Phy., **18**(09)(2021), 2150135. doi/abs/10.1142/S0219887821501358
7. I.S. Grant and W.R. Phillips: *Electromagnetism* (2nd Edition), Manchester Physics, John Wiley & Sons, (2008), ISBN 978-0-471-92712-9.
8. R. D. Hazeltine and J. D. Meiss: *Plasma Confinement*, Dover publications, inc. Mineola, New York, 2003.
9. C. Özgür: *On magnetic curves in* 3*-dimensional Heisenberg group*, Pro. of the Ins. of Math. and Mech., National Academy of Sciences of Azerbaijan, **43**(2)(2017), 278-286.
10. Z. Ozdemir, I. Gok, Y. Yayli and F.N. Ekmekci: *Killing magnetic flux surfaces in Euclidean 3-space*, Honam Math. J. **41**(2)(2019), 329-342. doi.org/10.5831/HMJ.2019.41.2.329
11. T.S. Pedersen and A. H. Boozer: *Confinement of nonneutral plasmas on magnetic surfaces*. Phys. Rev. Lett. **88**(2002), 205002.
12. Walter. A. Strauss: *Partial differential equations: An introduction*, ISBN 0-471-57364-7 (Wiley). The Mathematical Gazette, **77**(479), Lord, N. (1993), 286-287. doi:10.2307/3619758by
13. http://fusionwiki.ciemat.es/wiki/Flux_surface