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ON LACUNARY CONVERGENCE IN CREDIBILITY SPACE

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Abstract. In this paper, we present the notions of lacunary statistically convergent sequence for fuzzy variables, lacunary statistically Cauchy sequence in credibility space, and present a kind of lacunary statistical completeness for credibility space. Also, we present lacunary strong convergence concepts of sequences of fuzzy variables of different types.
Keywords:credibility measure, credibility theory, statistical convergence.

1. Introduction

Fuzzy theory is well advanced on the mathematics foundations of fuzzy set theory, initiated by Zadeh [50] and established in 1965. Fuzzy theory can be utilized in a comprehensive variety of real problems. For instance, possibility theory has been developed by many researchers, such as Dubois and Prade [6], Nahmias [34], Zadeh [51]. A fuzzy varible is a function from a credibility space (denoted with the credibility measure) to the set of $\mathbb{R}$. The convergence of fuzzy variables is significant component of credibility theory, which can be used into real problems in engineering and mathematical finance. Fuzzy variable, possibility distribution and membership function were examined by Kaufmann [14]. Possibility measure, which is usually determined as supremum preserving set function on the power set of a nonempty set, is a main concept in possibility theory but it is not self-dual. Since a self-dual measure is absolutely required in both theory and practice, Liu and

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Liu [19] have introduced a self-duality credibility measure. The credibility measure plays the role of possibility measure in fuzzy world because it shares some fundamental features with possibility measure. Specically, since Liu has begun the survey of credibility theory, and then many specific contents have been examined (see [15, 16, 17, 20, 21, 22, 45, 52]). Contemplating sequence convergence plays a key role in credibility theory, Liu [23] presented four kinds of convergence concept for fuzzy variables: convergence in credibility, convergence almost surely, convergence in mean, convergence in distribution. In addition, based upon credibility theory, several convergence features of credibility distribution for fuzzy variables were worked by Jiang [11] and Ma [26].

Wang and Liu [45] thought the relationships among convergence in mean, convergence in credibility, convergence almost uniformly, convergence in distribution, and convergence almost surely. Besides, numerous researchers emphasized convergence notions in classical measure theory, credibility theory, probability theory, and examined the connections between them. The concerned readers may examine Chen et al. [5], Lin [18], Liu and Wang [24], Xia [46] and You [48, 49].

Statistical convergence was first presented by Fast [7] and Steinhaus [38] as a generalization of ordinary convergence for real sequences. Statistical convergence turned out to be one of the most active areas of research in the summability theory after the works of Fridy [9] and Šalát [37]. Statistical convergence has also been studied in more general abstract spaces such as the fuzzy number space [35]. More investigations in this direction and more applications of statistical convergence can be seen in [4, 10, 13, 27, 28, 29, 30, 31, 32, 35, 36, 42, 43, 44]. Also, the readers should refer to the monographs [2], and [33], and recent papers [39], [40], [41] and [12] for the background on the sequence spaces.

The first study on lacunary sequence is examined in Freedman et al. [8]. Almost convergent sequences was defined by Lorentz [25]. For more details on almost convergence and certain summability methods one may refer to [1, 3, 47].

This paper is devoted to present a new kind of convergence for fuzzy variables sequences. In Section 2, some preliminary definitions and theorems related to fuzzy variables sequences, credibility space are presented. In Section 3, in addition, we plan to work the notion of lacunary statistical convergence of fuzzy variables and to construct fundamental features of the lacunary statistical convergence in credibility.

2. Preliminaries

A set function Cr is credibility measure if it supplies the subsequent axioms: Let Θ be a nonempty set, and $\mathcal{P}(\Theta)$ the power set of Θ (i.e., the largest algebra over Θ). Each element in $\mathcal{P}$ is called an event. For any $A \in \mathcal{P}(\Theta)$, Liu and Liu [19] presented a crebility measure $\mathrm{Cr}\{A\}$ to express the chance that fuzzy event A occurs. Li and Liu [16] proved that a set function $\mathrm{Cr}\{.\}$ a crebility measure if and only if

Axiom i. $\mathrm{Cr}\{\Theta\} = 1$;

Axiom ii. $\mathrm{Cr}\{A\} \leq \mathrm{Cr}\{B\}$ whenever $A \subset B$;

Axiom iii. Cr is self-dual, i.e., $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$, for any $A \in \mathcal{P}(\Theta)$;

Axiom iv. $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$ with $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

The triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is named a crebility space. A fuzzy variable was investigated by Liu and Liu [19] as function from the crebility space to the set of real numbers.

Example 2.1. Let $\Theta = \{\phi_1, \phi_2\}$. For this case, there are only four events: $\emptyset$, $\{\phi_1\}$, $\{\phi_2\}$, Θ. Determine $\mathrm{Cr}\{\Theta\} = 0$, $\mathrm{Cr}\{\phi_1\} = 0.7$, $\mathrm{Cr}\{\phi_2\} = 0.3$, and $\mathrm{Cr}\{\Theta\} = 1$. Then, the set function Cr is a credibility measure because it supplies the four axioms.

Definition 2.1. ([19]) The expected value of fuzzy variable μ is given by

$$E[\mu] = \int_0^{+\infty} \mathrm{Cr}\{\mu \geq r\}\, dr - \int_{-\infty}^{0} \mathrm{Cr}\{\mu \leq r\}\, dr$$

provided that at least one of the two integrals is finite.

If there is a $M > 0$ such that

$$\mathrm{Cr}\{\mu \leq -M\} = 0$$

and

$$\mathrm{Cr}\{\mu \leq M\} = 1,$$

then fuzzy variable μ is named as essentially bounded.

Theorem 2.1. *(Wang and Liu [45]) When the sequence $\{\mu_i\}$ convergence in credibility to μ, then $\{\mu_i\}$ converges a.s. to μ.*

Theorem 2.2. *(Liu, [23]) When the sequence $\{\mu_i\}$ convergence in mean to μ, then $\{\mu_i\}$ converges credibility to μ.*

A sequence $\{\mu_k\}$ of fuzzy variables is named as uniformly essentially bounded (UEB, shortly) provided that there is a $M > 0$ such that for all k, we get

$$\mathrm{Cr}\{\mu_k \leq -M\} = 0$$

and

$$\mathrm{Cr}\{\mu_k \leq M\} = 1.$$

Theorem 2.3. *(Bounded Convergence Theorem, [24]) Presume that $\{\mu_k\}$ is a sequence of UEB fuzzy variables. If $\{\mu_k\}$ is convergent in credibility to μ, then*

$$\lim_{k\to\infty} E[\mu_k] = E[\mu].$$

Theorem 2.4. *([18]) Take $f : \mathbb{R} \to \mathbb{R}$ as a convex function. Then, there is $k > 0$ such that*

$$|f(x_1) - f(x_2)| \leq k |x_1 - x_2|,$$

for any $x_1, x_2 \in \mathbb{R}$.

Theorem 2.5. *Let μ be a fuzzy variable. Then, for any given numbers $t > 0$ and $p > 0$, we have*

$$Cr\{|\mu| \geq t\} \leq \frac{E[|\mu|^p]}{t^p}. \tag{2.1}$$

3. Main results

In this section, based on existing lacunary statistical convergence, we study the lacunary statistical convergence in credibility and the lacunary statistical Cauchy sequence in credibility. In order to better explain our results, we will first present some significant definitions.

Definition 3.1. The sequence $\{\mu_i\}$ is said to be lacunary statistical convergent almost surely (a.s.) to μ if there exists $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : |\mu_i(\phi) - \mu(\phi)| \geq \eta\}| = 0,$$

for each $\eta > 0$ and every $\phi \in A$. In this instance, we write $\mu_i \stackrel{S_\theta}{\to} \mu$, a.s.

Definition 3.2. The sequence $\{\mu_i\}$ is called to be lacunary statistical convergent in credibility to μ if there is $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : \mathrm{Cr}\{|\mu_i - \mu| \geq \eta\} \geq \gamma\}| = 0,$$

for each $\eta > 0$ and $\gamma > 0$. In this case, we write $S_\theta(\mathrm{Cr}) - \lim \mu_i = \mu$.

Take $\mu, \mu_1, \mu_2, \ldots$ as fuzzy variables defined on credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$.

(H) The uniqueness of limit: If $st_\theta(\mathrm{Cr})$-$\lim \mu_i = \mu_1$ and $st_\theta(\mathrm{Cr})$-$\lim \mu_i = \mu_2$, at that case $\mu_1 = \mu_2$ in credibility.

