FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 4 (2022), 709 -719
https://doi.org/10.22190/FUMI220317049Y
Original Scientific Paper

CHARACTERIZATIONS OF SOME SPECIAL QUATERNIONIC CURVES

Ahmet Yücesan[1] and Ercan Tulum[2]

[1]**Süleyman Demirel University, Department of Mathematics, 32200 Isparta, Turkey**
[2]**Süleyman Demirel University, Graduate School of Natural and Applied Sciences, 32200 Isparta, Turkey**

Abstract. We derive a general differential equation satisfied by the distance function for quaternionic curves in Euclidean 4−space. By using this differential equation, we express characterizations of some special quaternionic curves such as spherical curves and rectifying curves. Lastly, we reconsider the characterization of a quaternionic general helix.
Keywords: Quaternionic curve, spherical curve, rectifying curve, general helix.

1. Introduction

As it is known, quaternions were discovered Irish mathematician William Rowan Hamilton $(1805-1865)$ in 1843 to generalize complex numbers. He discovered that the appropriate generalization is one in which the scalar (real) axis is left unchanged whereas the vector (imaginary) axis is supplemented by adding two further vector axes. It is therefore helpful to think of the scalar axis as representing "time" and the three vector axes as representing "space". In this case, the (real) quaternions have the algebraic form denoted by $q_0 + q_1 i + q_2 j + q_3 k$, where q_0, q_1, q_2 and q_3 are real numbers. The vector space is regarded as the usual 3−dimensional vector space with "unit vectors" i, j and k. So, the set of (real) quaternions is identified with the 4−dimensional Euclidean space [11].

Received March 17, 2022. accepted June 09, 2022.
Communicated by Uday Chand De
Corresponding Author: Süleyman Demirel University, Department of Mathematics, 32200 Isparta, Turkey | E-mail: ahmetyucesan@sdu.edu.tr
2010 *Mathematics Subject Classification.* Primary 53A04; Secondary 11R52

It is possible to encounter quaternions in many branches of science. Especially in applied mathematics, quaternions are used in calculations involving three-dimensional rotations such as in $3-$dimensional computer graphics, computer vision, and crystallographic texture analysis [8].

In differential geometry, the study of quaternionic curves begins with Bharathi and Nagaraj [2] obtaining the Frenet (Serret-Frenet) formulas for these curves. Since then, many articles have been made on quaternionic curves. Among these articles, articles about the characterization of some special quaternionic curves such as spherical, rectifying, and general helix (see [10], [1], [3] and [4] for these types of curves) have great interest. Güngör and Tosun [7] investigated quaternionic rectifying curves and given the characterizations of these curves. Sağlam [9] studied some characterizations of the osculating sphere of quaternionic curves and obtained the necessary sufficient condition for quaternionic curves to be spherical. And, Yoon [12] gave some characterizations for a quaternionic general helix.

In the present article, we give characterizations of some special quaternionic curves such as spherical, rectifying and general helix, inspired by [5] and [6]. In accordance with this purpose, we first derive a general differential equation that includes the distance function of quaternionic curves and its derivatives. Then, with the help of this differential equation, we can easily see the necessary and sufficient conditions for a quaternionic curve to be spherical or rectifying. We also give an example characterizing the spherical curve. Finally, we reconsider the characterization of a quaternionic general helix by considering [5].

2. Preliminaries

In this section, we recall the basic concepts about quaternions and quaternionic curves.

A quaternion q is of an expression of the form

$$q = q_0 + q_1 i + q_2 j + q_3 k,$$

such that q_A, $A \in \{1,2,3,4\}$ are real numbers and the basis $\{1, i, j, k\}$ has the following properties:

$$i \times i = j \times j = k \times k = -1,$$

$$i \times j = -j \times i = k,$$
$$j \times k = -k \times j = i,$$

and

$$k \times i = -i \times k = j,$$

where $\times$ is the quaternion product in the $4-$dimensional Euclidean space $\mathbb{R}^4$. Then a quaternion q can be given by

$$q = s_q + v_q,$$

where $s_q = q_0$ and $v_q = q_1 i + q_2 j + q_3 k$ are the scalar part and vector part of q, respectively. The set of all quaternions is denoted by $\mathbb{H}$.

For p and q are two quaternions in $\mathbb{H}$, the quaternion product of p and q is defined by

$$p \times q = s_p s_q - < v_p, v_q > + s_p v_q + s_q v_p + v_p \wedge v_q,$$

where $<,>$ and $\wedge$ denote the inner product and vector product of $3-$dimensional Euclidean space $\mathbb{R}^3$, respectively.

