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REMARKS ON SUBMANIFOLDS AS ALMOST η-RICCI-BOURGUIGNON SOLITONS

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Abstract. We give some characterizations for submanifolds admitting almost η-Ricci-Bourguignon solitons whose potential vector field is the tangential component of a concurrent vector field on the ambient manifold. We describe the particular cases of umbilical submanifolds and of hypersurfaces in a space with constant curvature.
Keywords: Ricci-Bourguignon soliton, concurrent vector field, submanifold

1. Introduction

Starting with Hamilton's paper, who introduced the Ricci flow around 1990s [7], the theory of Ricci solitons has been developed continuously. The Ricci flow is an evolution equation for a Riemannian metric, whose stationary solutions are the Ricci solitons. Their study is totally justified by the fact that their properties can provide useful information about the flow. Different notions of geometric solitons have recently been considered in the Riemannian setting, on manifolds carrying various structures or having certain types of potential vector fields. Throughout, almost η-Ricci solitons and almost η-Yamabe solitons with torse-forming potential vector field have been treated by the authors in [1]. Another generalization of Ricci soliton, namely, η-Ricci-Bourguignon soliton, will be further considered. Precisely, if (M, g) is a Riemannian manifold of dimension n, V is a vector field and η is a

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1-form on M, then the data $(g, V, \alpha, \beta, \gamma)$ define an *η-Ricci-Yamabe soliton* (which we shall call, *η-Ricci-Bourguignon soliton* and will denote it by η-RBS) if [8]

$$\frac{1}{2}\pounds_V g + Ric = (\alpha + \beta scal)g + \gamma\eta \otimes \eta, \tag{1.1}$$

where $\pounds_V g$ stands for the Lie derivative of g in the direction of V, Ric is the Ricci curvature, $scal$ is the scalar curvature and α, β, γ are real numbers. If α, β, γ are smooth functions on M, then we talk about an *almost η-RBS*. In particular, if $\gamma = 0$, (1.1) defines a β-Einstein soliton [2].

In this short paper, we point out some properties of almost η-RBS solitons whose potential vector field is concurrent, with a special view towards submanifolds. We prove that, under certain assumptions, if the manifold is Ricci symmetric, then its scalar curvature is constant. For the 3-dimensional case, we obtain some conditions on the defining functions α, β, γ. Also, we discuss the case when we have such type of soliton on a submanifold isometrically immersed into a Riemannian manifold, and whose potential vector field is the tangential component of a concurrent vector field on the ambient manifold. For the particular cases of an umbilical submanifold, of a pseudosymmetric hypersurface or isometrically immersed into a space of constant curvature, we formulate some conclusions. Also, we provide a necessary and sufficient condition for an orientable hypersurface immersed into the unit sphere to admit such type of soliton whose potential vector field is the tangential component of a concurrent vector field on the Euclidean space.

2. Solitons with concurrent vector field

Assume further that V is a *concurrent* vector field [10], that is, $\nabla V = I$, where ∇ is the Levi-Civita connection of g and I is the identity map. Then $\pounds_V g = 2g$, hence V is a conformal vector field, and from (1.1), we derive

$$Ric = (\alpha + \beta scal - 1)g + \gamma\eta \otimes \eta, \tag{2.1}$$

hence

$$Q = (\alpha + \beta scal - 1)I + \gamma\eta \otimes \tau, \tag{2.2}$$

where $g(QX_1, X_2) := Ric(X_1, X_2)$ and $i_\tau g = \eta$, and also

$$scal = \frac{1}{1 - n\beta}\left[n(\alpha - 1) + \gamma|\tau|^2\right], \tag{2.3}$$

provided $\beta \neq \frac{1}{n}$.

Therefore, in particular cases, the soliton equation is very similar to the equation that defines a quasi-Einstein manifold. Recall that a non-flat Riemannian manifold (M, g) $(n \geq 3)$ is said to be a *quasi-Einstein manifold* [3], if Ric is not identically zero and satisfies

$$Ric = a_1 g + a_2 A \otimes A,$$

for a_1 and a_2 non-zero smooth functions and A a non-zero 1-form. The functions a_1 and a_2 are called *associated functions*.