Theorem 3.1. *Lacunary statistical convergence in credibility satisfies the axiom **(H)**.*

Proof. Now, we examine that lacunary statistical convergence in credibility supplies the axiom (**H**). Presume that $st_\theta(\mathrm{Cr})$-$\lim \mu_i = \mu_1$ and $st_\theta(\mathrm{Cr})$-$\lim \mu_i = \mu_2$. Then, there is $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : \mathrm{Cr}\{|\mu_i - \mu_1| \geq \eta\} \geq \gamma\}| = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu_2| \geq \eta\right\} \geq \gamma\right\}\right| = 0$$

for each $\eta > 0$, $\gamma > 0$. We make the subsequent marks:

$$B_1 = \left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu_1| \geq \eta\right\} \geq \gamma\right\},$$

and

$$B_2 = \left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu_2| \geq \eta\right\} \geq \gamma\right\}.$$

Now let $i \in B_1 \cup B_2$. Then, we acquire

$$\mathrm{Cr}\left\{|\mu_i - \mu_1| \geq \eta\right\} < \gamma, \ \mathrm{Cr}\left\{|\mu_i - \mu_2| \geq \eta\right\} < \gamma.$$

Therefore

$$\mathrm{Cr}\left\{|\mu_1 - \mu_2| \geq \eta\right\} = \mathrm{Cr}\left\{|\mu_1 - \mu_i + \mu_i - \mu_2| \geq \eta\right\}$$
$$\leq \mathrm{Cr}\left\{|\mu_i - \mu_1| \geq \eta/2\right\} + \mathrm{Cr}\left\{|\mu_i - \mu_2| \geq \eta/2\right\} < 2\gamma.$$

Since $\gamma > 0$ is arbitrary, we acquire

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \mathrm{Cr}\left\{|\mu_1 - \mu_2| \geq \eta\right\} \geq \gamma\right\}\right| = 0,$$

which gives $\mu_1 = \mu_2$ in credibility. □

Definition 3.3. Take $\mu, \mu_1, \mu_2, ...$ as fuzzy variables with finite expected values determined on $(\Theta, \mathcal{P}, \mathrm{Cr})$. The sequence $\{\mu_i\}$ is called to be lacunary statistically convergent in mean to the fuzzy variable μ provided that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : E\left[|\mu_i - \mu|\right] \geq \eta\right\}\right| = 0$$

for each $\eta > 0$.

Theorem 3.2. *If the sequence $\{\mu_i\}$ lacunary statistical convergence in credibility to μ, then $\{\mu_i\}$ lacunary statistical converges a.s. to μ.*

Theorem 3.3. *When the sequence $\{\mu_i\}$ lacunary statistical converges in mean to μ, then $\{\mu_i\}$ lacunary statistical converges in credibility to μ.*

Proof. Let the fuzzy variable sequence $\{\mu_i\}$ be lacunary statistical convergent in mean to μ. For any taken $\eta, \gamma > 0$ with the aid of Markov inequality, we obtain

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu| \geq \eta\right\} \geq \gamma\right\}\right| \leq \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \frac{E\left[|\mu_i - \mu|\right]}{\eta} \geq \gamma\right\}\right| = 0.$$

Thus, $\{\mu_i\}$ lacunary statistical converges in credibility to μ. □

Theorem 3.4. *Take $\mu, \mu_1, \mu_2, ...$ as fuzzy variables. Then, $\{\mu_i\}$ lacunary statistical converges in credibility to μ if*

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ i \in I_r : \sum_{n\in I_r} \mathrm{Cr}\left\{ |\mu_n - \mu| \geq \eta \right\} \geq \gamma \right\} \right| = 0,$$

for any $\eta, \gamma > 0$.

Definition 3.4. The sequence $\{\mu_i\}$ is said to be lacunary statistical Cauchy sequence a.s. if for every $\eta > 0$, there is an event A with $\mathrm{Cr}\{A\} = 1$ and $N = N(\eta)$ such that for every $\phi \in A$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ i \in I_r : |\mu_i(\phi) - \mu_N(\phi)| \geq \eta \right\} \right| = 0.$$

Example 3.1. Contemplate the crebility space $(\Theta, \mathcal{P}, \mathrm{Cr})$ to be $\{\phi_1, \phi_2, ...\}$ with $\mathrm{Cr}\{\phi_t\} = \frac{1}{2}$ for $t = 1, 2, ...$ The fuzzy variables are given by

$$\mu_i(\phi_t) = \begin{cases} \frac{1}{t}, & \text{if } i = t \\ 0, & \text{otherwise.} \end{cases}$$

For any $\eta > 0$, taking $A = \Theta$ and $M = \left[\frac{1}{\eta}\right] + 1$, we get

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ i \in I_r : |\mu_i(\phi) - \mu_N(\phi)| \geq \frac{1}{M} > \eta \right\} \right| = 0,$$

for every $\phi \in A$. Then, the sequence $\{\mu_i\}$ is a lacunary statistical Cauchy sequence a.s.

Theorem 3.5. *The sequence $\{\mu_i\}$ lacunary statistical converges a.s. to μ iff $\{\mu_i\}$ is a lacunary statistical Cauchy sequence a.s.*

Proof. If $\{\mu_i\}$ lacunary statistical converges a.s. to μ, then there is a fuzzy event A with $\mathrm{Cr}\{A\} = 1$ such that for any $\eta > 0$, we acquire

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ i \in I_r : |\mu_i(\phi) - \mu(\phi)| \geq \frac{\eta}{2} \right\} \right| = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ i \in I_r : |\mu_N(\phi) - \mu(\phi)| \geq \frac{\eta}{2} \right\} \right| = 0$$

for every $\phi \in A$. Thus,

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ i \in I_r : |\mu_i(\phi) - \mu_N(\phi)| \geq \eta \right\} \right| < \lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ i \in I_r : |\mu_i(\phi) - \mu(\phi)| \geq \frac{\eta}{2} \right\} \right|$$

$$+ \lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ i \in I_r : |\mu_N(\phi) - \mu(\phi)| \geq \frac{\eta}{2} \right\} \right| = 0.$$

So, for every $\eta > 0$ we can select an $N = N(\eta)$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \left|\mu_i(\phi) - \mu_N(\phi)\right| \geq \eta\right\}\right| = 0,$$

i.e. $\{\mu_i\}$ is a lacunary statistical Cauchy sequence a.s.

On the contrary, if $\{\mu_i\}$ is a lacunary statistical Cauchy sequence a.s., then for any $\eta > 0$, there exists $N_1 = N_1(\eta)$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \left|\mu_i(\phi) - \mu_{N_1}(\phi)\right| \geq \eta\right\}\right| = 0.$$

If $\{\mu_i\}$ does not lacunary statistical converge a.s., then there is $\phi^* \in A$ and $\eta_0 > 0$, for any $N_2 \in \mathbb{N}$, we get

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \left|\mu_{N_3}(\phi^*) - \mu(\phi^*)\right| \geq \eta_0\right\}\right| = 1,$$

when $N_3 > N_2$. Let

$$M = \frac{\mu_{N_3}(\phi^*) + \mu_{N_1}(\phi^*)}{2}.$$

Considering the inequality $\mu_{N_1}(\phi^*) = 2M - \mu_{N_3}(\phi^*)$, we observe that

$$\begin{aligned} \left|\mu_{N_3}(\phi^*) - \mu_{N_1}(\phi)\right| &= \left|\mu_{N_3}(\phi^*) - 2M + \mu_{N_3}(\phi^*)\right| \\ &= 2\left|\mu_{N_3}(\phi^*) - M\right| > 2\eta_0, \end{aligned}$$

when $N_2 > N_1$. This means that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{n \in I_r : \left|\mu_{N_3}(\phi^*) - \mu_{N_1}(\phi)\right| \geq 2\eta_0\right\}\right| = 1,$$

i.e. $\{\mu_i\}$ is not a lacunary statistical Cauchy sequence a.s. A contradiction demonstrates proof of the theorem. So, $\{\mu_i\}$ lacunary statistical converges a.s. to μ. □

Now, we present the notion of lacunary statistical Cauchy sequence in credibility.

Definition 3.5. Take $\mu_1, \mu_2, ...$ as fuzzy variables. We say that the sequence $\{\mu_i\}$ is a lacunary statistical Cauchy sequence in credibility, if for any $\eta > 0$, $\gamma > 0$, there exists $N = N(\gamma)$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \mathrm{Cr}\left\{\left|\mu_i - \mu_N\right| \geq \eta\right\} \geq \gamma\right\}\right| = 0.$$

Example 3.2. Contemplate the crebility space $(\Theta, \mathcal{P}, \mathrm{Cr})$ to be $\{\phi_1, \phi_2, ...\}$ with $\mathrm{Cr}\{\phi_1\} = \frac{1}{2}$ and $\mathrm{Cr}\{\phi_t\} = \frac{1}{t}$, for $t = 2, 3, ...$ The fuzzy variables are denoted by

$$\mu_i(\phi_t) = \begin{cases} t, & \text{if } i = t \\ 0, & \text{otherwise.} \end{cases}$$

For any $\gamma > 0$, taking $\eta \in (0,1)$ and $N = \left[\frac{2}{\gamma}\right] + 1$, we acquire

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \mathrm{Cr}\left\{\left|\mu_i - \mu_N\right| \geq \eta\right\} \geq \gamma\right\}\right| = 0.$$

Therefore, the sequence $\{\mu_i\}$ is a lacunary statistical Cauchy sequence in credibility.

