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FUZZY DIFFERENTIAL SUBORDINATION INVOLVING GENERALIZED NOOR-RUSCHEWEYH OPERATOR

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Abstract. A new operator L^n_λ is introduced as a convex combination of Ruscheweyh derivative operator and Noor integral operator on the class $\mathcal{A}$ of analytic functions in the open unit disc E. The operator L^n_λ is studied using fuzzy set theory and fuzzy differential subordination. All the results proved are sharp. Some interesting special cases are derived as corollaries for particular choices of the functions acting as fuzzy best dominant.
Keywords: Banach space, nonexpansive mappings, iterative approximations.

1. Introduction

One of the most recent techniques of research in the field of geometric function theory is the method of differential subordination, which was introduced by Miller and Mocanu [4, 5]. A new research direction in this area has been launched by combining the concept of differential subordination with the complex function domain to the fuzzy set theory, see [10]. This notion is called fuzzy differential subordination. For more details, see [1, 2, 6, 7, 12].

In this paper, we shall use fuzzy subordination to obtain some interesting results in the context of geometric function theory for certain classes of analytic functions defined by generalized Noor-Ruscheweyh operator in the open unit disc.

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Let $E = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$. Let $H(E)$ be the space of functions which are analytic in E and let

$$H[a,n] = \big\{f \in H(E) : \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \quad z \in E, a \in \mathbb{C}, \big\},$$

where n is a positive integer.

We denote $H[1,n] = \mathcal{A}_n$ and $H[0,n] = \mathcal{A}$.

The class $C \subset \mathcal{A}$ consists of convex univalent functions of satisfying $Re\big(1 + \frac{zf''(z)}{f'(z)}\big) > 0, \quad z \in E$. Also the class $S^\star$ is defined as

$$S^\star = \big\{f \in \mathcal{A} : \quad Re\big(\frac{zf'(z)}{f(z)}\big) > 0, \quad z \in E\big\}.$$

The functions $f \in S^\star$ are called starlike functions in E.

Let $f, g \in \mathcal{A}$. we say that we say that f is subordinate to g, denoted by $f \prec g$ or $f(z) \prec g(z), \quad z \in E$, if there exists a Schawrz function w, which is analytic in E and satisfies the conditions $w(0) = 0, \quad |w(z)| < 1, \quad z \in E$, such that

$$f(z) = g(w(z)), \quad \forall z \in E.$$

If f is univalent, then $f \prec g$, if and only if, $f(0) = g(0)$ and $f(E) \subset g(E)$.

For f, g given by

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of these power series or the convolution $f \star g$ is defined as:

$$(f \star g)(z) = f(z) \star g(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \quad z \in E.$$

We now define the derivative operator $D^n : \mathcal{A} \to \mathcal{A}$ as:

$$\begin{aligned} D^n f(z) &= \frac{z}{(1-z)^{n+1}} \star f(z), \quad f \in \mathcal{A}, \quad n \in \mathbb{N} \cup \{0\} = \mathbb{N}_0 \\ &= z + \sum_{k=0}^{\infty} \frac{(k+n-1)!}{k!n!} a_k z^k. \end{aligned} \tag{1.1}$$

Note that

$$D^0 f(z) = f(z) \quad \text{and} \quad D' f(z) = zf'(z).$$

The operator $D^n f$ is called the Ruscheweyh derivative of f of order n, see Ruscheweyh [14].

Analogous to the operator $D^n f$, Noor defined an integral operator $\mathcal{I}_n : \mathcal{A} \to \mathcal{A}$ as follows:

$$\mathcal{I}_{n+1}(f(z)) = \frac{n+1}{n} \int_0^z t^{n-1} \mathcal{I}_n f(t) dt, \quad n \in \mathbb{N}_0. \tag{1.2}$$

We note that

$$\mathcal{I}_1^0 f(z) = f(z) \quad \text{and} \quad \mathcal{I}_0 f(z) = z f'(z), \quad f \in \mathcal{A}.$$

The operator $\mathcal{I}_n f$ is called the Noor integral of f of order n. For more details, see [3, 7, 8, 9] and the references therein.