The conjugate of $q = s_q + v_q \in \mathbb{H}$ is defined by $\overline{q} = s_q - v_q \in \mathbb{H}$. Then the quaternion inner product can be defined as follows:

$$\begin{array}{rccl} h: & \mathbb{H} \times \mathbb{H} & \to & \mathbb{R} \\ & (p,q) & \to & h(p,q) = \frac{1}{2}(p \times \overline{q} + q \times \overline{p}). \end{array}$$

The norm of a quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ is defined by

$$\|q\| = \sqrt{h(q,q)} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

If $\|q\| = 1$, then q is called unit quaternion.

A quaternion q is called a spatial quaternion whenever $q + \overline{q} = 0$ and the set of spatial quaternions denoted by $\mathbb{H}_S$ is identified with the $3-$dimensional Euclidean space $\mathbb{R}^3$. On the other hand, if $q - \overline{q} = 0$, then q is called a temporal quaternion. Therefore, any quaternion q can be written as the form $q = \frac{1}{2}(q + \overline{q}) + \frac{1}{2}(q - \overline{q})$. While $\frac{1}{2}(q + \overline{q})$ is the spatial part of q, $\frac{1}{2}(q - \overline{q})$ is the temporal part of q.

Let $I = [0, 1]$ be an interval in the real line $\mathbb{R}$. Then

$$\begin{array}{rccl} \gamma: & I \subset \mathbb{R} & \to & \mathbb{H}_S \\ & s & \to & \gamma(s) = \gamma_1(s) i + \gamma_2(s) j + \gamma_3(s) k \end{array}$$

is called a spatial quaternionic curve with the arclength parameter $s \in I$. The tangent vector $t(s) = \gamma^{'}(s) = \frac{d\gamma}{ds}$ has unit length $||t(s)|| = 1$ for all s. It follows $t^{'}(s) \times \overline{t}(s) + t(s) \times \overline{t}^{'}(s) = 0$ which implies $t^{'}(s)$ is orthogonal to $t(s)$ and $t^{'}(s) \times \overline{t}(s)$ is a spatial quaternion. Since $t^{'}(s)$ and $t^{'}(s) \times \overline{t}(s)$ are spatial quaternions, the unit spatial quaternions $n_1(s) = \frac{\gamma^{''}(s))}{\|\gamma^{''}(s))\|}$ and $n_2(s) = t(s) \times n_1(s)$ can be defined. Then $\{t(s), n_1(s), n_2(s)\}$ are the Frenet frame of $\gamma(s)$ and it is note that

$$t(s) \times t(s) = n_1(s) \times n_1(s) = n_2(s) \times n_2(s) = -1.$$

The Frenet formulas are given by

$$\begin{pmatrix} t^{'}(s) \\ n_1^{'}(s) \\ n_2^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n_1(s) \\ n_2(s) \end{pmatrix}$$

where $\kappa(s) = \left\| t^{'}(s) \right\|$ and $\tau(s)$ are the principal curvature and the torsion of γ, respectively.

As we mentioned above, the set of quaternions $\mathbb{H}$ is identical to 4−dimensional Euclidean space $\mathbb{R}^4$. Then

$$\begin{array}{llll} \beta: & I \subset \mathbb{R} & \to & \mathbb{H} \\ & s & \to & \beta(s) = \beta_0(s) + \beta_1(s)i + \beta_2(s)j + \beta_3(s)k, \end{array}$$

is called a quaternionic curve with the arclength parameter $s \in I$. The tangent vector $T(s) = \beta^{'}(s)$ has unit magnitude because s is the arclength parameter. So, $\{T, N_1, N_2, N_3\}$ forms the Frenet frame with the unit normal vectors $N_1(s)$, $N_2(s)$ and $N_3(s)$ perpendicular to $T(s)$. The Frenet formulas of β are given by

$$\begin{pmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \\ N_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & K(s) & 0 & 0 \\ -K(s) & 0 & \kappa(s) & 0 \\ 0 & -\kappa(s) & 0 & \tau(s) - K(s) \\ 0 & 0 & -(\tau(s) - K(s)) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \\ N_3(s) \end{pmatrix}, \tag{2.1}$$

where $K(s)$, $\kappa(s)$ and $\tau(s) - K(s)$ are the principal curvature, the torsion and the bitorsion of β, respectively [2].