Now, using (2.1), we obtain

Proposition 2.1. *If $(g, V, \alpha, \beta, \gamma)$ defines an almost η-RBS on M with concurrent vector field V, then M is a quasi-Einstein manifold and the associated functions are $(\alpha + \beta scal - 1)$ and γ.*

Now from $\nabla V = I$, we also obtain

$$R(X_1, X_2)V = 0 \tag{2.4}$$

for any $X_1, X_2 \in \chi(M)$. Differentiating covariantly (2.1), we infer

$$(\nabla_{X_1} Ric)(X_2, X_3) = [X_1(\alpha) + X_1(\beta)scal + \beta X_1(scal)]g(X_2, X_3) \tag{2.5}$$
$$+\gamma[\eta(X_3)(\nabla_{X_1}\eta)X_2 + \eta(X_2)(\nabla_{X_1}\eta)X_3] + X_1(\gamma)\eta(X_2)\eta(X_3),$$

for any $X_1, X_2, X_3 \in \chi(M)$ and we obtain

Proposition 2.2. *Let $(g, V, \alpha, \beta, \gamma)$ define an η-RBS on M $(\alpha, \beta, \gamma \in \mathbb{R},\ \beta \neq 0)$ with concurrent potential vector field V and $\nabla\eta = 0$. Then*

(a) $R(X_1, X_2)V = 0$, for any $X_1, X_2 \in \chi(M)$, in particular, $V \in kerQ$;

(b) M is Ricci symmetric (that is, $\nabla Ric = 0$) if and only if M has constant scalar curvature;

(c) Ric is a Codazzi tensor field (that is, $(\nabla_{X_1} Ric)(X_2, X_3) = (\nabla_{X_2} Ric)(X_1, X_3)$, for any $X_1, X_2, X_3 \in \chi(M)$) if and only if $d(scal) \otimes I = I \otimes d(scal)$.

Proof. If V is concurrent and α, β, γ are constant, from (2.4) follows (a). From (2.5), we deduce

$$(\nabla_{X_1} Ric)(X_2, X_3) = \beta X_1(scal)g(X_2, X_3),$$

for any $X_1, X_2, X_3 \in \chi(M)$ and we find (b) and (c). □

Next we shall make some remarks on the 3-dimensional case. It is known that the Riemann curvature tensor of a 3-dimensional Riemannian manifold (M, g) is given by

$$R(X_1, X_2)X_3 = Ric(X_2, X_3)X_1 - Ric(X_1, X_3)X_2 \tag{2.6}$$
$$+g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2 - \frac{scal}{2}[g(X_2, X_3)X_1 - g(X_1, X_3)X_2],$$

for any $X_1, X_2, X_3 \in \chi(M)$.

For $\nabla V = I$, (2.1), (2.2) and (2.6) imply

$$R(X_1, X_2)X_3 = \left[\left(2(\alpha - 1) + \frac{4\beta - 1}{2} scal\right) g(X_2, X_3) + \gamma\eta(X_2)\eta(X_3)\right] X_1 \tag{2.7}$$

$$-\left[\left(2(\alpha-1)+\frac{4\beta-1}{2}scal\right)g(X_1,X_3)+\gamma\eta(X_1)\eta(X_3)\right]X_2$$
$$+\gamma[\eta(X_1)g(X_2,X_3)-\eta(X_2)g(X_1,X_3)]V.$$

Therefore, by means of (2.4) and (2.7), we infer

$$\left[\left(2(\alpha-1)+\frac{4\beta-1}{2}scal\right)+\gamma|V|^2\right]\eta(X_2)X_1$$
$$=\left[\left(2(\alpha-1)+\frac{4\beta-1}{2}scal\right)+\gamma|V|^2\right]\eta(X_1)X_2,$$

for any $X_1, X_2 \in \chi(M)$, which implies

$$\left(2(\alpha-1)+\frac{4\beta-1}{2}scal\right)+\gamma|V|^2=0 \tag{2.8}$$

and we can state:

Proposition 2.3. *Let $(g, V, \alpha, \beta, \gamma)$ define an η-RBS on a 3-dimensional manifold M with concurrent potential vector field V.*

(a) If $\beta = \frac{1}{2}$, then $\alpha = 1$, $\gamma = -\frac{1}{2|V|^2}scal$.

(b) If $\beta \neq \frac{1}{2}$, then the scalar curvature equals to $\frac{2(\alpha-1)}{1-2\beta}$.

Proof. From (2.3) we have

$$\gamma|V|^2=(1-3\beta)scal-3(\alpha-1). \tag{2.9}$$

Combining (2.9) with (2.8), we find

$$(1-2\beta)scal=2(\alpha-1). \tag{2.10}$$

For $\beta = \frac{1}{2}$, (2.10) implies $\alpha = 1$ and we get (a) by means of (2.9). $\square$

Remark 2.1. From the previous proposition, we deduce that if $\beta = \frac{1}{2}$, then $\gamma scal \leq 0$ and if $\beta \neq \frac{1}{2}$, then $(\alpha-1)(2\beta-1)\gamma \geq 0$.