Using convolution, we can define this operator as follows:

$$\mathcal{I}_n f(z) = \big(f_n^{-1} \star f\big)(z),$$

where

$$f_n(z) = \frac{z}{(1-z)^{n+1}},$$

and $f_n^{-1}(z)$ is defined as

$$\big(f_n^{-1} \star f\big)(z) = \frac{z}{(1-z)^2}.$$

For the derivative operator D^n and integral operator $\mathcal{I}_n$, the following identities can easily be verified

$$z\big(D^n f(z)\big)' = (n+1) D^{n+1} f(z) - n D^n f(z) \tag{1.3}$$

and

$$(n+1)\mathcal{I}_n(f(z)) - n\mathcal{I}_{n+1} f(z) = z\big(\mathcal{I}_{n+1} f(z)\big)'. \tag{1.4}$$

For (1.3), we refer to [7].

Remark 1.1. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z$, then

$$\mathcal{I}_n f(z) = z + \sum_{k=2}^{\infty} \frac{(k!)(n!)}{(k+n-1)!} a_k z^k.$$

Definition 1.1. Let $\lambda \geq 0$, $f \in \mathcal{A}$ and $n \in \mathbb{N}_0$. Then the operator $L_\lambda^n : \mathcal{A} \to \mathcal{A}$ is defined as

$$L_\lambda^n f(z) = (1-\lambda) D^n f(z) + \lambda \mathcal{I}_n f(z), \quad z \in E, \tag{1.5}$$

where the operators D^n and $\mathcal{I}_n$ are defined by (1.1) and (1.2).

As special cases, we note that

$$L_0^n f(z) = D^n f(z), \quad L_1^n f(z) = \mathcal{I}_n f(z)$$

and

$$\begin{aligned} L_\lambda^0 f(z) &= (1-\lambda)f(z) + \lambda z f'(z) \\ L_\lambda^1 f(z) &= (1-\lambda)z f'(z) + \lambda f(z). \end{aligned}$$

To derive our results, we recall the basic concepts in fuzzy set theory.

Definition 1.2. [15] Let X be a nonempty set. A pair (A, F_A), where $F_A : X \to [0,1]$ and $A = \{x \in X : \quad 0 < F_A(x) \leq 1\}$ is called fuzzy subset. The function F_A is called membership function of the fuzzy subset (A, F_A).

Definition 1.3. [10] Let $D \subset \mathbb{C}$ and $z_0 \in D$ be a fixed point. We take the function $f, g \in H(D)$, where $H(D)$ is the class of analytic functions in D. The function f is said to be fuzzy subordination to g, written as $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z), \quad z \in E$, if there exists a function $F : \mathbb{C} \to [0,1]$, such that
(i). $f(z_0) = g(z_0)$
(ii). $F(f(z)) \leq F(g(z)), \quad \forall z \in D.$

Remark 1.2. (a). The function $F : \mathbb{C} \to [0,1]$ in Definition 1.3 can be considered as:
(i). $F(z) = \frac{|z|}{1+|z|}$
(ii). $F(z) = \frac{1}{1+|z|}$
or
(iii). $F(z) = \frac{\sqrt{|z|}}{1+\sqrt{|z|}}$.

(b). Relation (ii) in Definition 1.3 is equivalent to $f(D) \subset g(D)$.

Definition 1.4. [13] Let $\psi : \mathbb{C}^3 \times D \to \mathbb{C}, \quad a \in \mathbb{C}$ and let h be univalent in E with $h(z_o) = a$, g be univalent in D with $g(z_0) = a$ and p be analytic in D with $p(z_0) = a$. Likewise $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in D and $F : \mathbb{C} \to [0,1]$, $F(z) = \frac{|z|}{1+|z|}$.
If p is analytic in D and satisfies the second order differential subordination

$$F\big(\psi(p(z), zp'(z), z^2p''(z); z)\big) \leq F(h(z)), \quad z \in E, \tag{1.6}$$

that is,

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) \prec_{\mathcal{F}} \leq h(z), \quad z \in E,$$

or

$$\frac{|\psi\big(p(z), zp'(z), z^2p''(z); z\big)|}{1 + |\psi\big(p(z), zp'(z), z^2p''(z); z\big)|} \prec_{\mathcal{F}} \frac{|h(z)|}{1+|h(z)|}, \quad z \in E, \tag{1.7}$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant, if

$$\frac{|p(z)|}{1+|p(z)|} \leq \frac{|q(z)|}{1+|q(z)|},$$

that is,

$$p(z) \prec_{\mathcal{F}} q)z), \quad z \in E.$$

for all p satisfying (1.6) or (1.7).
A fuzzy dominant $\tilde{q}$ that satisfies $\quad \widetilde{q(z)} \prec_{\mathcal{F}} q(z), \quad z \in E$, for all fuzzy dominant q, is called the fuzzy best dominant of (1.6) or (1.7).