Note that we only examine quaternionic curves in the present article. However, we studied spatial quaternionic curves in [13].

3. A Differential Equation for Quaternionic Curves

In this section, we derive a general differential equation including the distance function and its derivatives for quaternionic curve with the non-zero curvatures. Then, with the help of this differential equation, we give characterizations for the quaternionic curve lying on the 3−sphere (spherical curve) and the rectifying quaternionic curve.

Proposition 3.1. *$\beta : I \to \mathbb{H}$ is a unit speed quaternionic curve parameterized by the arc length parameter $s \in I$ with non-zero curvatures and $d(s) = \|\beta(s)\|$ is the distance function of β. Then the function $f(s) = d(s)d^{'}(s)$ satisfies the differential equation*

$$\begin{aligned} &\tfrac{\rho\sigma\omega^2}{\omega-\sigma} f^{(iv)}(s) + \left(\left(\tfrac{\rho\sigma\omega^2}{\omega-\sigma}\right)' + \left(2\rho\omega^{'} + \rho^{'}\omega\right) \tfrac{\omega\sigma}{\omega-\sigma} \right) f'''(s) \\ &+ \left((3(\rho\omega^{'})' + \left(\rho^{'}\omega\right)^{'} + \tfrac{\rho}{\omega} + \tfrac{\omega}{\rho}) \tfrac{\omega\sigma}{\omega-\sigma} + \left(2\rho\omega^{'} + \rho^{'}\omega\right) \left(\tfrac{\omega\sigma}{\omega-\sigma}\right)' + \tfrac{\rho(\omega-\sigma)}{\sigma} \right) f''(s) \\ &+ \left(((\rho\omega^{'})'' + 2\left(\tfrac{\rho}{\omega}\right)' + \left(\tfrac{\omega}{\rho}\right)') \tfrac{\omega\sigma}{\omega-\sigma} + \left((\rho\omega^{'})' + \tfrac{\rho}{\omega} + \tfrac{\omega}{\rho}\right) \left(\tfrac{\omega\sigma}{\omega-\sigma}\right)' + \rho\omega^{'} \tfrac{\omega-\sigma}{\omega\sigma} \right) f'(s) \\ &+ \left(\left(\left(\tfrac{\rho}{\omega}\right)' \tfrac{\omega\sigma}{\omega-\sigma} \right)^{'} + \tfrac{\rho(\omega-\sigma)}{\omega^2\sigma} \right) f(s) - \left(\left((\rho\omega^{'})' + \tfrac{\omega}{\rho} \right) \tfrac{\omega\sigma}{\omega-\sigma} \right)^{'} - \rho\omega^{'} \tfrac{(\omega-\sigma)}{\omega\sigma} = 0 \end{aligned} \tag{3.1}$$

where $\rho = \frac{1}{\kappa(s)}$, $\sigma = \frac{1}{\tau(s)}$ and $\omega = \frac{1}{K(s)}$.

Proof. Let $\beta : I \to \mathbb{H}$ be the unit speed quaternionic curve parameterized by the arc length parameter $s \in I$ with non-zero curvatures. If we take the derivative of $d^2(s) = \|\beta(s)\|^2 = h(\beta(s), \beta(s))$, we get

$$f(s) = h(\beta(s), T(s)), \tag{3.2}$$

where $f(s)=d(s)d^{'}(s)$. By differentiating (3.2) and using (2.1), we obtain

$$\omega f^{'}(s)-\omega=h\left(\beta\left(s\right),N_{1}\left(s\right)\right). \tag{3.3}$$

Taking derivative of (3.3) and using (2.1) and (3.2), we find

$$\rho\omega f^{''}(s)+\rho\omega^{'}f^{'}(s)+\frac{\rho}{\omega}f(s)-\rho\omega^{'}=h\left(\beta\left(s\right),N_{2}\left(s\right)\right). \tag{3.4}$$

By differentiating (3.4) and using (2.1) and (3.3), we arrive

$$\begin{aligned}
&\tfrac{\rho\sigma\omega^{2}}{\omega-\sigma}f^{'''}(s)+\left(2\rho\omega^{'}+\rho^{'}\omega\right)\tfrac{\omega\sigma}{\omega-\sigma}f^{''}(s)+\left((\rho\omega^{'})^{'}+\tfrac{\rho}{\omega}+\tfrac{\omega}{\rho}\right)\tfrac{\omega\sigma}{\omega-\sigma}f^{'}(s)\\
&+(\tfrac{\rho}{\omega})^{'}\tfrac{\omega\sigma}{\omega-\sigma}f(s)-\left((\rho\omega^{'})^{'}+\tfrac{\omega}{\rho}\right)\tfrac{\omega\sigma}{\omega-\sigma}\\
&=h\left(\beta\left(s\right),N_{3}\left(s\right)\right).
\end{aligned} \tag{3.5}$$