Theorem 3.6. *Presume that* $\{\mu_i\}$ *is a lacunary statistical Cauchy sequence in credibility, then the sequence* $\{\mu_i\}$ *is lacunary statistical convergent a.s. to* μ.

Proof. Let $\{\mu_i\}$ be a lacunary statistical Cauchy sequence in credibility. Then, for any $\eta > 0$, $\gamma > 0$, there exists N such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{i \in I_r : \mathrm{Cr}\{|\mu_i - \mu_N| \geq \eta\} \geq \gamma\}\right| = 0.$$

If $\{\mu_i\}$ does not lacunary statistical converge a.s. to μ, then there is an element $\phi^* \in \Theta$ with $\gamma < \mathrm{Cr}\{\phi^*\} < 1$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{i \in I_r : |\mu_i(\phi^*) - \mu(\phi^*)| \geq \eta\}\right| = 1.$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{i \in I_r : \mathrm{Cr}(\mu_i(\phi^*) \geq \gamma)\}\right| = 1.$$

Another way of saying, there is $\eta > 0$ and subsequences $\mu_{i_k}(\phi^*)$ and $\mu_{N_k}(\phi^*)$ of $\mu_i(\phi^*)$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{i \in I_r : |\mu_{i_k}(\phi^*) - \mu_{N_k}(\phi^*)| \geq \eta\}\right| = 1.$$

for any k. From Axiom ii that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{i \in I_r : \mathrm{Cr}\{|\mu_{i_k} - \mu_{N_k}| \geq \eta\} \geq \gamma\}\right| \geq \lim_{r\to\infty} \frac{1}{h_r} \left|\{i \in I_r : \mathrm{Cr}\{\phi_i^*\} \geq \gamma\}\right|$$

for any k. As a result, $\{\mu_n\}$ is not a lacunary statistical Cauchy sequence in credibility. A contradiction finalizes the proof. So, $\{\mu_i\}$ is lacunary statistical convergent a.s. to μ. □

Theorem 3.7. *If* $\{\mu_i\}$ *lacunary statistical converges in credibility to* μ, *then* $\{\mu_i\}$ *is a lacunary statistical Cauchy sequence in credibility.*

Proof. When $\{\mu_i\}$ is lacunary statistical convergent in credibility to μ, then, there is $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ so that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\} \geq \frac{\gamma}{2}\right\}\right| = 0$$

for each $\eta > 0$ and $\gamma > 0$. Let

$$A = \left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\} \geq \frac{\gamma}{2}\right\}, \ B = \{i \in I_r : \mathrm{Cr}\{|\mu_i - \mu_N| \geq \eta\} \geq \gamma\}.$$

Thus

$$A^c = \left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\} < \frac{\gamma}{2}\right\}.$$

Next we prove $B \subset A$. Presume in contrast that $A \subseteq B$ and $i \in B \setminus A$. Then

$$\mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\} < \frac{\gamma}{2},\ \mathrm{Cr}\left\{|\mu_i - \mu_N| \geq \eta\right\} \geq \gamma.$$

Let $N \in A^c$, we get $\mathrm{Cr}\left\{|\mu_N - \mu| \geq \frac{\eta}{2}\right\} < \frac{\gamma}{2}$. Hence

$$\gamma \leq \mathrm{Cr}\left\{|\mu_i - \mu_N| \geq \eta\right\} \leq \mathrm{Cr}\left\{|\mu_N - \mu| \geq \frac{\eta}{2}\right\} + \mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\} < \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma,$$

which is a contradiction. Therefore, $B \subset A$. So, we get

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu_N| \geq \eta\right\} \geq \gamma\right\}\right| = 0.$$

Hence, $\{\mu_i\}$ is lacunary statistical Cauchy sequence in credibility. □

Definition 3.6. A credibility space is named as lacunary statistically complete in credibility if every lacunary statistical Cauchy sequence in credibility lacunary statistical converges in credibility.

Theorem 3.8. *Credibility space* $(\Theta, \mathcal{P}(\Theta), Cr)$ *is lacunary statistically complete in credibility.*

Proof. Take $\{\mu_n\}$ as a lacunary statistical Cauchy sequence in credibility. Then, there is a A with $\mathrm{Cr}\{A\} = 1$ and $N = N(\gamma)$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu_N| \geq \eta\right\} \geq \gamma\right\}\right| = 0$$

for each $\eta > 0$ and $\gamma > 0$. Assume in contrast that it is not lacunary statistical convergence in credibility. Then, there is a A with $Cr\{A\} = 1$ so that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\} \geq \frac{\gamma}{2}\right\}\right| \neq 0$$

for each $\eta > 0$ and $\gamma > 0$. Let

$$B = \left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\} \geq \frac{\gamma}{2}\right\}$$

and

$$C = \left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu_N| \geq \eta\right\} \geq \gamma\right\}.$$

Thus

$$B^c = \left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\} < \frac{\gamma}{2}\right\}.$$

Next we prove $B \subseteq C$. Assume $C \subseteq B$ and $i \in B^c \cap C$. Then

$$\mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\} < \frac{\gamma}{2},\ \mathrm{Cr}\left\{|\mu_i - \mu_N| \geq \eta\right\} \geq \gamma.$$

Let $N \in B^c$, we obtain

$$\mathrm{Cr}\left\{|\mu_N - \mu| \geq \frac{\eta}{2}\right\} < \frac{\gamma}{2}.$$

Hence, there is a $N = N(\gamma)$ such that

$$\gamma \leq \mathrm{Cr}\left\{|\mu_i - \mu_N| \geq \eta\right\} \leq \mathrm{Cr}\left\{|\mu_N - \mu| \geq \frac{\eta}{2}\right\} + \mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\}$$
$$\leq \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma,$$

which is impossible. Observe that $B \subseteq C$. This gives that

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{i \in I_r : \mathrm{Cr}\left\{|\mu_i - \mu| \geq \frac{\eta}{2}\right\} \geq \frac{\gamma}{2}\right\}\right| = 0.$$

Thus, the sequence $\{\mu_i\}$ have to be lacunary statistical convergent in credibility. This means that credibility space is lacunary statistically complete in credibility. □

Theorem 3.9. *Take $\mu, \mu_1, \mu_2, ...$ as fuzzy variables. Then, $\{\mu_i\}$ lacunary statistical converges a.s. to μ iff for any $\eta, \gamma > 0$, we get*

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{i \in I_r : \mathrm{Cr}\left(\bigcap_{r\in I_{r_i}} \bigcup_{i\in I_r} |\mu_i(\phi) - \mu(\phi)| > \eta\right) \geq \gamma\right\}\right| = 0.$$

Proof. According to the definition of lacunary statistical converges a.s., we have that there is $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ so that

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{i \in I_r : |\mu_i(\phi) - \mu(\phi)| \geq \eta\right\}\right| = 0$$

for each $\eta > 0$ and every $\phi \in A$. Then, for any $\eta > 0$, there exists m such that $|\mu_i(\theta) - \mu(\theta)| < \eta$ where $i > m$ and for any $A \in \mathcal{P}(\Theta)$, that is identical to

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{i \in I_r : \mathrm{Cr}\left(\bigcup_{r\in I_{r_i}} \bigcap_{i\in I_r} |\mu_i(\phi) - \mu(\phi)| > \eta\right) \geq \gamma\right\}\right| = 1.$$

From the duality axiom of crebility measure we obtain

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{i \in I_r : \mathrm{Cr}\left(\bigcap_{r\in I_{r_i}} \bigcup_{i\in I_r} |\mu_i(\phi) - \mu(\phi)| > \eta\right) \geq \gamma\right\}\right| = 0.$$

So, we acquire the result. □

Theorem 3.10. *If there is a sequence of numbers $\{\eta_i\}$ such that $\sum_{i=1}^{\infty} \eta_i < +\infty$ and*

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{i \in I_r : \sum_{i\in I_r} \mathrm{Cr}\left\{|\mu_{i+1} - \mu_i| \geq \eta_i\right\} \geq \gamma\right\}\right| = 0.$$

then $\{\mu_i\}$ lacunary statistical converges a.s. to μ.