2. Preliminaries Results

To establish our main results, we need the following Lemmas.

Lemma 2.1. *[1] Assume that h is a convex function with $h(0) = a$ and $\nu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with $Re(\nu) \geq 0$. If $p \in H[a, n]$ with $p(0) = a$, $\quad \Phi : \mathbb{C}^2 \times E \to \mathbb{C}$, such that*

$$\Phi\big(p(z), zp'(z); z)\big) = p(z) + \frac{1}{\nu} zp'(z) \prec_{\mathcal{F}} h(z), \quad z \in E,$$

then

$$p(z) \prec_{\mathcal{F}} q(z), \quad z \in E, \quad n \in \mathbb{N},$$

where

$$q(z) = \frac{\nu}{nz^{\nu}} \int_0^z h(t) t^{(-1+\frac{\nu}{n})} dt, \quad z \in E. \tag{2.1}$$

The function q is convex and it is the fuzzy best dominant.

Lemma 2.2. *[11] Let q be a convex function in E and let*

$$\psi(z) = g(z) + n\gamma z g'(z), \quad z \in E, n \in \mathbb{N}, \quad \gamma > 0.$$

If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots$$

is analytic in E and

$$p(z) + \gamma z p'(z) \prec_{\mathcal{F}} \psi(z), \quad z \in E,$$

then

$$p(z) \prec_{\mathcal{F}} g(z), \quad z \in E.$$

This result is sharp.

Lemma 2.3. *[13] Let $\delta, w \in \mathbb{C}$, $\quad w \neq 0$ and h be a convex function in D and $F : \mathcal{C} \to [0,1]$, $\quad F(z) = \frac{|z|}{1+|z|}$, $\quad z \in E$. We suppose*

$$q(z) + \frac{zq'(z)}{\delta + wq(z)} = h(z), \quad z \in D, \quad q(z_0) = h(z_0) = a,$$

has a solution $q \in H(D)$ which verifies

$$q(z) \prec_{\mathcal{F}} h(z), \quad z \in D,$$

or

$$\frac{|q(z)|}{1 + |q(z)|} \leq \frac{|h(z)|}{1 + |h(z)|}, \quad z \in D.$$

If the function $p \in H[0,1]$, $\quad \psi : \mathbb{C}^2 \times D \to \mathbb{C}$,

$$\psi\big(p(z), zp'(z); z\big) = p(z) + \frac{zp'(z)}{\delta + wp(z)}$$

is analytic in D with

$$\psi\big(p(z_0), z_0 p'(z_0)\big) = h(z_0), \quad z_0 \in D,$$

then

$$\psi\big(p(z), zp'(z); z\big) \prec_{\mathcal{F}} h(z), \quad z \in D \tag{2.2}$$

implies

$$p(z) \prec_{\mathcal{F}} q(z), \quad z \in D$$

and q is the fuzzy best dominant of (2.2).

3. Main Resullts

In this section, we prove our main results.

Theorem 3.1. *Let $f \in \mathcal{A}$ and q be univalent in E with $q(0) = 1$, $\quad \gamma \in \mathbb{C}^\star$ such that*

$$Re\Big\{\frac{zq''(z)}{q'(z)} + 1\Big\} \geq max\Big\{0, -Re\big(\frac{1}{\gamma}\big)\Big\}.$$

If

$$\frac{L_\lambda^{n+1} f(z)}{L_\lambda^n f(z)} + \gamma \left\{ \frac{z(L_\lambda^{n+1} f(z))'}{L_\lambda^n f(z)} - L_\lambda^{n+1} f(z) \frac{\big(z(L_\lambda^n f(z))\big)'}{(L_\lambda^n f(z))^2} \right\} \prec_{\mathcal{F}} h(z), \tag{3.1}$$
$$z \in E,$$

where $h(z) = q(z) + \gamma z q'(z)$, *then*

$$\frac{L_\lambda^{n+1} f(z)}{L_\lambda^n f(z)} \prec_{\mathcal{F}} q(z), \quad z \in E$$

and $q(z)$ *is the fuzzy best dominant of (3.2), given in (2.1), as*

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t) t^{\gamma - 1} dt. \tag{3.2}$$