Lastly, taking derivative of (3.5) and using (2.1) and (3.4), we have

$$\begin{aligned}
&\tfrac{\rho\sigma\omega^{2}}{\omega-\sigma}f^{(iv)}(s)+\left((\tfrac{\rho\sigma\omega^{2}}{\omega-\sigma})^{'}+\left(2\rho\omega^{'}+\rho^{'}\omega\right)\tfrac{\omega\sigma}{\omega-\sigma}\right)f^{'''}(s)\\
&+\left(\left(2\rho\omega^{'}+\rho^{'}\omega\right)^{'}\tfrac{\omega\sigma}{\omega-\sigma}+\left(2\rho\omega^{'}+\rho^{'}\omega\right)(\tfrac{\omega\sigma}{\omega-\sigma})^{'}+\left((\rho\omega^{'})^{'}+\tfrac{\rho}{\omega}+\tfrac{\omega}{\rho}\right)\tfrac{\omega\sigma}{\omega-\sigma}\right)f^{''}(s)\\
&+\left(\left((\rho\omega^{'})^{'}+\tfrac{\rho}{\omega}+\tfrac{\omega}{\rho}\right)^{'}\tfrac{\omega\sigma}{\omega-\sigma}+\left((\rho\omega^{'})^{'}+\tfrac{\rho}{\omega}+\tfrac{\omega}{\rho}\right)(\tfrac{\omega\sigma}{\omega-\sigma})^{'}+(\tfrac{\rho}{\omega})^{'}\tfrac{\omega\sigma}{\omega-\sigma}\right)f^{'}(s)\\
&+\left((\tfrac{\rho}{\omega})^{'}\tfrac{\omega\sigma}{\omega-\sigma}\right)^{'}f(s)-\left((\rho\omega^{'})^{'}+\tfrac{\omega}{\rho}\right)^{'}\tfrac{\omega\sigma}{\omega-\sigma}-\left((\rho\omega^{'})^{'}+\tfrac{\omega}{\rho}\right)(\tfrac{\omega\sigma}{\omega-\sigma})^{'}\\
&=-(\tfrac{\omega-\sigma}{\sigma\omega})\left(\rho\omega f^{''}(s)+\rho\omega^{'}f^{'}(s)+\tfrac{\rho}{\omega}f(s)-\rho\omega^{'}\right)
\end{aligned}$$

and thus we obtain (3.1). $\square$

Now, we can give below the known characterizations for spherical and quaternionic rectifying curves as consequences of Proposition 3.1.

Corollary 3.1. [9] *A unit speed quaternionic curve $\beta:I\to\mathbb{H}$ is a spherical if and only if*

$$\left(\frac{\omega\sigma}{\omega-\sigma}\left((\rho\omega^{'})^{'}+\frac{\omega}{\rho}\right)\right)^{2}+(\rho\omega^{'})^{2}+\omega^{2}=r^{2}, \tag{3.6}$$

where $r>0$ is a constant.

Proof. Let $\beta\left(s\right)$ be a spherical curve that is a quaternionic curve lying on $3-$sphere with radius r. Then the distance function of $\beta\left(s\right)$ satisfies $d\left(s\right)=r$ which implies $f=dd^{'}=0$. The differential equation (3.1) reduces to

$$\left(\left((\rho\omega^{'})^{'}+\frac{\omega}{\rho}\right)\frac{\omega\sigma}{\omega-\sigma}\right)^{'}+\rho\omega^{'}\frac{(\omega-\sigma)}{\omega\sigma}=0. \tag{3.7}$$

Multiplying both sides of (3.7) by

$$2\frac{\omega\sigma}{\omega-\sigma}\left((\rho\omega^{'})^{'}+\frac{\omega}{\rho}\right)$$

gives

$$(3.8)\; 2\frac{\omega\sigma}{\omega-\sigma}\left((\rho\omega^{'})^{'}+\frac{\omega}{\rho}\right)\left(\left((\rho\omega^{'})^{'}+\frac{\omega}{\rho}\right)\frac{\omega\sigma}{\omega-\sigma}\right)^{'}+2\rho\omega^{'}(\rho\omega^{'})^{'}+2\omega\omega^{'}=0.$$

By integrating (3.8) we get (3.6).