Also, differentiating covariantly (2.10) and (2.9) with respect to X_1, we consequently obtain

$$\beta X_1(scal)+X_1(\beta)scal=-X_1(\alpha) \tag{2.11}$$

and

$$\gamma X_1(|V|^2)+X_1(\gamma)|V|^2=-3[\beta X_1(scal)+X_1(\beta)scal+X_1(\alpha)] \tag{2.12}$$

and we can state:

Proposition 2.4. *Let $(g, V, \alpha, \beta, \gamma)$ define an η-RBS on a 3-dimensional manifold M with concurrent potential vector field V.*

(a) If M is connected and α, β are constant, then the scalar curvature of M is constant provided $\beta \neq \frac{1}{2}$.

(b) If α, β and γ are constants, then the vector field V is collinear with $grad(scal)$ provided $\gamma \neq 0$.

Proof. (a) follows from (2.11) and for (b), notice that, if α, β and γ are constants, then, for any $X_1 \in \chi(M)$, we get

$$\gamma X_1(|V|^2) = -3\beta X_1(scal),$$

from (2.12), which is equivalent to

$$2\gamma g(X_1, V) = -3\beta g(X_1, grad(scal)),$$

hence the conclusion. $\square$

3. Submanifolds as almost η-RBS

Let $\left(\widetilde{M}, \widetilde{g}\right)$ be a Riemannian manifold, let M be an n-dimensional isometrically immersed submanifold and denote by g the induced Riemannian metric on M. Then the two Levi-Civita connections, ∇ and $\widetilde{\nabla}$, on (M, g) and $\left(\widetilde{M}, \widetilde{g}\right)$, respectively, satisfy the Gauss and Weingarten equations:

$$\widetilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + h(X_1, X_2),$$

$$\widetilde{\nabla}_{X_1} N = -B_N(X_1) + \nabla^{\perp}_{X_1} N,$$

where h is the second fundamental form and B_N is the shape operator in the direction of the normal vector field N, that is, $\widetilde{g}(B_N(X_1), X_2) = \widetilde{g}(h(X_1, X_2), N)$ for X_1, $X_2 \in \chi(M)$.

The submanifold M is said to be N-*umbilical* [4] (with respect to a normal vector field N), if its shape operator satisfies $B_N = \varphi I$, for φ a function on M and I is the identity map.

Now we shall describe almost η-RBS on a submanifold M having as potential vector field the tangential component of a concurrent vector field V on $\widetilde{M}$, that is, $\widetilde{\nabla} V = I$.

Let X_1, $X_2 \in \chi(M)$. Then

$$X_1 = \widetilde{\nabla}_{X_1} V = \nabla_{X_1} V^T + h(X_1, V^T) - B_{V^{\perp}}(X_1) + \nabla^{\perp}_{X_1} V^{\perp}$$

and we infer

$$\nabla_{X_1} V^T = X_1 + B_{V^\perp}(X_1).$$

Then we get

$$\begin{aligned}\frac{1}{2}(\pounds_{V^T} g)(X_1, X_2) &= \frac{1}{2}[g(\nabla_{X_1} V^T, X_2) + g(\nabla_{X_2} V^T, X_1)] \\ &= g(X_1, X_2) + \widetilde{g}(h(X_1, X_2), V^\perp)\end{aligned}$$

and we can state:

Theorem 3.1. *If V is a concurrent vector field on $\left(\widetilde{M}, \widetilde{g}\right)$, then $(g, V^T, \alpha, \beta, \gamma)$ defines an almost η-RBS on M if and only if*

$$(3.1)\qquad \begin{aligned} Ric(X_1, X_2) = {} & (\alpha + \beta scal - 1) g(X_1, X_2) + \gamma \eta(X_1)\eta(X_2) \\ & - \widetilde{g}(h(X_1, X_2), V^\perp), \end{aligned}$$

for any $X_1, X_2 \in \chi(M)$, where Ric is the Ricci curvature tensor of M.

Contracting now (3.1), if we denote by $\rho_{\overrightarrow{H}}$ the restriction of $\widetilde{g}(\overrightarrow{H}, V^\perp)$ to M, we get the expression of the scalar curvature of M

$$scal = \frac{n(\alpha - 1 - \rho_{\overrightarrow{H}}) + \gamma |\tau|^2}{1 - n\beta},$$

provided $\beta \neq \frac{1}{n}$.