Proof. Since

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{i\in I_r:\sum_{i\in I_r}\mathrm{Cr}\left\{|\mu_{i+1}-\mu_i|\geq\eta_i\right\}\geq\gamma\right\}\right|=0,$$

we have

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{i\in I_r:\mathrm{Cr}\left\{|\mu_{i+1}-\mu_i|\geq\eta_i\right\}\geq\gamma\right\}\right|=0.$$

From

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{i\in I_r:\mathrm{Cr}\left(\bigcap_{r\in I_{r_i}}\bigcup_{i\in I_r}\left\{|\mu_{i+1}(\phi)-\mu_i(\phi)|>\eta_i\right\}\right)\geq\gamma\right\}\right|$$

$$<\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{i\in I_r:\mathrm{Cr}\left(\bigcup_{i\in I_r}\left\{|\mu_{i+1}(\phi)-\mu_i(\phi)|>\eta_i\right\}\right)\geq\gamma\right\}\right|$$

$$<\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{i\in I_r:\sum_{i\in I_r}\mathrm{Cr}\left\{|\mu_{i+1}(\phi)-\mu_i(\phi)|>\eta_i\right\}\geq\gamma\right\}\right|.$$

By taking the limit $r\to\infty$ on both side of aforementioned inequality, we acquire

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{n\in I_r:\mathrm{Cr}\left(\bigcap_{r\in I_{r_i}}\bigcup_{i\in I_r}\left\{|\mu_{i+1}(\phi)-\mu_i(\phi)|>\eta_i\right\}\right)\geq\gamma\right\}\right|=0.$$

Therefore, we obtain

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{i\in I_r:\mathrm{Cr}\left(\bigcap_{r\in I_{r_i}}\bigcup_{i\in I_r}\left\{|\mu_{i+1}(\phi)-\mu_i(\phi)|\leq\eta_i\right\}\right)<\gamma\right\}\right|=1.$$

Let

$$T=\bigcap_{r\in I_{r_i}}\bigcup_{i\in I_r}\left\{\phi\in\Theta:|\mu_{i+1}(\phi)-\mu_i(\phi)|\leq\eta_i\right\}.$$

Since $\sum_{i=1}^{\infty}\eta_i<+\infty$, for any $\eta>0$, there is M_0 such that

$$\eta_{M_0}+\eta_{M_0+1}+\eta_{M_0+2}+...=\sum_{i=M_0}^{\infty}\eta_i\leq\eta.$$

If $\phi\in T$, there is m_0 such that

$$\phi\in\bigcap_{i=m_0}^{\infty}\left\{\phi\in\Theta:|\mu_{i+1}(\phi)-\mu_i(\phi)|\leq\eta_i\right\}.$$

Getting $M \geq \max\{m_0, M_0\}$, we get

$$\eta_M + \eta_{M+1} + ... < \eta$$

and

$$\phi \in \bigcap_{i=M}^{\infty} \{\phi \in \Theta : | \, |\mu_{i+1}(\phi) - \mu_i(\phi)| \leq \eta_i\}.$$

Indicate

$$S = \bigcap_{i=m}^{\infty} \{\phi \in \Theta : |\mu_{i+1}(\phi) - \mu_i(\phi)| \leq \eta_i\},$$

then $T \subseteq S$. Hence, $\mathrm{Cr}\{S\} = 1$. Namely, as long as $i \geq M$, we get $|\mu_{i+1} - \mu_i| \leq \eta_i$, for any $\phi \in S$, then

$$|\mu_M - \mu_{M+i}| \leq |\mu_M - \mu_{M+1}| + ... + |\mu_{M+i-1} - \mu_{M+i}| \leq \eta_M + ... + \eta_{M+i-1} \leq \eta.$$

Thus, $\{\mu_i\}$ is a lacunary statistical Cauchy sequence a.s. Based on Theorem 3.5, $\{\mu_i\}$ lacunary statistical converges a.s. to μ. □

Theorem 3.11. *Take $\mu, \mu_1, \mu_2, ...$ as fuzzy variables and take $f : \mathbb{R} \to \mathbb{R}$ as a convex function. If $\{\mu_i\}$ is lacunary statistical convergent a.s. to μ, then $\{f(\mu_i)\}$ is lacunary statistical convergent a.s. to $f(\mu)$.*

Proof. Considering f is a convex function, it is obvious from Theorem 2.4 that there is a constant w such that

$$|f(x) - f(y)| \leq w|x - y|,$$

for any $x, y \in \mathbb{R}$. Replacing x with μ_i and y with μ, we acquire

$$|f(\mu_i) - f(\mu)| \leq w|\mu_i - \mu|.$$

Since $\{\mu_i\}$ lacunary statistical converges a.s. to μ, then for any $\eta > 0$, there is an event A with $\mathrm{Cr}\{A\} = 1$ such that for every $\phi \in A$

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ i \in I_r : |\mu_i - \mu| < \frac{\eta}{w} \right\} \right| = 1.$$

Then

$$|f(\mu_i) - f(\mu)| \leq w|\mu_i - \mu| < w.\frac{\eta}{w} = \eta.$$

Therefore

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : |f(\mu_i) - f(\mu)| < \eta\}| = 1.$$

Hence, there is an event A with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : |f(\mu_i) - f(\mu)| \geq \eta\}| = 0$$

for every $\eta > 0$ which gives $\{f(\mu_i)\}$ lacunary statistical converges a.s. to $f(\mu)$. □

Theorem 3.12. *If $\{\mu_n\}$ is lacunary statistical convergent to μ in credibility and $f:\mathbb{R}\to\mathbb{R}$ is a convex function, then $\{f(\mu_n)\}$ is lacunary statistical convergent in credibility to $f(\mu)$.*

Proof. Since $\{\mu_i\}$ lacunary statistical converges to μ in credibility, we acquire

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{i\in I_r:\mathrm{Cr}\left\{|\mu_i-\mu|\geq\frac{\eta}{w}\right\}\geq\gamma\right\}\right|=0,$$

for every $\eta,\gamma>0$. Thus

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{i\in I_r:\mathrm{Cr}\left\{|\mu_i-\mu|<\frac{\eta}{w}\right\}<\gamma\right\}\right|=1.$$

For that reason f is a convex function, we write

$$|f(\mu_i)-f(\mu)|\leq w\,|\mu_i-\mu|<w.\frac{\eta}{w}=\eta.$$

Therefore

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{i\in I_r:\mathrm{Cr}\{|f(\mu_i)-f(\mu)|<\eta\}<\gamma\}\right|=1.$$

Thus

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{i\in I_r:\mathrm{Cr}\{|f(\mu_i)-f(\mu)|\geq\eta\}\geq\gamma\}\right|=0.$$

Hence, $\{f(\mu_i)\}$ lacunary statistical converges to $f(\mu)$ in credibility. □

Theorem 3.13. *Let $\mu,\mu_1,\mu_2,...$ be fuzzy variables, and take $f:\mathbb{R}\to\mathbb{R}$ as a convex function. If $\{\mu_i\}$ is lacunary statistical convergent in mean to μ, then $\{f(\mu_i)\}$ is lacunary statistical convergent in mean to $f(\mu)$.*

Proof. If $\{\mu_i\}$ lacunary statistical converges in mean to μ, then for every $\eta>0$, we get

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{i\in I_r:E[|\mu_i-\mu|]\geq\eta\}\right|=0.$$

Utilizing Theorem 2.2, for any $\eta,\gamma>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{i\in I_r:\mathrm{Cr}\{|\mu_i-\mu|\geq\eta\}\geq\gamma\}\right|=0.$$

In view of the fact that f is a convex function, from Theorem 3.12 we can write

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{i\in I_r:\mathrm{Cr}\{|f(\mu_i)-f(\mu)|\geq\eta\}\geq\gamma\}\right|=0.$$

Simultaneously, we can deduce that $|f(\mu_i)-f(\mu)|$ is bounded. That is $|f(\mu_i)-f(\mu)|$ is uniformly essentially bounded. Therefore we get, we acquire

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{i\in I_r:E[|f(\mu_i)-f(\mu)|]\geq\varepsilon\}\right|=0.$$

As a consequence, $\{f(\mu_i)\}$ lacunary statistical converges in mean to $f(\mu)$. □

The following results are obtained from Theorems 2.1 and 3.12 and Theorems 2.2 and 3.13, respectively.