Conversely, we assume that $\beta(s)$ is a unit speed quaternionic curve with non-zero curvatures satisfies (3.6). From (2.1) and (3.7), we have

$$\left(\beta(s)+\omega N_1+(\omega^{'}\rho)N_2+\frac{\omega\sigma}{\omega-\sigma}\left((\rho\omega^{'})^{'}+\frac{\omega}{\rho}\right)N_3\right)^{'}=0.$$

Thus, we obtain

$$h(\beta(s)-M,\beta(s)-M)=r^2$$

which implies $\beta(s)$ lies on $3-$sphere with center M and radius $r>0$ that is $\beta(s)$ is a spherical curve. $\square$

Corollary 3.2. [7] *A unit speed quaternionic curve $\beta : I \to \mathbb{H}$ is a rectifying if and only if*

$$(3.9)\qquad \left(\frac{\rho\sigma}{\omega-\sigma}+\frac{\omega\sigma}{\omega-\sigma}(s+c)\left(\frac{\rho}{\omega}\right)^{'}\right)^{'}+\frac{\rho(\omega-\sigma)}{\sigma\omega^2}(s+c)=0,$$

where c is a constant.

Proof. Assume that $\beta(s)$ is a quaternionic rectifying curve. Then the distance function $d(s)=\|\beta(s)\|$ is given by

$$d(s)=\sqrt{s^2+c_1s+c_2},$$

where c_1 and c_2 are constants. Hence, from $f(s)=d(s)\,d^{'}(s)$ we get

$$f(s)=\sqrt{s^2+c_1s+c_2}\frac{2s+c_1}{2\sqrt{s^2+c_1s+c_2}}=s+c,$$

where $c=c_1/2$. Substituting $f(s)=s+c$, $f^{'}(s)=1$ and $f^{''}(s)=f^{'''}(s)=f^{(iv)}(s)=0$ into (3.1), we obtain

$$\begin{aligned}&((\rho\omega^{'})^{''}+2(\tfrac{\rho}{\omega})^{'}+(\tfrac{\omega}{\rho})^{'})\tfrac{\omega\sigma}{\omega-\sigma}+\left((\rho\omega^{'})^{'}+\tfrac{\rho}{\omega}+\tfrac{\omega}{\rho}\right)(\tfrac{\omega\sigma}{\omega-\sigma})^{'}+\rho\omega^{'}\tfrac{\omega-\sigma}{\omega\sigma}\\ &+(\left((\tfrac{\rho}{\omega})^{'}\tfrac{\omega\sigma}{\omega-\sigma}\right)^{'}+\tfrac{\rho(\omega-\sigma)}{\omega^2\sigma})(s+c)-\left(\left((\rho\omega^{'})^{'}+\tfrac{\omega}{\rho}\right)\tfrac{\omega\sigma}{\omega-\sigma}\right)^{'}-\rho\omega^{'}\tfrac{(\omega-\sigma)}{\omega\sigma}=0\end{aligned}$$

or

$$2(\tfrac{\rho}{\omega})'\tfrac{\omega\sigma}{\omega-\sigma}+\tfrac{\rho}{\omega}(\tfrac{\omega\sigma}{\omega-\sigma})'+(\left((\tfrac{\rho}{\omega})'\tfrac{\omega\sigma}{\omega-\sigma}\right)'+\tfrac{\rho(\omega-\sigma)}{\omega^2\sigma})\,(s+c)=0. \tag{3.10}$$

If (3.10) is arranged, then we get (3.9).

Conversely, we suppose that Equation (3.9) satisfies and we consider

$$X(s)=\beta(s)-(s+c)\,T(s)-\frac{\rho}{\omega}(s+c)\,N_2(s)-\frac{\sigma\omega}{\omega-\sigma}\left((s+c)\left(\frac{\rho}{\omega}\right)'+\frac{\rho}{\omega}\right)N_3(s). \tag{3.11}$$

