FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 2 (2022), 439-452
https://doi.org/10.22190/FUMI220323030T
Original Scientific Paper

ON CUBIC (α, β)-METRICS IN FINSLER GEOMETRY

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Abstract. In this paper, we study the class of cubic (α, β)-metrics. We show that every weakly Landsberg cubic (α, β)-metric has vanishing S-curvature. Using it, we prove that cubic (α, β)-metric is a weakly Landsberg metric if and only if it is a Berwald metric. This yields an extension of the Matsumoto's result for Landsberg cubic Finsler metrics.
Keywords: Cubic metric, weakly Landsberg metric, S-curvature

1. Introduction

Let (M, F) be an n-dimensional Finsler manifold, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let $F = F(x, y)$ be a scalar function on TM defined by $F = \sqrt[m]{A}$, where $A := \mathfrak{a}_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ and $\mathfrak{a}_{i_1 \dots i_m}$ is symmetric in all its indices. F is called an m-th root Finsler metric. The theory of m-th root metrics has been developed by Shimada [13], and applied to Biology as an ecological metric by Antonelli [1]. The third root metrics are called the cubic metric $F = \sqrt[3]{\mathfrak{a}_{ijk}(x) y^i y^j y^k}$. The class of cubic metrics first were considered by Wegener in 1935 and then by Kropina in 1961 (see [27] and [6]). In [27], Wegener studied cubic Finsler metrics of dimensions two and three. Wegener's paper is only an abstract of his PhD thesis without almost all calculations. In [8], Matsumoto gave an improved version of Wegener's paper. Also, Matsumoto studied two-dimensional cubic metrics and based on the main scalar, he found the necessary and sufficient

Received March 23, 2022. accepted April 11, 2022.
Communicated by Uday Chand De
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2010 *Mathematics Subject Classification.* Primary 53B40; Secondary 53C60

condition under which a Finsler metric be a cubic metric. For more progress on m-th root Finsler metrics one can see [14]-[21] and [23]-[26].

In order to understand the structure of m-th root metrics, one can study the non-Riemannian curvatures of these metrics. Let (M, F) be a Finsler manifold. The second derivatives of $\mathcal{F} := \frac{1}{2}F_x^2$ at a non-zero vector $y \in T_xM_0$ is an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of $\mathcal{F}$ at $y \in T_xM_0$ is a symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. Taking a trace of Cartan torsion give us mean Cartan torsion $\mathbf{I} := trace(\mathbf{C})$. The horizontal derivative of $\mathbf{I}$ along geodesics is called the mean Landsberg curvature $\mathbf{J} := \mathbf{I}_{|k}y^k$. Finsler metrics with $\mathbf{J} = 0$ are called weakly Landsberg metrics. The mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y \in T_xM_0$. In [2], Bao-Shen proved that on a weakly Landsberg manifold, the volume function $Vol(x)$ is a constant. In [11], Shen showed that for a weakly Landsberg manifold all the slit tangent spaces T_xM_0 are minimal in slit tangent bundle TM_0. Some rigidity problems also lead to weakly Landsberg manifolds. For example, Shen proved that every closed Finsler manifold of non-positive flag curvature and constant S-curvature is weakly Landsbergian [12]. Apparently, weakly Landsberg manifolds deserve further investigation.

In [27], Wegener proved that two and three-dimensional cubic metrics with vanishing Landsberg curvature are Berwald metrics. It seems that his proof is complicated. Then Matsumoto showed that cubic metrics with vanishing Landsberg curvature are Berwald metrics. Every Landsberg metric is a weakly Landsberg metric. It is natural to study cubic Finsler metric with vanishing mean Landsberg curvature. Then we prove the following.

Theorem 1.1. *Let $F = F(x, y)$ be a cubic (α, β)-metric on a manifold M. Then F is a weakly Landsberg metric if and only if it is a Berwald metric.*

Thus Theorem 1.1 can be consider as a generalization of Matsumoto's result.

By Theorem 1.1, one can obtain the following.