Corollary 3.1. *Take $f:\mathbb{R}\to\mathbb{R}$ as a convex function. If $\{\mu_i\}$ is lacunary statistical convergent in credibility to μ, then $\{f(\mu_i)\}$ is lacunary statistical convergent a.s. to $f(\mu)$.*

Corollary 3.2. *Take $f:\mathbb{R}\to\mathbb{R}$ as a convex function. If $\{\mu_i\}$ is lacunary statistical convergent in mean to μ, then $\{f(\mu_i)\}$ is lacunary statistical convergent in credibility to $f(\mu)$.*

Theorem 3.14. *Take $f:\mathbb{R}\to\mathbb{R}$ as a continuous function. If $\{\mu_i\}$ is lacunary statistical convergent a.s. to μ, then $\{f(\mu_i)\}$ is lacunary statistical convergent a.s. to $f(\mu)$.*

Proof. Considering f is a continuous function, for every $\varepsilon>0$, there exists $\delta_1>0$ such that $|\mu_i-\mu|<\delta_1$ implies $|f(\mu_i)-f(\mu)|<\varepsilon$. Therefore,

$$\{i\in\mathbb{N}:|f(\mu_i)-f(\mu)|\geq\varepsilon\}\subset\{i\in\mathbb{N}:|\mu_i-\mu|\geq\delta_1\}.$$

If $\{\mu_i\}$ lacunary statistical converges a.s. to μ, then for any $\delta>0$, there is an event A with $\mathrm{Cr}\{A\}=1$ such that for every $\phi\in A$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{n\in I_r:\mathrm{Cr}\{|\mu_n-\mu|\geq\varepsilon\}\geq\delta\}\right|=0,$$

thus

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{n\in I_r:\mathrm{Cr}\{|f(\mu_n)-f(\mu)|\geq\varepsilon\}\geq\delta\}\right|=0.$$

Therefore, for any $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{n\in I_r:|f(\mu_n)-f(\mu)|\geq\varepsilon\}\right|=0.$$

As a consequence, $\{f(\mu_n)\}$ lacunary statistical converges a.s. to $f(\mu)$. □

Theorem 3.15. *If $\{\mu_i\}$ is lacunary statistical convergent to μ in credibility and $f:\mathbb{R}\to\mathbb{R}$ is a continuous function, then $\{f(\mu_i)\}$ is lacunary statistical convergent to $f(\mu)$ in credibility.*

Proof. If $\{\mu_i\}$ lacunary statistical converges to μ in credibility, then for every $\eta>0$ and $\delta>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{i\in I_r:\mathrm{Cr}\{|\mu_i-\mu|\geq\eta\}\geq\delta\}\right|=0. \tag{3.1}$$

For the reason that f is a continuous function, for every $\eta > 0$, there exists $\delta_1 > 0$ such that $|\mu_i - \mu| < \delta_1$ implies $|f(\mu_i) - f(\mu)| < \eta$. Therefore, $|f(\mu_i) - f(\mu)| \geq \eta$ implies $|\mu_i - \mu| \geq \delta_1$. For that reason one can write,

$$\{|f(\mu_i) - f(\mu)| \geq \eta\} \subset \{|\mu_i - \mu| \geq \delta\}.$$

Take credibility from the both sides,

$$\mathrm{Cr}\{|f(\mu_i) - f(\mu)| \geq \eta\} \leq \mathrm{Cr}\{|\mu_i - \mu| \geq \delta_1\},$$

which implies

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : \mathrm{Cr}\{|f(\mu_i) - f(\mu)| \geq \eta\} \geq \delta\}| \leq \lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : \mathrm{Cr}\{|\mu_i - \mu| \geq \delta_1\} \geq \delta\}|.$$

From (3.1), we have

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : \mathrm{Cr}\{|f(\mu_i) - f(\mu)| \geq \eta\} \geq \delta\}| = 0.$$

That means, $\{f(\mu_n)\}$ lacunary statistical converges to $f(\mu)$ in credibility. □

Theorem 3.16. *Take $\mu, \mu_1, \mu_2, ...$ as fuzzy variables and take $f : \mathbb{R} \to \mathbb{R}$ as a continuous function. If $\{\mu_n\}$ is lacunary statistical convergent in mean to μ, then $\{f(\mu_n)\}$ is lacunary statistical convergent in mean to $f(\mu)$.*

Proof. If $\{\mu_n\}$ lacunary statistical converges in mean to μ, then

$$\lim_{r\to\infty} \frac{1}{h_r} |\{n \in I_r : E[|\mu_n - \mu|] \geq \varepsilon\}| = 0.$$

By utilizing Theorem 2.1, for any $\varepsilon > 0$ and $\delta > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{n \in I_r : \mathrm{Cr}\{|\mu_n - \mu| \geq \varepsilon\} \geq \delta\}| = 0.$$

For the reason that f is a continuous function, it is obvious from Theorem 3.15 that

$$\lim_{r\to\infty} \frac{1}{h_r} |\{n \in I_r : \mathrm{Cr}\{|f(\mu_n) - f(\mu)| \geq \varepsilon\} \geq \delta\}| = 0.$$

Simultaneously, we can deduce that $|f(\mu_n) - f(\mu)|$ is UEB. So, we obtain

$$\lim_{r\to\infty} \frac{1}{h_r} |\{n \in I_r : E[|f(\mu_n) - f(\mu)|] \geq \varepsilon\}| = 0.$$

Consequently, $\{f(\mu_n)\}$ lacunary statistical converges in mean to $f(\mu)$. □

Using similar arguments as in Theorems 2.1 and 3.15 and Theorems 2.2 and 3.16, respectively, we get the following results.

Corollary 3.3. *Let $\mu, \mu_1, \mu_2, ...$ be fuzzy variables, and take $f : \mathbb{R} \to \mathbb{R}$ as a continuous function. If $\{\mu_n\}$ is lacunary statistical convergent in credibility to μ, then $\{f(\mu_n)\}$ is lacunary statistical convergent a.s. to $f(\mu)$.*

Corollary 3.4. *Let $\mu, \mu_1, \mu_2, ...$ be fuzzy variables, and take $f : \mathbb{R} \to \mathbb{R}$ as a continuous function. If $\{\mu_n\}$ is lacunary statistical convergent in mean to μ, then $\{f(\mu_n)\}$ is lacunary statistical convergent in credibility to $f(\mu)$.*

Now, we investigate the space $|\sigma_1|$ of strongly Cesàro summable and the space N_θ of strongly lacunary summable fuzzy variable sequences by

$$|\sigma_1| = \left\{ \mu = (\mu_i(\phi)) : \text{there exists } \mu(\phi) \text{ such that } \frac{1}{n}\sum_{i=1}^{n} \|\mu_i(\phi) - \mu(\phi)\| \to 0, \text{ as } n \to \infty \right\},$$

and

$$N_\theta = \left\{ \mu = (\mu_i(\phi)) : \text{there exists } \mu(\phi) \text{ such that } \nu_r \equiv \frac{1}{h_r}\sum_{i \in I_r} \|\mu_i(\phi) - \mu(\phi)\| \to 0, \text{ as } r \to \infty \right\}.$$

Theorem 3.17. $|\sigma_1| \subseteq N_\theta$*, it is necessary and sufficient that* $\lim_r \inf q_r > 1$.

Proof. For the sufficiency we presume $\lim_r \inf q_r > 1$, then there exists $\xi(\phi) \in (\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ and $\mathrm{Cr}(\xi(\phi)) > 0$ such that $1 + \mathrm{Cr}(\xi(\phi)) \le q_r$ for each $r \ge 1$. Now, for $\mu(\phi) \in |\sigma_1|^0$ we acquire

$$\nu_r = \frac{1}{h_r}\sum_{i=1}^{k_r} \|\mu_i(\phi)\| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}} \|\mu_i(\phi)\|$$

$$= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{i=1}^{k_r} \|\mu_i(\phi)\|\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}} \|\mu_i(\phi)\|\right).$$

Since $h_r = k_r - k_{r-1}$, we get $\frac{k_r}{h_r} \le \frac{1+\mathrm{Cr}(\xi(\phi))}{\mathrm{Cr}(\xi(\phi))}$ and $\frac{k_{r-1}}{h_r} \le \frac{1}{\mathrm{Cr}(\xi(\phi))}$; as $\mathrm{Cr}(\xi(\phi)) > 0$ and $q_r = \frac{k_r}{k_{r-1}}$.

The terms $\frac{1}{k_r}\sum_{i=1}^{k_r} \|\mu_i(\phi)\|$ and $\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}} \|\mu_i(\phi)\|$ both converges to 0. Hence, ν_r converge to 0, namely, $\eta_i(\phi) \in N_\theta^0$. Hence, $|\sigma_1| \subseteq N_\theta$.

For the sufficiency we presume, $\lim_r \inf q_r = 1$. Since θ is lacunary sequence, we can select a subsequence k_{r_j} of θ providing,

$$\frac{k_{r_j}}{k_{r_j-1}} < 1 + \frac{1}{j} \text{ and } \frac{k_{r_j-1}}{k_{r_{j-1}}} > j, \text{ where } r_j \ge r_{j-1} + 2.$$

Identify $\mu = (\mu_i(\phi))$ by

$$\mu_i(\phi) = \begin{cases} 1, & \text{if } i \in I_{r_j} \text{ for some } j = 1, 2, ...; \\ 0, & \text{otherwise.} \end{cases}$$

Then, for any $\mu(\phi)$,

$$\frac{1}{h_{r_j}} \sum_{i \in I_{r_j}} \|\mu_i(\phi) - \mu(\phi)\| = \|1 - \mu(\phi)\|; \ j = 1, 2, ...$$

and

$$\frac{1}{h_r} \sum_{i \in I_r} \|\mu_i(\phi) - \mu(\phi)\| = \|\mu(\phi)\| \ \text{ for } r \neq r_j.$$

It gives that $(\mu_i(\phi)) \in N_\theta$.