Differentiating (3.11) and using (2.1) and (3.9), we find

$$\begin{aligned}
X'(s) &= -(s+c)\tfrac{1}{\omega}N_1(s)-\left(\tfrac{\rho}{\omega}\right)'(s+c)\,N_2(s)\\
&\quad -\tfrac{\rho}{\omega}N_2(s)-\tfrac{\rho}{\omega}(s+c)\left(-\tfrac{1}{\rho}N_1(s)+\tfrac{\omega-\sigma}{\sigma\omega}N_3(s)\right)\\
&\quad -(\tfrac{\sigma\omega}{\omega-\sigma})'\left((s+c)\left(\tfrac{\rho}{\omega}\right)'+\tfrac{\rho}{\omega}\right)N_3(s)-\tfrac{\sigma\omega}{\omega-\sigma}\left((s+c)\left(\tfrac{\rho}{\omega}\right)'+\tfrac{\rho}{\omega}\right)'N_3(s)\\
&\quad +\left((s+c)\left(\tfrac{\rho}{\omega}\right)'+\tfrac{\rho}{\omega}\right)N_2(s)\\
&=0
\end{aligned}$$

which implies that X is a constant vector. Therefore, $\beta(s)$ is a quaternionic rectifying curve. □

Corollary 3.3. *Let $\beta : I \to \mathbb{H}$ be a unit speed quaternionic curve. Then for a constant c*

$$h^2(\beta(s),N_1(s))+h^2(\beta(s),N_2(s))+h^2(\beta(s),N_3(s))=c^2 \tag{3.12}$$

holds if and only if either $\beta(s)$ is a spherical curve or a rectifying curve.

Proof. Assume that $\beta(s)$ is a unit speed quaternionic curve satisfying the equation (3.12). By differentiating (3.12) and using (2.1), we get

$$\frac{1}{\omega}h(\beta(s),T(s))\,h(\beta(s),N_1(s))=0. \tag{3.13}$$

On the other hand, since $\beta(s)$ can be written as

$$\begin{aligned}
\beta(s)= \; & h(\beta(s),T(s))\,T(s)+h(\beta(s),N_1(s))\,N_1(s)\\
&+h(\beta(s),N_2(s))\,N_2(s)+h(\beta(s),N_3(s))\,N_3(s),
\end{aligned}$$

the distance functional becomes

$$d(s)=\sqrt{h^2(\beta(s),T(s))+h^2(\beta(s),N_1(s))+h^2(\beta(s),N_2(s))+h^2(\beta(s),N_3(s))}. \tag{3.14}$$

Substituting (3.12) into (3.14), we obtain

$$d^2(s)=h^2(\beta(s),T(s))+c^2. \tag{3.15}$$

After differentiating (3.15) and using (2.1), (3.13) and (3.2), we arrive

$$\begin{aligned} d(s)\,d'(s) &= h(\beta(s),T(s))\left(h(T(s),T(s))+\tfrac{1}{\omega}h(\beta(s),N_1(s))\right)\\ &= h(\beta(s),T(s))+\tfrac{1}{\omega}h(\beta(s),T(s))\,h(\beta(s),N_1(s))\\ &= f(s). \end{aligned} \tag{3.16}$$

Also, from (3.2) and (3.15) we have

$$d^2(s) = f^2(s) + c^2. \tag{3.17}$$

If we take the derivative of (3.17) and use (3.16), we find

$$d(s)\,d'(s) = f(s)\,f'(s) = f(s).$$

Hence, either $f(s) = 0$ or $f'(s) = 1$ that is $f(s) = s + c$, where c is a constant. Therefore, either β is a spherical curve or a rectifying curve.

Conversely, it is clear that if either unit quaternionic curve $\beta(s)$ is a spherical or a rectifying curve, then it is clear that (3.12) holds. □

We now give an example for the results. For example, let's take the following example for Corollary 3.1 and Corollary 3.3.

Example 3.1. Let $\beta(s) = \cos(\sqrt{\frac{2}{3}}s)+\sin(\sqrt{\frac{2}{3}}s)i+\cos(\frac{1}{\sqrt{3}}s)j+\sin(\frac{1}{\sqrt{3}}s)k$ be a unit speed quaternionic curve in $\mathbb{H}$ (see, [5]). Then we have $\rho = \frac{3\sqrt{10}}{2}$, $\sigma = \frac{3\sqrt{5}}{3\sqrt{2}+5}$ and $w = \frac{3\sqrt{5}}{5}$. So, we see that the differential equation (3.6) satisfies and β lies on $3-$sphere with radius $r = \sqrt{2}$, that is, it is a spherical curve. Also, we obtain