Now, for a $V^\perp$-umbilical submanifold, by means of (3.1), we state:

Corollary 3.1. *If V is a concurrent vector field on $\left(\widetilde{M}, \widetilde{g}\right)$ and M is $V^\perp$-umbilical, then $(g, V^T, \alpha, \beta, \gamma)$ defines an almost η-RBS on M if and only if M is a quasi-Einstein manifold with associated functions $(\alpha + \beta scal - 1 - \varphi)$ and γ.*

In the particular case of a hypersurface, since

$$\widetilde{g}\left(h(X_1, X_2), V^\perp\right) = g(B(X_1), X_2)\widetilde{g}(N, V^\perp) = H(X_1, X_2)\widetilde{g}(N, V^\perp),$$

where N is the unit normal vector field of M and H is the second fundamental tensor field, if we denote by ρ the restriction of $\widetilde{g}(N, V^\perp)$ to M, then we obtain:

Corollary 3.2. *If V is a concurrent vector field on $\left(\widetilde{M}, \widetilde{g}\right)$ and M is a hypersurface of $\widetilde{M}$, then $(g, V^T, \alpha, \beta, \gamma)$ defines an almost η-RBS on M if and only if the equality*

$$(3.2)\qquad Ric = (\alpha + \beta scal - 1) g - \rho H + \gamma \eta \otimes \eta$$

holds on M.

If the ambient space is of constant curvature, we prove the next result.

Theorem 3.2. *If V is a concurrent vector field on an $(n+1)$-dimensional Riemannian manifold $\left(\widetilde{M}(c), \widetilde{g}\right)$ of constant curvature c, M is a hypersurface of $\widetilde{M}$ and $(g, V^T, \alpha, \beta, \gamma)$ defines an almost η-RBS on M, then the second fundamental tensor field H of M satisfies*

$$H^2 = [\rho + trace(H)]\, H - \gamma\eta \otimes \eta + \left[(n-1)c - \alpha - \beta\left((trace(H))^2 - trace\left(H^2\right) + n(n-1)c\right) - 1)\right] g. \tag{3.3}$$

Proof. From the Gauss equation, we have

$$Ric(X_1, X_2) = trace(H)H(X_1, X_2) - H^2(X_1, X_2) + (n-1)cg(X_1, X_2) \tag{3.4}$$

and

$$scal = (trace(H))^2 - trace\left(H^2\right) + n(n-1)c.$$

Comparing the right hand sides of (3.2) and (3.4), we find

$$\left[\alpha + \beta\left((trace(H))^2 - trace\left(H^2\right) + n(n-1)c\right) - 1\right] g(X_1, X_2)$$
$$-\rho H(X_1, X_2) + \gamma\eta(X_1)\eta(X_2)$$
$$= trace(H)H(X_1, X_2) - H^2(X_1, X_2) + (n-1)cg(X_1, X_2),$$

thus we obtain (3.3). □

It is known that the position vector field of a hypersurface isometrically immersed into an Euclidean space is a concurrent vector field. Therefore, Theorem 3.2 can be given, in this case, like follows.

Proposition 3.1. *If V is the position vector field of an isometrically immersed hypersurface M in an Euclidean space and $(g, V^T, \alpha, \beta, \gamma)$ defines an almost η-RBS on M, then the second fundamental tensor field H of M satisfies*

$$H^2 = [\rho + trace(H)]\, H - \gamma\eta \otimes \eta - \left[\alpha + \beta\left((trace(H))^2 - trace\left(H^2\right)\right) - 1)\right] g.$$

Moreover, it is a gradient soliton with the potential vector field $V^T = grad(f)$, for $f := \frac{1}{2}(|V^T|^2 + \rho^2)$.