But, $(\mu_i(\phi))$ is strongly summable, since if we contemplate w is sufficiently large, there is a unique j for which $k_{r_j - 1} < w \leq k_{r_{j+1} - 1}$ and we obtain

$$\frac{1}{w} \sum_{i=1}^{w} \|\mu_i(\phi)\| \leq \frac{k_{r_{j-1}} + h_{r_j}}{k_{r_j} - 1} \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

If $w \to \infty$, it follows that $j \to \infty$. Hence, we acquire $(\mu_i(\phi)) \in |\sigma_1|^0$. □

Theorem 3.18. *$N_\theta \subseteq |\sigma_1|$, it is necessary and sufficient that* $\lim_r \sup q_r < \infty$.

Proof. For the sufficiency we contemplate $\lim_r \sup q_r < \infty$, there exist $H(\phi) \in (\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ and $\mathrm{Cr}(H(\phi)) > 0$ such that $q_r < \mathrm{Cr}(H(\phi))$ for all $r \geq 1$. Thinking $(\mu_i(\phi)) \in N_\theta^0$ and $\gamma > 0$ we can select $T > 0$ such that $\nu_i < m$ for all $i = 1, 2, ...$

Then, if s is any integer with $k_{r-1} < s \leq k_r$, where $r > T$, we can write,

$$\begin{aligned}
&\tfrac{1}{s} \sum_{i=1}^{s} \|\mu_i(\phi)\| \leq \tfrac{1}{k_{r-1}} \sum_{i=1}^{s} \|\mu_i(\phi)\| \\
&= \tfrac{1}{k_{r-1}} \left(\sum_{I_1} \|\mu_i(\phi)\| + ... + \sum_{I_r} \|\mu_i(\phi)\| \right) \\
&= \tfrac{1}{k_{r-1}} \nu_1 + \tfrac{k_2 - k_1}{k_{r-1}} \nu_2 + ... + \tfrac{k_T - k_{T-1}}{k_{r-1}} \nu_T \\
&+ \tfrac{k_{T+1} - k_T}{k_{r-1}} \nu_{T+1} + ... + \tfrac{k_r - k_{r-1}}{k_{r-1}} \nu_r \\
&\leq \left(\sup_{i \geq 1} \nu_i \right) \tfrac{k_T}{k_{r-1}} + \left(\sup_{i \geq T} \nu_i \right) \tfrac{k_r - k_T}{k_{r-1}} \\
&= m . \tfrac{k_T}{k_{r-1}} + \gamma . \mathrm{Cr}(H(\phi)).
\end{aligned}$$

Since $k_{r-1} \to \infty$, as $s \to \infty$, it follows that $\frac{1}{s} \sum_{i=1}^{s} \|\mu_i(\phi)\| \to 0$ and as a result $\eta(\phi) \in |\sigma_1|^0$.

For the necessity part we think $\lim_r \sup q_r = \infty$ and create a sequence in N_θ that is not strongly Cesàro Summable.

We select a subsequence $\left(k_{r_j}\right)$ of θ so that $q_{r_j} > j$ and establish $\eta = (\eta(\phi))$ by

$$\mu_i(\phi) = \begin{cases} 1, & \text{if } k_{r_{j-1}} < i \leq 2k_{r_j-1} \text{ for some } j = 1, 2, ...; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\nu_{r_j} = \frac{k_{r_j-1}}{k_{r_j} - k_{r_{j-1}}} < \frac{1}{j-1}$$

if $r = r_j$, $\nu_r = 0$. Thus $(\mu_i(\phi)) \in N_\theta^0$.

Any sequence in $|\sigma_1|$ consisting of only 0's and 1's has an strong limit $\eta(\phi) \in (\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$, where $\mathrm{Cr}(\eta(\phi)) = 1$ or $\mathrm{Cr}(\eta(\phi)) = 0$.

For the sequence $\mu = (\mu_i(\phi))$ and $i = 1, 2, ..., k_{r_j}$.

$$\begin{aligned} \tfrac{1}{k_{r_j}} \sum \|\mu(\phi) - 1\| &\geq \tfrac{1}{k_{r_j}} \left(k_{r_j} - 2k_{r_j-1}\right) \\ &= 1 - \tfrac{2k_{r_j}-1}{k_r} \\ &> 1 - \tfrac{2}{j} \end{aligned}$$

which convergence to 1 and for $i = 1, 2, ..., 2k_{r_j} - 1$

$$\frac{1}{2k_{r_j} - 1} \sum_i \|\mu_i(\phi)\| \geq \frac{k_{r_j-1}}{2k_{r_j} - 1} = \frac{1}{2}.$$

Thus, $(\mu_i(\phi)) \in |\sigma_1|$. □

Now, we investigate the strong almost convergence in respect of fuzzy variables in the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$.

Definition 3.7. The sequence $\{\mu_i\}$ in the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is called to be strongly almost convergent if there exists a fuzzy variable $\mu(\phi) \in (\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ for which

$$\frac{1}{p} \sum_{i=s+1}^{s+p} \|\mu_i(\phi) - \mu(\phi)\| \to 0 \;\; (p \to \infty),$$

uniformly in $s = 0, 1, 2, ...$

Theorem 3.19. $|AC| \subset N_\theta$.

Proof. Let $(\mu_i(\phi)) \in |AC|$ and take $\gamma > 0$, then there are $P > 0$ and μ such that

$$\frac{1}{p} \sum_{i=s+1}^{s+p} \|\mu_i(\phi) - \mu(\phi)\| < \gamma$$

for $p > P$, $s = 0, 1, 2, \ldots.$

As ϕ is lacunary sequence we can select $T > 0$ such that $r \geq T$ means $h_r > P$ and as a result $\nu_r < \gamma$. So $(\mu_i(\phi)) \in N_\theta$. Hence to acquire a sequence in N_θ but not in $|AC|$ establish $\mu = (\mu_i(\phi))$ by

$$\mu_i(\phi) = \begin{cases} 1, & \text{if for some } r,\ k_{r-1} < i \leq k_{r-1} + \sqrt{h_r}; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, μ involves arbitrarily long strings of 0's and 1's, from which it argues that μ is not strongly almost convergent. But

$$\nu_r = \frac{1}{h_r} \sum_{i \in I_r} \|\mu_i\| = \frac{1}{h_r} \sqrt{h_r} = \frac{1}{\sqrt{h_r}}$$

which converges to 0 as $r \to \infty$. Hence, $(\mu_i(\phi)) \in N_\theta$. □

Now, we present the lacunary convergence notions of fuzzy variable sequences in credibility space and acquire the relations between them.

Definition 3.8. The fuzzy variable sequence $\{\mu_i\}$ is called to be lacunary strongly almost surely (l.s.a.s.) to the fuzzy variable μ iff there exists $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} \|\mu_i(\phi) - \mu(\phi)\| = 0,$$

for every $\phi \in A$.

Definition 3.9. The fuzzy variable sequence $\{\mu_i\}$ is called to be lacunary strongly convergent in credibility to μ if

$$\lim_{r \to \infty} \mathrm{Cr}\left(\left\{\phi \in A : \frac{1}{h_r} \sum_{i \in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\}\right) = 0,$$

for every $\phi \in A$, each $\gamma > 0$.

Definition 3.10. The fuzzy variable sequence $\{\mu_i\}$ is called to be lacunary strongly convergent in mean to μ if

$$\lim_{r \to \infty} E\left[\frac{1}{h_r} \sum_{i \in I_r} \|\mu_i(\phi) - \mu(\phi)\|\right] = 0,$$

for every $\phi \in A$.

Definition 3.11. Assume that $\Phi, \Phi_1, \Phi_2, ...$ are the credibility distributions of fuzzy variables $\mu, \mu_1, \mu_2, ...$ respectively. We say that the fuzzy variable sequence $\{\mu_i\}$ lacunary strong convergent in distribution to μ if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \|\Phi_i(c) - \Phi(c)\| = 0.$$

for all c at which $\Phi(c)$ is continuous

Definition 3.12. The fuzzy variable sequence $\{\mu_i\}$ is called to be convergent uniformly almost surely (u.a.s.) to μ if there is an sequence of events $\{A_i\}$, $\mathrm{Cr}\{A_i\} \to 0$ such that $\{\mu_i\}$ converges uniformly to μ in $\mathcal{P}(\Theta) - A_i$, for any fixed $i \in \mathbb{N}$.

Here, we examine the relations among the convergence of fuzzy variable sequences.