$$T = -\sqrt{\frac{2}{3}}\sin(\sqrt{\frac{2}{3}}s) + \sqrt{\frac{2}{3}}\cos(\sqrt{\frac{2}{3}}s)i - \frac{1}{\sqrt{3}}\sin(\frac{1}{\sqrt{3}}s)j + \frac{1}{\sqrt{3}}\cos(\frac{1}{\sqrt{3}}s)k,$$

$$N_1 = -\frac{2}{\sqrt{5}}\cos(\sqrt{\frac{2}{3}}s) - \frac{2}{\sqrt{5}}\sin(\sqrt{\frac{2}{3}}s)i - \frac{1}{\sqrt{5}}\cos(\frac{1}{\sqrt{3}}s)j - \frac{1}{\sqrt{5}}\sin(\frac{1}{\sqrt{3}}s)k,$$

$$N_2 = \frac{1}{\sqrt{3}}\sin(\sqrt{\frac{2}{3}}s) - \frac{1}{\sqrt{3}}\cos(\sqrt{\frac{2}{3}}s)i - \sqrt{\frac{2}{3}}\sin(\frac{1}{\sqrt{3}}s)j + \sqrt{\frac{2}{3}}\cos(\frac{1}{\sqrt{3}}s)k,$$

and

$$N_3 = \frac{1}{\sqrt{5}}\cos(\sqrt{\frac{2}{3}}s) + \frac{1}{\sqrt{5}}\sin(\sqrt{\frac{2}{3}}s)i - \frac{2}{\sqrt{5}}\cos(\frac{1}{\sqrt{3}}s)j - \frac{2}{\sqrt{5}}\sin(\frac{1}{\sqrt{3}}s)k.$$

For $c = \sqrt{2}$, we see that Equation (3.12) is confirmed.

4. Characterization of Quaternionic General Helices

Helices are the simplest geometric shapes that can be observed in the molecular structures of nature. Also, helices arise in nanosprings, carbon nanotubes, DNA double, bacterial flagella, aerial hyphae in actynomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, and helical stairs [4]. Due to the presence of helices in these structures, it becomes very attractive to examine helices by scientists in different fields. From the view of differential geometry, a general helix is defined by the property that its tangent vector makes a constant angle with a fixed straight line (the axis of the general helix) and a general helix is characterized by its curvature and torsion by means of a moving frame along the curve. Lancret's theorem states that a space curve is a general helix if and only if its torsion τ and its curvature κ satisfy $\tau = c\kappa$ for some $c \in \mathbb{R}$ [1]. The general helix concept for quaternionic curves was studied by Yoon [12]. Yoon gave characterizations for quaternionic general helices with three interrelated theorems. Inspired by [5], we rearrange the characterization of a quaternionic general helix with the following theorem.

Theorem 4.1. *Let $\beta : I \to \mathbb{H}$ be a unit speed quaternionic curve with non-zero curvatures. Then $\beta(s)$ is a general helix if and only if*

$$\frac{\rho}{\omega} = \delta C \sin\Big(\int_0^s \frac{\omega - \sigma}{\sigma\omega} ds\Big), \tag{4.1}$$

for C is a non-zero constant and δ is 1 or -1.

Proof. Let $\beta\,(s)$ be a unit speed quaternionic curve with non-zero curvatures and the axis of $\beta(s)$ be the unit vector U. So, for a constant a, we have

$$h\,(T, U) = \cos\theta = a \tag{4.2}$$

along $\beta(s)$. By differentiating (4.2) and using (2.1), we get

$$h\,(N_1, U) = 0. \tag{4.3}$$

Again, by differentiating (4.3) and using (2.1), we obtain

$$h(N_2, U) = \frac{\rho}{\omega} a. \tag{4.4}$$

Finally, if we take the derivative of (4.4) and use (2.1), then we have

$$h(N_3, U) = \frac{\sigma\omega}{\omega - \sigma}\left(\frac{\rho}{\omega}\right)' a.$$

Then, we can write the unit vector U as follows

$$U = aT\,(s) + \frac{\rho}{\omega} aN_2\,(s) + \frac{\sigma\omega}{\omega - \sigma}\left(\frac{\rho}{\omega}\right)' aN_3\,(s)\,. \tag{4.5}$$