Corollary 1.1. *Let $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$ be a cubic metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Then the following are equivalent:*
(a) F is a Landsberg metric;
(b) F is a weakly Landsberg metric;
(c) F is a Berwald metric;
Also, (a)-(c) hold if and only if only if there exist functions $f_i = f_i(x)$ on M satisfy following

$$b_{i|j} = 3(c_1 + c_2b^2)b_if_j + (c_1 + 3c_2b^2)b_jf_i - b_kf^k(c_1a_{ij} + 3c_2b_ib_j), \tag{1.1}$$

where $b^2 = b_ib^i$. In this case, f_i are given by following

$$f_i = \frac{1}{6c_1}\frac{\partial}{\partial x^i}\left[\frac{\log(b^2)}{c_1 + c_2b^2}\right]. \tag{1.2}$$

A Finsler metric $F = F(x, y)$ on a manifold M is called relatively isotropic mean Landsberg metric if its mean Landsberg curvature satisfies $\mathbf{J} = cF\mathbf{I}$, where $c = c(x)$ is a scalar function on M. As a conclusion of Theorem 1.1, one can get the following.

Corollary 1.2. *Let $F = F(x, y)$ be a cubic (α, β)-metric on a manifold M. Then F is a relatively isotropic mean Landsberg metric if and only if it is a Berwald metric.*

2. Preliminary

Let M be a n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent bundle and $TM_0 := TM - \{0\}$ the slit tangent bundle. Suppose that (M, F) is a Finsler manifold. The following quadratic form $\mathbf{g}_y$ on T_xM is called fundamental tensor

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]\big|_{s,t=0}, \quad u, v \in T_xM.$$

Let $\{\partial_i := \partial/\partial x^i\}$ be a basis for T_xM at $x \in M$ and $\{dx^i\}$ is its dual. Put

$$g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} = FF_{y^iy^j} + F_{y^i}F_{y^j}.$$

Then

$$\mathbf{g}_y(u, v) = g_{ij}(y)u^iv^j, \quad u = u^i\partial_i, \quad v = v^j\partial_j.$$

By homogeneity of F, it follows that $F^2 = g_{ij}y^iy^j$.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $g^{ij} := (g_{ij})^{-1}$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

For a Finsler manifold (M, F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4}g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big], \quad y \in T_xM. \tag{2.1}$$

In this case, $\mathbf{G}$ is called the spray associated to (M, F). In local coordinates, a curve $c = c(t)$ is called a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in R^n \ \big| \ F\big(y^i \frac{\partial}{\partial x^i}|_x\big) < 1\Big\}}.$$

Let G^i denote the geodesic coefficients of F in the same local coordinate system. Then for $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, the S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}\big[\ln \sigma_F(x)\big], \tag{2.2}$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. The S-curvature is introduced by Shen for a comparison theorem on Finsler manifolds [11]. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric.

For a non-zero vector $y \in T_xM$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature and F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

Define the mean of Berwald curvature by $\mathbf{E}_y : T_xM \times T_xM \to \mathbb{R}$, where

$$\mathbf{E}_y(u, v) := \frac{1}{2}\sum_{i=1}^{n} g^{ij}(y)\mathbf{g}_y\Big(\mathbf{B}_y(u, v, e_i), e_j\Big).$$

The family $\mathbf{E} = \{\mathbf{E}_y\}_{y \in TM\setminus\{0\}}$ is called the *mean Berwald curvature* or *E-curvature*. In local coordinates, $\mathbf{E}_y(u, v) := E_{ij}(y)u^i v^j$, where

$$E_{ij} := \frac{1}{2}B^m{}_{mij}.$$

$\mathbf{E}$ is called the mean Berwald curvature. F is called a weakly Berwald metric if $\mathbf{E} = \mathbf{0}$. By (2.2), one can get the following

$$\mathbf{S}_{y^i y^j} = \big[G^m\big]_{y^i y^j y^m} = E_{ij}.$$

Thus $\mathbf{S} = 0$ implies that $\mathbf{E} = 0$.

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := I_{i|s}y^s.$$

$\mathbf{J}$ is called the mean Landsberg curvature or J-curvature [4]. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J}_y = 0$. Mean Landsberg curvature can be defined as following

$$J_i := y^m \frac{\partial I_i}{\partial x^m} - I_m \frac{\partial G^m}{\partial y^i} - 2G^m \frac{\partial I_i}{\partial y^m}.$$

By definition, we get

$$\mathbf{J}_y(u) := \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\big(U(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U = U(t)$ is a linearly parallel vector field along the geodesic σ with $U(0) = u$.

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. First, we are going to remark the relation between an m-th root metric and an (α,β)-metric. For this aim, we remark that the (α,β)-metric $F = \alpha^{m+1}\beta^{-m}$ is called m-Kropina metric. In [9], Matsumoto-Numata studied the class of cubic metrics and proved the following.