Theorem 3.20. *Take $\{\mu_i\}$ as a sequence of fuzzy variables. If the sequence $\{\mu_i\}$ lacunary strongly convergent in mean to a fuzzy variable μ, then $\{\mu_i\}$ lacunary strongly converges in credibility to μ.*

Proof. Let the fuzzy variable sequence $\{\mu_i\}$ be lacunary strongly convergent in mean to μ. For any taken $\gamma > 0$, with the aid of Markov inequality, we obtain

$$\lim_{r\to\infty} \mathrm{Cr}\left(\left\{\phi \in A : \frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\}\right) \leq \lim_{r\to\infty} \frac{E\left[\frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\|\right]}{\gamma} = 0.$$

As a result, $\{\mu_i\}$ lacunary strongly converges in credibility to μ. □

But the converse of the above theorem need not be true in general i.e. lacunary strongly convergence in credibility does not imply lacunary strongly convergence in mean. This can be denoted by the following example.

Example 3.3. Contemplate the crebility space $(\Theta, \mathcal{P}, \mathrm{Cr})$ to be $\{\phi_1, \phi_2, ...\}$ with power set and

$$\mathrm{Cr}\{A\} = \begin{cases} \sup\limits_{\phi_i \in A} \frac{1}{i}, & \text{if } \sup\limits_{\phi_i \in A} \frac{1}{i} < 0.5; \\ 1 - \sup\limits_{\phi_i \in A^c} \frac{1}{i}, & \text{if } \sup\limits_{\phi_i \in A^c} \frac{1}{i} < 0.5; \\ 0.5, & \text{otherwise} \end{cases}$$

and the fuzzy variables are identified by

$$\mu_i(\phi_j) = \begin{cases} i, & \text{if } j = i; \\ 0, & \text{otherwise} \end{cases}$$

for $i \in I_r$ and $\mu \equiv 0$. For $\gamma > 0$, we acquire

$$\lim_{r\to\infty} \mathrm{Cr}\left(\left\{\phi \in A : \tfrac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\}\right)$$
$$= \lim_{r\to\infty} \mathrm{Cr}\left(\left\{\phi \in A : \tfrac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi)\| > \gamma\right\}\right)$$
$$= \lim_{r\to\infty} \mathrm{Cr}\left(\{\phi_i\}\right) = \lim_{r\to\infty} \tfrac{1}{i} = 0 \text{ (as } i \in I_r).$$

The sequence $\{\mu_i\}$ lacunary strongly convergent in credibility to μ. But for each $i \in I_r$, we obtain the distribution of fuzzy variable $\|\mu_i - \mu\| = \|\mu_i\|$ is

$$\Phi_i(c) = \begin{cases} 0, & \text{if } c < 0; \\ 1 - \frac{1}{i}, & \text{if } 0 \le c < i; \\ 1, & \text{otherwise.} \end{cases}$$

$$E\left[\tfrac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\|\right] = \int_0^{+\infty} \mathrm{Cr}\{\phi \ge c\}\, dc$$
$$-\int_{-\infty}^{0} \mathrm{Cr}\{\phi \le c\}\, dc = \int_0^{i} 1 - \left(1 - \tfrac{1}{i}\right) dc = 1.$$

That is, the $\{\mu_i\}$ does not lacunary strongly converge in mean to μ.

Lacunary strongly convergent in distribution does not mean lacunary strongly convergence in credibility. Subsequent example denotes this.

Example 3.4. Contemplate the crebility space $(\Theta, \mathcal{P}, \mathrm{Cr})$ to be $\{\phi_1, \phi_2, ...\}$ with $\mathrm{Cr}\{\phi_1\} = \mathrm{Cr}\{\phi_2\} = \frac{1}{2}$. We think the fuzzy variable as

$$\mu(\phi) = \begin{cases} 1, & \text{if } \phi = \phi_1; \\ -1, & \text{if } \phi = \phi_2. \end{cases}$$

We also take $\{\mu_i\} = -\mu$ for $i \in I_r$. Then, $\{\mu_i\}$ and μ have the same distribution and so $\{\mu_i\}$ converges in distribution to μ. But, However, for any given $\gamma > 0$, we obtain

$$\lim_{r\to\infty} \mathrm{Cr}\left(\left\{\phi \in A : \tfrac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\}\right)$$
$$= \lim_{r\to\infty} \mathrm{Cr}\left(\left\{\phi \in A : \tfrac{1}{h_r}\sum_{i\in I_r} \|2\mu_i(\phi)\| > \gamma\right\}\right) \neq 0.$$

Therefore, the sequence $\{\mu_i\}$ does not lacunary strongly converge in credibility to μ.

Now, we present the relation between statistical convergence uniformly a.s. and statistical convergence a.s. of fuzzy variable sequence $\{\mu_i\}$ in credibility space.

Theorem 3.21. *Take $\mu, \mu_1, \mu_2, ...$ as fuzzy variables. Then, $\{\mu_i\}$ is lacunary strongly a.s. to μ iff for any $\gamma > 0$, we have*

$$\mathrm{Cr}\left(\bigcap_{r\in I_{r_i}} \bigcup_{i\in I_r} \left\{\phi \in A : \frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\}\right) = 0.$$

Proof. As stated in the definition of lacunary strongly a.s., we get that there is an $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| = 0$$

for every $\phi \in A$. Then, for any $\gamma > 0$ there exists t such that $\frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| < \gamma$, where $i > t$ and for any $\phi \in A$, that is identical to

$$\mathrm{Cr}\left(\bigcup_{r\in I_{r_i}} \bigcap_{i\in I_r} \left\{\phi \in A : \frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\}\right) = 1.$$

From the duality axiom of crebility measure we obtain

$$\mathrm{Cr}\left(\bigcap_{r\in I_{r_i}} \bigcup_{i\in I_r} \left\{\phi \in A : \frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\}\right) = 0.$$

Hence, the theorem is finalized. □

Theorem 3.22. *Take $\mu, \mu_1, \mu_2, ...$ as fuzzy variables. Then, $\{\mu_i\}$ lacunary strongly convergent u.a.s. to the fuzzy variable μ iff for any $\gamma > 0$, we have*

$$\lim_{r\to\infty} \mathrm{Cr}\left(\bigcup_{i\in I_r} \left\{\phi \in A : \frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\}\right) = 0.$$

Proof. If $\{\mu_i\}$ is lacunary strongly convergent uniformly a.s. to μ then for any $\delta > 0$ there is a A such that $\mathrm{Cr}\{A\} < \delta$ and $\{\mu_i\}$ converges uniformly to μ on $\mathcal{P}(\Theta) - A$. For this reason, for any $\gamma > 0$, there exists $t > 0$ such that $\frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| < \gamma$ where $i > t$ and $\phi \in \mathcal{P}(\Theta) - A$. That is,

$$\bigcup_{i\in I_r} \left\{\phi \in A : \frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\} \subset A.$$

From the subadditivity axiom that

$$\mathrm{Cr}\left(\bigcup_{i\in I_r} \left\{\phi \in A : \frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\}\right) \le \mathrm{Cr}(A) < \delta.$$

Thus

$$\lim_{r\to\infty} \mathrm{Cr}\left(\bigcup_{i\in I_r} \left\{\phi \in A : \frac{1}{h_r}\sum_{i\in I_r} \|\mu_i(\phi) - \mu(\phi)\| > \gamma\right\}\right) = 0.$$

Conversely if,

$$\lim_{r\to\infty} \mathrm{Cr}\left(\bigcup_{i\in I_r}\left\{\phi\in A:\frac{1}{h_r}\sum_{i\in I_r}\|\mu_i(\phi)-\mu(\phi)\|>\gamma\right\}\right)=0$$

for any $\gamma > 0$, then for given $\delta > 0$ and $i \geq 1$, there is m_i such that

$$\mathrm{Cr}\left(\bigcup_{i\in I_r}\left\{\phi\in A:\frac{1}{h_r}\sum_{i\in I_r}\|\mu_i(\phi)-\mu(\phi)\|\geq\frac{1}{i}\right\}\right)<\frac{\delta}{2^i}.$$

Let

$$A=\bigcup_{i\in I_r}\bigcap_{m_i\in I_{r_i}}\left\{\phi\in A:\frac{1}{h_r}\sum_{i\in I_r}\|\mu_i(\phi)-\mu(\phi)\|\geq\frac{1}{i}\right\}.$$

Then $\mathrm{Cr}(A) < \delta$.