Proof. Just remark that, since for any vector field X_1 on M, we have

$$\nabla_{X_1} V^T = X_1 + \rho B_N(X_1) \quad \text{and} \quad grad(\rho) = -B_N(V^T),$$

we infer

$$X_1(f) = \frac{1}{2}\left(X_1(|V^T|^2) + X_1(\rho^2)\right) = g(\nabla_{X_1} V^T, V^T) + \rho X_1(\rho) = g(X_1, V^T).$$

□

Now we will look at pseudosymmetric hypersurfaces of a Riemannian manifold of constant curvature and prove that, under certain assumptions, if the hypersurface admits an η-RBS having as potential vector field the tangential component of a concurrent vector field on the ambient space, then it must be a β-Einstein soliton. Let (M, g) be a Riemannian manifold and let R be its Riemann curvature tensor field. Then the tensor fields $R \cdot R$ and $Q(g, R)$ are defined by

$$(R(X_1, X_2) \cdot R)\,(U_1, U_2, U_3, U_4) := -R(R(X_1, X_2)U_1, U_2, U_3, U_4)$$
$$-... - R(U_1, U_2, U_3, R(X_1, X_2)U_4)$$

and

$$Q(g, R)(U_1, U_2, U_3, U_4; X_1, X_2) := -R((X_1 \wedge X_2)U_1, U_2, U_3, U_4)$$
$$-... - R(U_1, U_2, U_3, (X_1 \wedge X_2)U_4),$$

respectively, where $X_1 \wedge X_2$ is given by

$$(X_1 \wedge X_2)X_3 := g(X_2, X_3)X_1 - g(X_1, X_3)X_2,$$

for $X_1, X_2, X_3, U_1, U_2, U_3, U_4 \in \chi(M)$ [5].

An n-dimensional Riemannian manifold (M, g) is said to be *pseudosymmetric,* if the condition

$$R \cdot R = fQ(g, R)$$

holds on the set $\mathcal{U}$ defined by

$$\mathcal{U} := \left\{p \in M : \left(R - \frac{scal}{n(n-1)}G\right) \neq 0 \text{ at } p\right\},$$

where f is some smooth function on $\mathcal{U}$ and the $(0, 4)$-tensor field G is defined by

$$G(U_1, U_2, U_3, U_4) := g((U_1 \wedge U_2)U_3, U_4),$$

for $U_1, U_2, U_3, U_4 \in \chi(M)$ [5]. In [6], it was proved the following:

Lemma 3.1. *Let $\left(\widetilde{M}(c), \widetilde{g}\right)$ be an $(n+1)$-dimensional Riemannian manifold of constant curvature c $(n \geq 3)$ and let M be a hypersurface of $\widetilde{M}$. If at a point $p \in \mathcal{U} \subset M$, the second fundamental tensor field H satisfies*

$$H^2 = \alpha H + \beta g,$$

with α, β smooth functions, then the relation

$$R \cdot R = \left(\frac{\widetilde{scal}}{n(n-1)} - \beta\right) Q(g, R),$$

holds at p, where $\widetilde{scal}$ is the scalar curvature of $\widetilde{M}$ restricted to M.

Then we can state the following result:

Theorem 3.3. *Let* $\left(\widetilde{M}(c),\widetilde{g}\right)$ *be an* $(n+1)$*-dimensional Riemannian manifold of constant curvature* c $(n\geq 3)$*, let* M *be a pseudosymmetric hypersurface of* $\widetilde{M}$*, whose second fundamental tensor field* H *satisfies*

$$H^2=[\rho+trace(H)]\,H \tag{3.5}$$
$$+\left[(n-1)c-\alpha-\beta\left((trace(H))^2-trace\left(H^2\right)+n(n-1)c\right)-1\right]g$$

and let V *be a concurrent vector field on* $\widetilde{M}$*. If* $(g,V^T,\alpha,\beta,\gamma)$ *defines an almost* η*-RBS on* M*, then it is a* β*-Einstein soliton.*

Proof. Notice that (3.3) combined with (3.5) implies

$$\gamma\eta\otimes\eta=0,$$

hence $\gamma=0$, which means that the soliton is a β-Einstein soliton. $\square$

Further, a similar result as the one obtained in Theorem 3.3 given in [9], can be formulated in our setting.

Let M be an immersed hypersurface of the sphere $\mathbb{S}^{n+1}(1)$. Denote by $\langle\cdot,\cdot\rangle$ the Euclidean metric on the Euclidean space $\mathbb{E}^{n+2}$ and by g the induced metric on M, as well as, on $\mathbb{S}^{n+1}(1)$. Consider a concurrent vector field V on $\mathbb{E}^{n+2}$. If N and B are the unit normal vector field and the shape operator of M in $\mathbb{S}^{n+1}(1)$, and $N_{\mathbb{S}}$ is the unit normal vector field of $\mathbb{S}^{n+1}(1)$ in $\mathbb{E}^{n+2}$, then we define the functions δ,ϱ on M by $\delta=\langle V,N\rangle\mid_M$ and $\varrho=\langle V,N_{\mathbb{S}}\rangle\mid_M$ and the restriction of V to M can be represented as $V\mid_M=U+\delta N+\varrho N_{\mathbb{S}}$, for $U\in\chi(M)$. Then we prove the following result:

Theorem 3.4. *Let* V *be a concurrent vector field on the Euclidean space* $\mathbb{E}^{n+2}$ *and let* M *be an orientable hypersurface of the unit sphere* $\mathbb{S}^{n+1}(1)$*. Denote by* ξ *the component of* V *tangent to* $\mathbb{S}^{n+1}(1)$ *and by* U *the component of* ξ *tangent to* M*. Then* $(g,U,\alpha,\beta,\gamma)$ *defines an almost* η*-RBS on* M *if and only if the equality*

$$H^2=[trace(H)+\delta]\,H \tag{3.6}$$
$$+\left[n-\varrho-\alpha-\beta\left(n(n-1)+(trace(H))^2-trace(H^2)\right)\right]g-\gamma\eta\otimes\eta$$

holds on M*.*

Proof. Let $\nabla,\overline{\nabla}$ and D be the Levi-Civita connections on M, $\mathbb{S}^{n+1}(1)$ and $\mathbb{E}^{n+2}$, respectively. We have

$$V\mid_{\mathbb{S}^{n+1}(1)}=\xi+\varrho N_{\mathbb{S}},$$

and for any $X_1 \in \chi(M)$, by taking the covariant derivative with respect to X_1, we obtain

$$X_1 = D_{X_1} V = D_{X_1}\xi + X_1(\varrho)N_{\mathbb{S}} + \varrho D_{X_1} N_{\mathbb{S}}.$$

By means of Gauss and Weingarten formulas, we get

$$X_1 = \overline{\nabla}_{X_1}\xi - g(X_1, \xi)N_{\mathbb{S}} + X_1(\varrho)N_{\mathbb{S}} + \varrho X_1.$$

Looking at the tangential and the normal components, we deduce

$$\overline{\nabla}_{X_1}\xi + (\varrho - 1)X_1 = 0 \tag{3.7}$$

and

$$X_1(\varrho) - g(X_1, \xi) = 0.$$

The vector field ξ on $\mathbb{S}^{n+1}(1)$ can be written as

$$\xi = U + \delta N$$

which combined with (3.7), gives

$$\overline{\nabla}_{X_1}(U + \delta N) + (\varrho - 1)X_1 = 0.$$

Again, from Gauss and Weingarten formulas, we infer

$$-(\varrho - 1)X_1 = \nabla_{X_1} U + g(B(X_1), U)N + X_1(\delta)N - \delta B(X_1).$$

Looking again at the tangential and the normal components, we deduce

$$\nabla_{X_1} U = -(\varrho - 1)X_1 + \delta B(X_1) \tag{3.8}$$

and

$$0 = g(B(X_1), U) + X_1(\delta).$$

Therefore, for any $X_1, X_2 \in \chi(M)$, we have

$$\begin{aligned} (\pounds_U g)(X_1, X_2) &= g(\nabla_{X_1} U, X_2) + g(\nabla_{X_2} U, X_1) \\ &= -2(\varrho - 1)g(X_1, X_2) + 2\delta H(X_1, X_2). \end{aligned} \tag{3.9}$$

Since the Gauss equation for a hypersurface M in $\mathbb{S}^{n+1}(1)$ gives

$$Ric(X_1, X_2) = (n-1)g(X_1, X_2) + trace(H)H(X_1, X_2) - H^2(X_1, X_2), \tag{3.10}$$

by means of (3.9) and (3.10), we infer

$$\begin{aligned} \frac{1}{2}(\pounds_U g)(X_1, X_2) + Ric(X_1, X_2) &= (n - \varrho)\, g(X_1, X_2) \\ &+ [trace(H) + \delta]\, H(X_1, X_2) - H^2(X_1, X_2). \end{aligned}$$

Contracting now (3.10), we get

$$scal = n(n-1) + (trace(H))^2 - trace(H^2). \tag{3.11}$$

If there exist smooth functions α, β, γ on M such that (3.6) holds, then we get

$$\frac{1}{2}\left(\pounds_U g\right)(X_1, X_2) + Ric(X_1, X_2)$$

$$= \left[\alpha + \beta\left(n(n-1) + (trace(H))^2 - trace(H^2)\right)\right] g\left(X_1, X_2\right) + \gamma\eta(X_1)\eta(X_2),$$

hence (M, g) is an almost η-RBS. The other implication immediately follows. $\square$

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