Proof. Let

$$\frac{L_\lambda^{n+1} f(z)}{L_\lambda^n f(z)} = p(z), \tag{3.3}$$

p is analytic in E with $p(0) = 1$.
Logarithmic differentiation of (3.3) together with some simple computation yield to us

$$\frac{zp'(z)}{p(z)} = \frac{z\big(L_\lambda^{n+1} f(z)\big)'}{L_\lambda^{n+1} f(z)} - \frac{z\big(L_\lambda^n f(z)\big)'}{L_\lambda^n f(z)}.$$

That is

$$\begin{aligned} & p(z) + \gamma z p'(z) \\ = \quad & \frac{L_\lambda^{n+1} f(z)}{L_\lambda^n f(z)} + \gamma \frac{L_\lambda^{n+1} f(z)}{L_\lambda^n f(z)} \left\{ \frac{z\big(L_\lambda^{n+1} f(z)\big)'}{L_\lambda^{n+1} f(z)} - \frac{z\big(L_\lambda^n f(z)\big)'}{L_\lambda^n f(z)} \right\}. \end{aligned} \tag{3.4}$$

From (3.1) and (3.4), we obtain

$$p(z) + \gamma z p'(z) \prec_{\mathcal{F}} q(z) + \gamma z q'(z).$$

Now, using Lemma 2.2, we have $p(z) \prec_{\mathcal{F}} q(z), \quad z \in E$ and q, given by (3.2), is fuzzy bet dominant of (3.1) by Lemma 2.1. This completes the proof. □

We consider some applications of Theorem 3.1 as:

Corollary 3.1. *Let* $\lambda = 0, f \in \mathcal{A}$ *and let the function* q *satisfy the conditions given in Theorem 3.1 with* $Re(\gamma) > 0$.
If

$$\begin{aligned} & \frac{D^{n+1} f(z)}{D^n f(z)} + \gamma \left\{ \frac{z\big(D^{n+1} f(z)\big)'}{D^n f(z)} - \frac{D^{n+1} f(z)}{D^n f(z)} \frac{z\big(D^n f(z)\big)'}{D^n f(z)} \right\} \\ \prec_{\mathcal{F}} \quad & q(z) + \gamma z q'(z), z \in E, \end{aligned} \tag{3.5}$$

then

$$\frac{D^{n+1} f(z)}{D^n f(z)} \prec_{\mathcal{F}} q(z), \quad z \in E$$

and $q(z)$ *is the fuzzy best dominant of (3.5).*

3.1. Special cases

(i). Take $n = 0$. Then $D^0 f(z) = f(z)$ and $D' f(z) = zf' f(z)$ and from (3.5), we have

$$\frac{zf'(z)}{f(z)} + \gamma\left\{\frac{z(zf'(z))'}{f(z)} - \frac{zf'(z)}{f(z)} \cdot \frac{zf'(z)}{f(z)}\right\} \prec_{\mathcal{F}} q(z) + \gamma z g'(z), \quad z \in E,$$

which implies

$$\frac{zf'(z)}{f(z)} \prec_{\mathcal{F}} q(z), \quad z \in E.$$

(ii). For $n = 1$, we have from (3.5).

$$\begin{aligned} 1 + (1+3\gamma)\frac{zf''(z)}{f'(z)} \quad &+ \quad \gamma\left\{1 - (1 + \frac{zf''(z)}{f'(z)})^2 + \frac{z^2 f'''(z)}{f'(z)}\right\} \\ &\prec_{\mathcal{F}} \quad q(z) + \gamma z q'(z), \quad z \in E, \end{aligned}$$

implies

$$\frac{zf''(z)}{f'(z)} \prec_{\mathcal{F}} q(z), \quad z \in E.$$

This result is sharp.