Additionally,we acquire

$$\sup_{\phi\in\mathcal{P}(\Theta)-A}\frac{1}{h_r}\sum_{i\in I_r}\|\mu_i(\phi)-\mu(\phi)\|\geq\frac{1}{i}$$

for any $i = 1, 2, ...$ and $i > m_i$. Hence the result is proved. □

Theorem 3.23. *If $\{\mu_i\}$ is lacunary strongly convergent u.a.s. to μ, then $\{\mu_i\}$ is lacunary strongly convergent in credibility to μ.*

Proof. If $\{\mu_i\}$ is lacunary strongly convergent uniformly a.s. to μ, then

$$\lim_{r\to\infty} \mathrm{Cr}\left(\bigcup_{i\in I_r}\left\{\phi\in A:\frac{1}{h_r}\sum_{i\in I_r}\|\mu_i(\phi)-\mu(\phi)\|>\gamma\right\}\right)=0$$

from the above theorem. But we have

$$\begin{aligned}&\mathrm{Cr}\left(\left\{\phi\in A:\tfrac{1}{h_r}\sum_{i\in I_r}\|\mu_i(\phi)-\mu(\phi)\|>\gamma\right\}\right)\\ &\leq \mathrm{Cr}\left(\bigcup_{i\in I_r}\left\{\phi\in A:\tfrac{1}{h_r}\sum_{i\in I_r}\|\mu_i(\phi)-\mu(\phi)\|>\gamma\right\}\right).\end{aligned}$$

As a result, $\{\mu_i\}$ is lacunary strongly convergent in credibility to μ. □

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REFERENCES

1. F. Başar: *Infinite matrices and almost boundedness*, Boll. Unione Mat. Ital. **6** (7) (1992), 395–402.
2. F. Başar: *Summability Theory and its Applications*, Second edition, CRC Press/Taylor & Francis Group, 2022.
3. F. Başar and M. Kirişçi: *Almost convergence and generalized difference matrix*, Comput. Math. Appl. **61** (3) (2011), 602–611.
4. C. Belen and S. A. Mohiuddine: *Generalized weighted statistical convergence and application*, Appl. Math. Comput. **219** (2013), 9821–9826.
5. X. Chen, Y. Ning and X. Wang: *Convergence of complex uncertain sequence*, J. Intell. Fuzzy Syst. **30** (6) (2016), 3357–3366.
6. D. Dubois and H. Prade: *Possibility theory: An approach to computerized processing of uncertainty*, New York, Plenum, 1998.
7. H. Fast: *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.
8. A. R. Freedman, J. J. Sember and M. Raphael: *Some Cesàro-Type summability spaces*, Proc. London Math. Soc. **3** (3) (1978), 508–520.
9. J. A. Fridy: *On statistical convergence*, Analysis **5** (1985), 301–313.
10. B. Hazarika, A. Alotaibi and S. A. Mohiuddine: *Statistical convergence in measure for double sequences of fuzzy-valued functions*, Soft Comput. **24** (2020), 6613–6622.
11. Q. Jiang: *Some remarks on convergence in cedibility distribution of fuzzy variable*, In: Proceedings of a Conference on Intelligence Science and Information Engineering, Wuhan, China, 2011, pp. 446–449.
12. U. Kadak and F. Başar: *Power series with real or fuzzy coefficients*, Filomat **25** (3) (2012), 519–528.
13. U. Kadak and S. A. Mohiuddine: *Generalized statistically almost convergence based on the difference operator which includes the (p,q)-Gamma function and related approximation theorems*, Results Math. **73** (9) (2018), 1–31.
14. A. Kaufmann: *Introduction to the theory of fuzzy subsets*, New York: Academic Press, 1975.
15. H. Kwakernaak: *Fuzzy random variables-I: definition and theorem*, Inf. Sci. **15** (1) (1978), 1–29.
16. X. Li and B. Liu: *A sufficient and necessary condition for credibility measures*, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems **14** (2006), 527–535.
17. X. Li and B. Liu: *Chance measure for hybrid events with fuzziness and randomness*, Soft Comput. **13** (2) (2008), 105–115.
18. X. Lin: *Characteristics of convex function*, Journal of Guangxi University for Nationalities (Natural Science Edition) **6** (4) (2000), 250–253.
19. B. Liu and Y. K. Liu: *Expected value of fuzzy variable and fuzzy expected value models*, IEEE Trans. Fuzzy Syst. **10** (4) (2002), 445–450.
20. B. Liu: *Theory and Practice of Uncertain Programming*, Physica-Verlag, Heidelberg, 2002.
21. B. Liu: *Uncertainty Theory*, 2nd ed., Springer-Verlag, Berlin, 2007.

22. B. Liu: *A survey of credibility theory*, Fuzzy Optim. Decis. Mak. **5** (4) (2006), 387–408.

23. B. Liu: *Inequalities and convergence concepts of fuzzy and rough variables*, Fuzzy Optim. Decis. Mak. **2** (2) (2003), 87–100.

24. Y. K. Liu and S. M. Wang: *Theory of fuzzy random optimization*, China Agricultural University Press, Beijing, 2006, 280.

25. G. G. Lorentz: *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

26. S. Ma: *The convergences properties of the credibility distribution sequence of fuzzy variables*, J. Modern Math. Frontier. **3** (1) (2014), 24–27.

27. S. A. Mohiuddine, H. Şevli and M. Cancan: *Statistical convergence in fuzzy 2-normed space*, J. Comput. Anal. Appl. **12** (4) (2010), 787–798.

28. S. A. Mohiuddine, A. Asiri and B. Hazarika: *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*, Int. J. Gen. Syst. **48** (5) (2019), 492–506.

29. S. A. Mohiuddine, B. Hazarika and A. Alotaibi: *On statistical convergence of double sequences of fuzzy valued functions*, J. Intell. Fuzzy Syst. **32** (2017), 4331–4342.

30. S. A. Mohiuddine and B. A. S. Alamri: *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM **113** (3) (2019), 1955–1973.

31. S. A. Mohiuddine, B. Hazarika and B. M. A. Alghamdi: *Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems*, Filomat **33** (14) (2019), 4549–4560.

32. M. Mursaleen and S. A. Mohiuddine: *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*, J. Comput. Appl. Math. **233** (2009), 142–149.

33. M. Mursaleen and F. Başar: *Sequence Spaces: Topics in Modern Summability Theory*, CRC Press, Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton, London, New York, 2020.

34. S. Nahmias: *Fuzzy variables*, Fuzzy Sets Syst. **1** (1978), 97–110.

35. F. Nuray and E. Savaş: *Statistical convergence of sequences of fuzzy numbers*, Math. Slovaca **45** (1995), 269–273.

36. D. Rath and B. C. Tripathy: *On statistically convergent and statistical Cauchy sequences*, Indian J. Pure Appl. Math. **25** (4) (1994), 381–386.

37. T. Šalát: *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (2) (1980), 139–150.

38. H. Steinhaus: *Sur la convergence ordinarie et la convergence asymptotique*, Colloq. Math. **2** (1951), 73–74.

39. Ö. Talo and F. Başar: *On the space $bv_p(F)$ of sequences of p-bounded variation of fuzzy numbers*, Acta Math. Sin. Eng. Ser. **24** (7) (2008), 1205–1212.

40. Ö. Talo and F. Başar: *Certain spaces of sequences of fuzzy numbers defined by a modulus function*, Demonstratio Math. **43** (1) (2010), 139–149.

41. Ö. Talo and F. Başar: *Quasilinearity of the classical sets of sequences of fuzzy numbers and some related results*, Taiwanese J. Math. **14** (5) (2010), 1799–1819.
42. B. C. Tripathy and A. Baruah: *Lacunary statistically convergent and lacunary strongly convergent generalized difference sequences of fuzzy real numbers*, Kyungpook Math. J. **50** (4) (2010), 565–574.
43. B. C. Tripathy and H. Dutta: *On some lacunary difference sequence spaces defined by a sequence of Orlicz functions and q-lacunary Δ_m^n -statistical convergence*, Analele Stiint. ale Univ. Ovidius Constanta Ser. Mat. **20** (1) (2012), 417–430.
44. B. C. Tripathy and A. J. Dutta: *Lacunary bounded variation sequence of fuzzy real numbers*, J. Intell. Fuzzy Syst. **204** (1) (2013), 185–189.
45. G. Wang and B. Liu: *New theorems for fuzzy sequence convergence*, In: Proceedings of a Conference on Information and Management Science, Chengdu, China, 2003, pp. 100–105.
46. Y. Xia: *Convergence of uncertain sequences*, M.S. Thesis, Suzhou University of Science and Technology, 2011.
47. M. Yeşilkayagil and F. Başar: *Spaces of A-almost null and A-almost convergent sequences*, J. Egypt. Math. Soc. **23** (2) (2015), 119–126.
48. C. You: *On the convergence of uncertain sequences*, Math. Comput. Model. **49** (3-4) (2009), 482–487.
49. C. You, R. Zhang and K. Su: *On the convergence of fuzzy variables*, J. Intell. Fuzzy Syst. **36** (2) (2019), 1663–1670.
50. L. A. Zadeh: *Fuzzy set*, Inf. Control. **8** (3) (1965), 338–353.
51. L. A. Zadeh: *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets Syst. **1** (1978), 3–28.
52. R. Zhao, W. Tang and H. Yun: *Random fuzzy renewal process*, Eur. J. Oper. Res. **14** (2006), 189–201.