The differentiation of (4.5) and in view of the equation (2.1) gives

$$\left[(\frac{\rho}{\omega})\frac{\omega - \sigma}{\omega\sigma} + \left(\frac{\sigma\omega}{\omega - \sigma}\left(\frac{\rho}{\omega}\right)'\right)' \right] aN_3 = 0$$

which implies either $a = 0$ or

$$(\frac{\rho}{\omega})\frac{\omega-\sigma}{\omega\sigma} + \left(\frac{\sigma\omega}{\omega-\sigma}\left(\frac{\rho}{\omega}\right)'\right)' = 0. \tag{4.6}$$

$a = 0$ means $U = 0$ which is a contradiction. Hence, only the equation (4.6) holds. Multiplying both sides of (4.6) by

$$2\tfrac{\sigma\omega}{\omega-\sigma}\left(\tfrac{\rho}{\omega}\right)'$$

gives

$$(\frac{\rho}{\omega})^2 + \left(\frac{\sigma\omega}{\omega-\sigma}\left(\frac{\rho}{\omega}\right)'\right)^2 = C^2, \tag{4.7}$$

where C is a constant. If we arrange (4.7), then we get

$$\left(\frac{\rho}{\omega}\right)' = \mp\frac{\omega-\sigma}{\sigma\omega}\sqrt{C^2-(\frac{\rho}{\omega})^2}. \tag{4.8}$$

Integrating (4.8), we obtain (4.1). □

Conversely, we suppose that $\beta(s)$ is a quaternionic curve satisfying (4.1). We take

$$U = \frac{1}{\sqrt{1+C^2}}T(s) + \frac{\delta C}{\sqrt{1+C^2}}\sin(\int_0^s \frac{\omega-\sigma}{\sigma\omega}ds)N_2(s) + \frac{\delta C}{\sqrt{1+C^2}}\cos(\int_0^s \frac{\omega-\sigma}{\sigma\omega}ds)N_3(s). \tag{4.9}$$

By differentiating (4.9) and using (2.1) and (4.1), we get

$$U' = \tfrac{1}{\rho\sqrt{1+C^2}}\left(\tfrac{\rho}{\omega} - \delta C\sin(\int_0^s \tfrac{\omega-\sigma}{\sigma\omega}ds)\right)N_1(s) = 0.$$

This means that U is a constant vector that satisfies

$$h(T,U) = \tfrac{1}{\sqrt{1+C^2}} = constant.$$

Therefore, $\beta(s)$ is a quaternionic general helix.

REFERENCES

1. M. Barros: *General helices and a theorem of Lancret.* Proc. Amer. Math. Soc. **125(5)** (1997), 1503–1509.
2. K. Bharathi and M. Nagaraj: *Quaternion valued function of a real variable Serret-Frenet formula.* Indian J. Pure Appl. Math. **18(6)** (1987), 507–511.
3. B. Y. Chen: *When does the position vector of a space curve always lie in its rectifying plane?.* Amer. Math. Monthly **110(2)** (2003), 147–152.

4. N. Chouaieb, A. Goriely, J. H. Maddocks: *Helices.* PNAS **103(25)** (2006), 9398–9403.
5. S. Deshmukh, I. Al-Dayel, K. İlarslan: *Frenet curves in Euclidean 4-space.* Int. Electron. J. Geom. **10(2)** (2017), 56–66.
6. S. Deshmukh, B. Y. Chen, N. B. Turki: *A differential equation for Frenet curves in Euclidean 3-space and its applications*: Rom. J. Math. Comput. Sci. **8(1)** (2018), 1–6.
7. M. A. Gungor and M. Tosun: *Some characterizations of quaternionic rectifying curves.* Differ. Geom. Dyn. Syst. **13** (2011), 89–100.
8. K. Kunze and H. Schaeben: *The bingham distribution of quaternions and its spherical radon transform in texture analysis.* Math. Geol. **36(8)** (2004), 917–943.
9. D. Sağlam: *On the osculating spheres of a real quaternionic curve in the Euclidean space E^4.* Int. J. Math. Comb. **3** (2012), 46–53.
10. D. J. Struik: *Lectures on Classical Differential Geometry.* Addison-Wesley Publishing Company Inc, Massachusetts, USA, 1950.
11. J. P. Ward: *Quaternions and Cayley Numbers.* Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1997.
12. D. W. Yoon: *On the quaternionic general helices in Euclidean 4-space.* Honam Math. J. **34(3)** (2012), 381–390.
13. A. Yücesan and E. Tulum: *On the characterizations of some special spatial quaternionic curves.* In: Proceedings of a Conference on Istanbul Int. Mod. Sci. Res. (V. Akman, R. Arora, eds.), Istanbul, 2021, pp. 893–899.