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SOME CHARACTERIZATIONS OF α-COSYMPLECTIC MANIFOLDS ADMITTING $*$-CONFORMAL RICCI SOLITIONS

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Abstract. The object of the present paper is to give some characterizations of α-cosymplectic manifolds admitting $*$-conformal Ricci solitons. Such manifolds with gradient $*$-conformal Ricci solitons have also been considered.
Keywords: Almost contact manifolds, cosymplectic manifolds, Ricci solitons, conformal Ricci solitons.

1. Introduction

Most of the geometric properties of a manifold are controlled by the Ricci tensor of the manifold. In [9], the notion of Ricci curvature has been extended to $*$-Ricci tensor. The idea of Ricci flow was coined by R. S. Hamilton [10]. A Ricci flow is a pseudo parabolic heat type partial differential equation where the unknown variable is the metric tensor. The theory of Ricci flow has also been developed in [7], in some different perspective to address some issues in relativistic mechanics. The theory of Ricci flow has become popular in the past years due to its application by Perelman [14] to solve the well known Poincare conjecture. A fixed solution of a Ricci flow, up to diffeomorphisms and scaling, is known as Ricci soliton. The notion of Ricci soliton has been generalized and extended by several geometers in several contexts. $*$-Ricci solitons have been studied in the papers [12, 13]. Theory

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of conformal Ricci solitons was introduced by N. Basu [1] in 2015 in the context of Kenmotsu manifolds. Later the study has been enriched in the papers [6, 10].

On the other hand, α-cosymplectic manifolds form an important class of almost contact manifolds which are receiving intensive attentions nowadays. Geometry of α-cosymplectic manifolds has been investigated in the papers [2, 4, 15]. The expression of $*$-Ricci tensor for α-cosymplectic manifolds has been determined in the paper [11]. Motivated by these works, in the present paper, we are interested to study $*$-conformal Ricci solitons on α-cosymplectic manifolds.

The present paper is organized as follows: After the introduction, in Section 2, we report some well-known results as preliminary information which will be required for subsequent calculations. Section 3 contains the study of α-cosymplectic manifolds with $*$-conformal Ricci solitons. Section 4 is devoted to gradient $*$-conformal Ricci solitons. The last section contains an example.

2. Preliminaries

A $(2n+1)$-dimensional connected differentiable manifold is called an almost contact manifold [3] if there exist a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where $X \in \chi(M)$, $\chi(M)$ is the set of all vector fields on M. The manifold is called almost contact metric manifold if there exists a Riemannian metric g on M such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$X, Y \in \chi(M)$. For such a manifold, we have

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi),$$

for all $X, Y \in \chi(M)$. On an almost contact metric manifold, we also have

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.3}$$

An almost contact metric manifold is said to be normal if the Nijenhuis tensor of ϕ vanishes. For a real number α, a normal almost contact metric manifold is said to be α-cosymplectic if [2, 4, 5]

$$d\eta = 0, \;\; d\Phi = 2\alpha\eta \wedge \Phi, \tag{2.4}$$

where

$$\Phi(X, Y) = g(X, \Phi Y). \tag{2.5}$$

For an α-cosymplectic manifold we also have

$$(\nabla_X \phi)Y = \alpha(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.6}$$

$$\nabla_X \xi = \alpha(X - \eta(X)\xi). \tag{2.7}$$

$$(\nabla_Z \eta)X = \alpha(g(X,Z) - \eta(Z)\eta(X)). \tag{2.8}$$

If $\alpha = 0$, the manifold is cosymplectic. For $\alpha = 1$, it is Kenmotsu. For an α-cosymplectic manifold, we also know

$$R(X,Y)\xi = \alpha^2(\eta(X)Y - \eta(Y)X), \tag{2.9}$$

$$R(X,\xi)Y = \alpha^2(g(X,Y)\xi - \eta(Y)X), \tag{2.10}$$

$$S(X,\xi) = -2n\alpha^2\eta(X). \tag{2.11}$$

The expression of $*$-Ricci tensor on α-cosymplectic manifolds has been determined in the paper [11]. In a $(2n+1)$-dimensional α-cosymplectic manifold the $*$-Ricci tensor is given by

$$S^*(X,Y) = S(X,Y) + \alpha^2(2n-1)g(X,Y) + \alpha^2\eta(X)\eta(Y). \tag{2.12}$$

The $*$-Ricci operator Q^* is given by $S^*(X,Y) = g(Q^*X,Y)$. For details we refer [11].

$$S^*(X,\xi) = 0. \tag{2.13}$$

Following the similar method as in [8], one can easily establish the following:

Lemma 2.1. If M is an α-cosymplectic manifold of dimension $(2n+1)$, then for any vector field $Y \in \chi(M)$, $(\nabla_\xi Q)Y = -2\alpha QY - 4n\alpha^3 Y$.