Corollary 3.2. *If*

$$\begin{aligned} \frac{\mathcal{I}_{n+1} f(z)}{\mathcal{I}_n f(z)} \quad &+ \quad \gamma\{\frac{z(\mathcal{I}_{n+1} f(z))'}{\mathcal{I}_n f(z)} - \frac{\mathcal{I}_{n+1} f(z)}{\mathcal{I}_n f(z)} \cdot \frac{z(\mathcal{I}_n f(z))'}{\mathcal{I}_n f(z)}\} \\ &\prec_{\mathcal{F}} h(z), \end{aligned} \tag{3.6}$$

where $h(z) = q(z) + \gamma z q'(z), z \in E,$
then

$$\frac{\mathcal{I}_{n+1} f(z)}{\mathcal{I}_n f(z)} \prec_{\mathcal{F}} q(z), \quad z \in E.$$

This result is sharp in the sense that q *is fuzzy best dominant for (3.6),*
For $n = 0$, *in (3.6), we obtain*

$$\frac{f(z)}{zf'(z)} \prec_{\mathcal{F}} q(z), \quad z \in E.$$

Remark 3.1. Let $A, B \in \mathbb{C}, \quad A \neq B$ such that $|B| \leq 1$. Then the function $(\frac{1+Az}{1+Bz})^\beta, \quad 0 < \beta \leq 1$ is convex and univalent in E.
Let

$$\begin{aligned} \phi_\beta(A, B; z) \quad &= \quad (\frac{1+Az}{1+Bz})^\beta \\ &= \quad 1 + \beta(A-B)z \\ &\quad + \Big[- \beta(A-B)B + \frac{1}{2}\beta(\beta-1)(A-B)^2 z^2 + \ldots \Big]. \end{aligned}$$

Then $\phi_\beta(A,B;z)$ is analytic in E and $\phi_\beta(0)=1$.
We note that
(i).

$$\begin{aligned} Re\Big\{\phi'_\beta(A,B;z)\Big\} &= Re\Big\{\beta(A-B)\frac{(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}}\Big\} \\ &\geq \beta|(A-B)|\frac{(1-|A|z)^{\beta-1}}{(1-|B|)^{\beta+1}} \\ &> 0, \quad \forall z \in E. \end{aligned}$$

This shows, by Noshiro-Warchawski Theorem, that $\phi_\beta(A,B;z)$ is univalent in E.
(ii). Also, by simple computation,

$$\begin{aligned} Re\Big\{\frac{(z\phi'_\beta(A,B;z))'}{\phi'_\beta(A,B;z)}\Big\} &\geq \Big\{\frac{1-\beta(A-B)r-ABr^2}{(1+Ar)(1+Br)}\Big\} \\ &= \Big\{\frac{1-\beta(A-B)r-ABr^2}{(1+Ar)(1+Br)}\Big\} \\ &= \frac{T(r)}{(1+Ar)(1+Br)}. \end{aligned}$$

Since $T(r)=1-\beta(A-B)r-ABr^2$ is decreasing in $(0,1)$ and $T(0)=1$, which implies

$$\left\{\frac{\Big(z\phi'_\beta(z)\Big)'}{\phi'_\beta(z)}\right\} \geq 0.$$

This proves the assertion that $\phi_\beta(A,B;z)$ is convex univalent in E.

Using Remark 3.1, we have

Corollary 3.3. *Let $A,B,\gamma \in \mathbb{C}$, $A \neq B$ such that $|B| \leq 1$ and $Re(\gamma) > 0$. If, for $f \in \mathcal{A}$,*

$$\begin{aligned} \frac{L_\lambda^{n+1}f(z)}{L_\lambda^n f(z)} &+ \gamma\Big\{\frac{z(L_\lambda^{n+1}f(z))'}{L_\lambda^n f(z)} - \frac{L^{n+1}f(z)\big(z(L_\lambda^n f(z))\big)'}{(L_\lambda^n f(z))^2}\Big\} \\ &\prec_{\mathcal{F}} \Big(\frac{1+Az}{1+Bz}\Big)^\beta + \frac{\gamma\beta z(A-B)(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}} \\ &= h(z), \quad z \in E, \end{aligned} \tag{3.7}$$

then

$$\frac{L_\lambda^{n+1}f(z)}{L_\lambda^n f(z)} \prec_{\mathcal{F}} \Big(\frac{1+Az}{1+Bz}\Big)^\beta, \quad z \in E$$

and $q(z)=\big(\frac{1+Az}{1+Bz}\big)^\beta$ is the fuzzy best dominant of (3.7).