3. α-cosymplectic manifolds with $*$-conformal Ricci solitons

Definition 3.1. A $(2n+1)$-dimensional α-cosymplectic manifold is said to have $*$-conformal Ricci soliton if

$$(\pounds_V g)(X,Y) + 2S^*(X,Y) = (2\lambda - (p + \frac{2}{2n+1}))g(X,Y), \tag{3.1}$$

where S^* is the $*$-Ricci tensor of the manifold given in (2.12). Here $\pounds_V$ denotes Lie derivative with respect to the vector field V, λ and p are real numbers and $p \geq 0$. The soliton is called expanding, steady or shrinking according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$. Suppose an α-cosymplectic manifold admits a $*$-conformal Ricci soliton. Covariant differentiation of (3.1) gives

$$(\nabla_Z(\pounds_V g))(X,Y) = -2(\nabla_Z S^*)(X,Y). \tag{3.2}$$

By the relation between covariant derivative and Lie derivative, it is known from Yano [16]

$$(\nabla_X(\pounds_V g))(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y). \tag{3.3}$$

By symmetry of $\pounds_V\nabla$, and using some combinoterial computation, we have from (3.3)

$$g((\pounds_V\nabla)(X,Y),Z) = \frac{1}{2}(\nabla_X(\pounds_V g))(Y,Z) + \frac{1}{2}(\nabla_Y(\pounds_V g))(X,Z) - \frac{1}{2}(\nabla_Z(\pounds_V g))(X,Y). \tag{3.4}$$

By virtue of (3.2), (3.3) and (3.4), for $Y=\xi$, we get

$$g((\pounds_V\nabla)(X,\xi),Z) = (\nabla_Z S^*)(X,\xi) - (\nabla_X S^*)(\xi,Z) - (\nabla_\xi S^*)(X,Z). \tag{3.5}$$

In view of (2.12), we have

$$(\nabla_Z S^*)(X,Y) = (\nabla_Z S)(X,Y) + \alpha^2(\nabla_Z\eta)X\eta(Y) + \alpha^2\eta(X)(\nabla_Z\eta)Y. \tag{3.6}$$

By virtue of the above equation, we obtain

$$(\nabla_Z S^*)(X,\xi) - (\nabla_X S^*)(Z,\xi) = (1-2n)(\alpha^2(\nabla_Z\eta)X - (\nabla_X\eta)Z) + \alpha^2(\eta(X)Z - \eta(Z)X). \tag{3.7}$$

Again from (2.12)

$$(\nabla_\xi S^*)(X,Y) = (\nabla_\xi S)(X,Y) + \alpha^2(\nabla_\xi\eta)X\eta(Y) + \alpha^2\eta(X)(\nabla_\xi)Y. \tag{3.8}$$

By virtue of (3.7) and (3.8), (3.5) takes the form

$$g((\pounds_V\nabla)(X,\xi),Z) = (1-2n)\alpha^2((\nabla_Z\eta)X - (\nabla_X\eta)Z) - (\nabla_\xi S)(X,Z) - \alpha^2(\nabla_\xi\eta)X\eta(Z) + \alpha^2\eta(X)(\nabla_\xi\eta)Z. \tag{3.9}$$

In view of Lemma 2.1, the above equation takes the form

$$g((\pounds_V\nabla)(X,\xi),Z) = (1-2n)\alpha^2((\nabla_Z\eta)X - (\nabla_X\eta)Z) + 2\alpha S(X,Z) + 4n\alpha^3 g(X,Z) - \alpha^2(\nabla_\xi\eta)X\eta(Z) + \alpha^2\eta(X)(\nabla_\xi\eta)Z. \tag{3.10}$$

Using (2.8) in the above, one obtains

$$g((\pounds_V\nabla)(X,\xi),Z) = 2\alpha S(X,Z) + 4n\alpha^3 g(X,Z).$$

As a consequence of the above equation, we get

$$(\pounds_V\nabla)(X,\xi) = 2\alpha QX + 4n\alpha^3 X. \tag{3.11}$$

The above equation yields

$$(\nabla_\xi(\pounds_V\nabla))(X,\xi) = -4\alpha^2 QX - 4n\alpha^3 X. \tag{3.12}$$

A formula from Yano [16] gives

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z).$$

Putting $Y = Z = \xi$ we get

$$(\pounds_V R)(X,\xi)\xi = (\nabla_X \pounds_V)(\xi,\xi) - (\nabla_\xi \pounds_V \nabla)(X,\xi). \tag{3.13}$$

In view of (3.12) and (3.13) we get

$$(\pounds_V R)(X,\xi)\xi = 0. \tag{3.14}$$

But from (2.9) and (2.10)

$$(\pounds_V R)(X,\xi)\xi = \alpha^2(\pounds_V g)(X,\xi)\xi - \alpha^2(\pounds_V g)(\xi,\xi)X. \tag{3.15}$$