We now discuss some special cases.
(i). Let $\beta=1$, $A=0$, $B=-1$ and $\gamma=1$. Then

$$h(z)=\frac{1}{1-z}+\frac{z}{(1-z)^2}=\frac{1}{(1-z)^2}, \quad \text{and} \quad q(z)=\frac{1}{1-z}.$$

(ii). Let $\beta = 1, \quad A = 1, \quad B = -1$. Then

$$h(z) = \frac{1 + 2\gamma z - z^2}{(1-z)^2}, \quad q(z) = \frac{1+z}{1-z}.$$

(iii). For $A = 1 - 2\rho, \quad B = -1, \quad \beta = \gamma = 1, \quad \rho \in [0, 1)$, we have

$$\begin{aligned} h(z) &= \frac{1 - (1-2\rho)z}{1-z} + \frac{2(1-\rho)z}{(1-z)^2} \\ q(z) &= \frac{1 + (1-2\rho)z}{1-z}. \end{aligned}$$

Theorem 3.2. *Let $f \in \mathcal{A}$ and q be a convex function in E. Let $F : \mathbb{C} \to [0, 1]$ be defined as*

$$F(z) = \frac{|z|}{1 + |z|}, \quad z \in E.$$

Then

$$\frac{z(L_0^{n+1} f(z))'}{L_0^{n+1} f(z)} \prec_{\mathcal{F}} \frac{(1+z)(1+n++z) + 2z}{(1-z)(1+n+z)}, \quad z \in E, \tag{3.8}$$

implies

$$\frac{z(L_0^{n+1} f(z))'}{L_0^{n+1} f(z)} \prec_{\mathcal{F}} \frac{1+z}{1-z}, \quad z \in E$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant of (3.8).

Proof. For $\lambda = 0, \quad L_0^{n+1} f(z) = D^{n+1} f(z)$.
Let

$$\frac{z(L_0^n f(z))'}{L_0^n f(z)} = \frac{z(D^n f(z))'}{D^n f(z)} = p(z). \tag{3.9}$$

It can b noted that p is analytic in E and $p(0) = 1$.
Using identity (1.3) and relation (3.9), we obtain

$$\frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} = p(z) + \frac{zp'(z)}{p(z) + n}. \tag{3.10}$$

Let the fuzzy function $F : \mathbb{C} \to [0, 1]$ be given by

$$F(z) = \frac{|z|}{1 + |z|}, \quad z \in E.$$

and consider an analytic function h in E be defined as

$$h(z) = q(z) + \frac{zp'(z)}{p(z) + n}, \quad h(0) = q(0),$$

where q is a univalent solution in E which satisfies

$$\frac{|q(z)|}{1+|q(z)|} \leq \frac{|h(z)|}{1+|h(z)|},$$

that is,

$$q(z) \prec_{\mathcal{F}} h(z), \quad z \in E.$$

Now $(p(z) + \frac{zp'(z)}{p(z)+n})$ is analytic in E with

$$\frac{|p(z) + \frac{zp'(z)}{p(z)+n}|}{1+|p(z) + \frac{zp'(z)}{p(z)+n}|} \leq \frac{|h(z)|}{1+|h(z)|},$$

that is

$$\{p(z) + \frac{zp'(z)}{p(z)+n}\} \prec_{\mathcal{F}} h(z) = \frac{(1+z)(1+z+n)+2z}{(1-z)(1+z+n)}, \quad z \in E.$$

This completes the proof. □

A special case, with $n = 0$, we obtain from (3.9) and (3.10) that

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) = \{p(z) + \frac{zp'(z)}{p(z)}\} \prec_{\mathcal{F}} \frac{1+4z+z^2}{1-z^2}, \quad z \in E$$

implies

$$p(z) = \frac{zf'(z)}{f(z)} \prec_{\mathcal{F}} \frac{1+z}{1-z}, \quad z \in E.$$

With similar technique, we can prove the following.

Theorem 3.3. *Let $f \in \mathcal{A}$, $q(z) = \frac{1+z}{1-z}$ and $F : \mathbb{C} \to [0,1]$ be defined as $F(z) = \frac{|z|}{1+|z|}, z \in E$. Then, with*

$$\frac{z(L_1^n f(z))'}{L_1^n f(z)} \prec_{\mathcal{F}} \frac{(1+z)(1+z+n)+2z}{(1-z)(1+z+n)}, \quad z \in E \tag{3.11}$$

implies

$$\frac{z(L_1^n f(z))'}{L_1^n f(z)} \prec_{\mathcal{F}} \frac{1+z}{1-z}, \quad z \in E.$$

where $q(z) = \frac{1+z}{1-z}$ is the fuzzy best dominant of (3.11).

Remark 3.2. In Theorem 3.3, we choose fuzzy function $F : \mathbb{C} \to [0,1]$ as $F(z) = \frac{1}{1+|h(z)|}, \quad z \in E$. Then

$$\frac{1}{1+\left|p(z)+\frac{zp'(z)}{p(z)+n}\right|} \leq \frac{1}{1+|h(z)|}, \quad z \in E, \tag{3.12}$$

where

$$h(z) = \frac{(n+1)(n-1)z}{(1-z)((n+1)-nz)}, \quad q(z) = \frac{1}{1-z}, \quad z \in E.$$

Thus

$$\frac{z(L_1^n f(z))'}{L_1^n f(z)} \prec_{\mathcal{F}} h(z)$$

implies

$$\frac{z(L_1^n f(z))'}{L_1^{n+1} f(z)} \prec_{\mathcal{F}} q(z) = \frac{1}{1-z}, \quad z \in E.$$

Here the function $q(z) = \frac{1}{1-z}$ is the fuzzy best dominant of (3.12).

Theorem 3.4. *Let q be a convex function in E with $q(0) = 1$ and let $Re(\gamma) > 0$. If $f \in \mathcal{A}$ and*

$$\begin{aligned} &(1+\gamma)\left\{\frac{zL_\lambda^n f(z)}{(L_\lambda^{n+1} f(z))^2}\right\} \\ &+\gamma z^2 \frac{L_\lambda^n f(z)}{(L_\lambda^{n+1} f(z))^2}\left\{\frac{L_\lambda^n f(z))'}{L_\lambda^n f(z)} - 2(\frac{L_\lambda^{n+1} f(z))'}{L_\lambda^{n+1} f(z)}\right\} \prec_{\mathcal{F}} h(z), \end{aligned} \tag{3.13}$$

where h is analytic in E and

$$h(z) = q(z) + \gamma z q'(z), \quad z \in E, \tag{3.14}$$

then

$$\frac{zL_\lambda^n f(z)}{(L_\lambda^{n+1} f(z))^2} \prec_{\mathcal{F}} q(z), \quad z \in E.$$

The function

$$q(z) = \frac{\gamma}{z^\gamma}\int_0^z h(t) t^{\gamma-1} dt,$$

given by (3.14), is the fuzzy best dominant of (3.13).

Proof. Let

$$\frac{zL_\lambda^n f(z)}{(L_\lambda^{n+1} f(z))^2} = p(z). \tag{3.15}$$

Logarithmic differentiation of (3.15) with some computation, we have

$$zp'(z) = p(z)\left\{\frac{z(L_\lambda^n f(z))'}{L_\lambda^n f(z)} - 2z(\frac{L_\lambda^{n+1} f(z))'}{L_\lambda^{n+1} f(z)} + 1\right\}. \tag{3.16}$$

Using (3.13), (3.14), (3.15) and (3.16), we get

$$p(z) + \gamma p'(z) \prec_{\mathcal{F}} h(z), \quad z \in E.$$

We apply Lemma 2.1 to obtain $P(z) \prec_{\mathcal{F}} q(z), z \in E$ and $q(z)$ is given by (3.14).
□

We discuss some applications of Theorem 3.4.

Corollary 3.4. *In Theorem 3.4, we take*

$$h(z) = \big(\frac{1+Az}{1+Bz}\big)^{\beta}\big[1 + \frac{\gamma\beta z(A-B)}{(1+Az)(1+Bz)}\big]$$

in fuzzy differential subordination (3.13).
Now $h(z)$ with $\beta \in (0,1]$, $\quad -1 \leq B < A \leq 1$ is analytic in E.
From (3.13), we have

$$p(z) + \gamma zp'(z) \prec_{\mathcal{F}} h(z) = q(z) + \gamma zq'(z), \quad z \in E.$$

and

$$\frac{zL_\lambda^n f(z)}{\big(L_\lambda^{n+1} f(z)\big)^2} = p(z) \prec_{\mathcal{F}} q(z) = \big(\frac{1+Az}{1+Bz}\big)^{\beta}, \quad z \in E,$$

and $q(z)$ is best fuzzy best dominant.