Thus, by (3.14) and (3.15), we have

$$\alpha^2(\pounds_V g)(X,\xi)\xi = \alpha^2(\pounds_V g)(\xi,\xi)X. \tag{3.16}$$

In (3.1) putting $Y = \xi$ and using (2.11) and (2.13), we deduce that

$$(\pounds_V g)(X,\xi) = (2\lambda - (p + \frac{2}{2n+1}))\eta(X).$$

Putting $X = \xi$, in the above equation, we obtain

$$(\pounds_V g)(\xi,\xi) = -(\lambda - (\frac{p}{2} + \frac{1}{2n+1})). \tag{3.17}$$

In view of (3.16), for $\alpha \neq 0$, we get

$$\big(2\lambda - (p + \frac{2}{2n+1})\big) = 0.$$

Thus, $\lambda = \frac{p}{2} + \frac{1}{2n+1}$ Since $p \geq 0$ we get $\lambda > 0$. Thus, we state the following

Theorem 3.1. A non-cosymplectic α-cosymplectic manifold as a $*$-conformal Ricci soliton is shrinking.

4. Gradient $*$-conformal Ricci solitons on α-cosymplectic manifolds

Definition 4.1. A Ricci soliton on a Riemannian manifold is called gradient $*$-conformal Ricci soliton if

$$\nabla\nabla f = S^* - (2\lambda - (p + \frac{2}{2n+1}))g \tag{4.1}$$

holds in the manifold. Here f is the potential function. The above equation has following alternative form

$$\nabla_X Df = Q^*X - (2\lambda - (p + \frac{2}{2n+1}))X. \tag{4.2}$$

The above equation yilds

$$R(X,Y)Df = (\nabla_X Q^*)Y - (\nabla_Y Q^*)X.$$

In the above equation taking $Y = \xi$ and using (2.11) we have

$$R(X,\xi)Df = 0.$$

Replacing X by Df in the above equation, we see that

$$R(Df,\xi)Df = 0. \tag{4.3}$$

But putting $X = Df$ in (2.10), we have

$$R(Df,\xi)Df = \alpha^2(g(Df,Df)\xi - \eta(Df)Df). \tag{4.4}$$

Comparing (4.3) and (4.4) we get $\alpha = 0$. Thus, we state the following

Theorem 4.1. A α-cosymplectic manifold admitting gradient $*$-conformal Ricci soliton is cosymplectic.

5. Example

Example 5.1. Consider the 3-dimensional manifold $M = \{(x_1, x_2, z) \in \mathbb{R}^3\}$, where (x_1, x_2, z) are the standard coordinates in $\mathbb{R}^3$. Let e_1, e_2, e_3 be the vector fields on M given by

$$e_1 = e^{\alpha z}\frac{\partial}{\partial x_1}, \;\; e_2 = e^{\alpha z}\frac{\partial}{\partial x_2}, \;\; e_3 = -\frac{\partial}{\partial z} = \xi.$$

Define the metric g by

$$\begin{aligned} g(e_i, e_j) &= 0, i \neq j, i, j = 1, 2, 3 \\ &= 1, i = j. \end{aligned}$$

Let η be the 1-form on M defined by $\eta(X) = g(X, e_3)$ and ϕ be the $(1,1)$-tensor field on M defined by $\phi e_1 = -e_2, \phi e_e = e_1$ and $\phi e_3 = 0$. It is a routine calculation to check that the manifold is almost contact with $\xi = e_3$. Here

$$\begin{array}{lll} \nabla_{e_1} e_3 = \alpha e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = -\alpha e_3, \\ \nabla_{e_2} e_3 = \alpha e_2 & \nabla_{e_2} e_2 = -\alpha e_3, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

Obviously, the manifold is α-cosymplectic.

Here, we can easily calculate the non-vanishing components of the curvature tensor as follows;

$$\begin{array}{ll} R(e_1,e_2)e_2 = -\alpha^2 e_1, & R(e_1,e_3)e_3 = -\alpha^2 e_1 \\ R(e_1,e_2)e_1 = \alpha^2 e_2, & R(e_2,e_3)e_3 = -\alpha^2 e_2. \\ R(e_1,e_3)e_2 = \alpha^2 e_3, & R(e_1,e_3)e_1 = \alpha^2 e_3. \end{array}$$

Here

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = -2\alpha^2.$$

The components of the $*$-Ricci tensor S^* are given by $S^*(e_i,e_j) = 0$ for all $i,j = 1,2,3$. The $*$-scalar curvature $r^* = 0$. If we take, $V = e_3$, $\lambda = \frac{1}{2}$ and $p = \frac{1}{3}$, then the manifold is a $*$- conformal Ricci soliton.

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