We consider the following special cases.
(i). For $A = 0$, $\quad \beta = 1$ and $B = -1$, we have

$$p(z) = \frac{zL_\lambda^n f(z)}{\big(L_\lambda^{n+1} f(z)\big)^2} \prec_{\mathcal{F}} \frac{1}{1-z}, \quad z \in E,$$

where

$$p(z) + \gamma zp'(z) \prec_{\mathcal{F}} h(z) = \frac{1}{1-z}\big[1 + \frac{\gamma z}{(1-z)}\big] = \frac{1+(\gamma-1)z}{(1-z)^2}$$

and $q(z) = \frac{1}{1-z}$ is the fuzzy best dominant in this case.

(ii). Takin $\beta = 1, A = 1, \quad B = -1$, we get

$$\begin{aligned} q(z) &= \frac{1+z}{1-z} \\ h(z) &= \frac{1+z}{1-z}\big[1 + \frac{2\gamma z}{1-z^2}\big] = q(z) + \gamma zq'(z) \end{aligned}$$

and $p(z) \prec_{\mathcal{F}} q(z), \quad z \in E$.

Corollary 3.5. *Let $\beta = 1$, $A, B, \gamma \in \mathbb{C}$, $A \neq B$ such that $|B| \leq 1$ and $Re(\gamma) > 0$.*
For $\lambda = 0$, $L_0^n f(z) = D^n f(z)$ and $D^0 f(z) = f(z)$, $D' f(z) = z f'(z)$.
If $f \in \mathcal{A}$ and $n = 0$, then, it follows from Theorem 3.4 that

$$(1-\gamma)\frac{f(z)}{z(f'(z))^2} + \gamma\left\{\frac{1}{f'(z)} - \left(\frac{2f(z)f''(z)}{(f'(z))^2}\right)\right\} \prec_{\mathcal{F}} h(z), \quad z \in E, \tag{3.17}$$

where

$$q(z) = q(z) + \gamma z q'(z) = \frac{1+Az}{1+Bz} + \gamma\frac{(A-B)z}{(1+Bz)^2} \tag{3.18}$$

and

$$q(z) = \frac{1+Az}{1+Bz}, \quad z \in E.$$

Remark 3.3. Since $L_1^n f(z) = \mathcal{I}_n f(z)$, $\mathcal{I}_0 = z f'(z)$ and $\mathcal{I}_1 f(z) = f(z)$, we can easily obtain a similar fuzzy differential subordination result using the operator $\mathcal{I}_n$. In this direction, we have the following result.

Corollary 3.6. *Let $\lambda = 1$ and $n = 0$ in (3.13) and let $h(z)$ be given by (3.18). Then, for $f \in \mathcal{A}$,*

$$\begin{aligned} &\left\{\frac{(1+\gamma)f'(z) + \gamma(zf'(z))'}{(\frac{f(z)}{z})^2} - \frac{2\gamma(f'(z))^2}{(\frac{f(z)}{z})^3}\right\} \\ \prec_{\mathcal{F}} \quad &\left\{\frac{1+Az}{1+Bz} + \frac{\gamma(A-B)z}{(1+Bz)^2}\right\}, \quad z \in E \end{aligned} \tag{3.19}$$

implies

$$\frac{f'(z)}{(\frac{f(z)}{z})^2} \prec_{\mathcal{F}} \frac{1+Az}{1+Bz}, \quad z \in E.$$

The function $q(z) = \frac{1+Az}{1+Bz}$, $z \in E$ is the fuzzy best dominant of (3.19).

4. Conclusion

In this paper, we have introduced a new operator involving the convex combination of Ruscheweyh derivative operator and Noor integral operator. Using the fuzzy differential subordination and fuzzy set theory, we obtained several new results. It is shown that all the obtained results are sharp. Several important special cases, which can be obtained, are also highlighted. It is an open problem to investigate the applications of this new operator in various branches of pure and applied sciences. The ideas and techniques of this paper should be starting point for future results.

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Competing interests

The authors declare that they have no competing interests.

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