Lemma 3.1. ([9]) *Let $F = \sqrt[3]{A}$ be a cubic Finsler metric on an n-dimensional manifold M, where $A = a_{ijk}y^iy^jy^k$. Then the following hold:*

(1) *If $dim(M) = 2$, then by choosing suitable quadratic form $\alpha = \sqrt{\mathfrak{a}_{ij}(x)y^iy^j}$ and one form $\beta = b_i(x)y^i$, F is a $(-\frac{1}{3})$-Kropina metric*

$$F = \sqrt[3]{\alpha^2\beta}, \tag{3.1}$$

where α^2 may be degenerate.

(2) *If $dim(M) \geq 3$ and F is a function of a non-degenerate quadratic form $\alpha = \sqrt{\mathfrak{a}_{ij}(x)y^iy^j}$ and a one-form $\beta = b_i(x)y^i$ which is homogeneous in α and β of degree one, then it is written in the following form*

$$F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}, \tag{3.2}$$

where c_1 and c_2 are real constants.

Lemma 3.1 explains that the complete form of a cubic (α,β)-metric is written by (3.2). In [5], Kim-Park used the strategy of Matsumoto for indication of structure of m-th root (α,β)-metrics. More precisely, they showed that an m-th root (α,β)-metric is given by following

$$F = \sum_{r=0}^{m} c_{m-2r}\alpha^{2r}\beta^{m-2r}, \qquad s \leq \frac{m}{2}.$$

In this paper, we focus on (3.2) which is more complete than (3.1). Also, we suppose that the associated Riemannian metric α is positive-definite.

Let $F=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$ be an (α,β)-metric, where $\phi=\phi(s)$ is a C^∞ on $(-b_0,b_0)$ with certain regularity, $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on a manifold M. For an (α,β)-metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j:=db_i-b_j\theta_i^j$, where $\theta^i:=dx^i$ and $\theta_i^j:=\Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let

$$r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),\quad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),\quad r_{i0}:=r_{ij}y^j,\quad r_{00}:=r_{ij}y^iy^j,$$
$$r_j:=b^ir_{ij},\quad s_{i0}:=s_{ij}y^j,\quad s_j:=b^is_{ij},\quad r_0:=r_jy^j,\quad s_0:=s_jy^j.$$

In order to prove Theorem 1.1, we need the following.

Proposition 3.1. *Let $F=F(x,y)$ be a weakly Landsberg cubic (α,β)-metric on an n-dimensional manifold M. Then $\mathbf{S}=0$.*

Proof. For an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, the mean Landsberg curvature is given by

$$\begin{aligned}J_i=&-\frac{1}{2\Delta\alpha^4}\Bigg\{\frac{2\alpha^3}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big](r_0+s_0)h_i\\&+\frac{\alpha^2}{b^2-s^2}(\Psi_1+s\frac{\Phi}{\Delta})(r_{00}-2\alpha Qs_0)h_i+\alpha\Big[-\alpha^2Q's_0h_i+\alpha Q(\alpha^2s_i-y_is_0)\\&+\alpha^2\Delta s_{i0}+\alpha^2(r_{i0}-2\alpha Qs_i)-(r_{00}-2\alpha Qs_0)y_i\Big]\frac{\Phi}{\Delta}\Bigg\}.\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}Q&:=\frac{\phi'}{\phi-s\phi'},\\\Delta&:=1+sQ+(b^2-s^2)Q',\\\Psi&:=\frac{Q'}{2\Delta},\\\Phi&:=-(Q-sQ')\{n\Delta+1+sQ\}-(b^2-s^2)(1+sQ)Q'',\\\Psi_1&:=\sqrt{b^2-s^2}\Delta^{\frac{1}{2}}\Big[\frac{\sqrt{b^2-s^2}\Phi}{\Delta^{\frac{3}{2}}}\Big]',\\h_i&:=b_i-\alpha^{-1}sy_i.\end{aligned}$$

For more details, one can see [7]. It is easy to see that the function of cubic (α,β)-metric $F=\sqrt[3]{c_1\beta\alpha^2+c_2\beta^3}$ is given by $\phi=\sqrt[3]{c_1s+c_2s^3}$. Let us put

$$\lambda:=\frac{1}{2c_1s^3(b^2-s^2)(-3c_2b^2s^2-4c_1s^2)^3},$$

By putting $\phi=\sqrt[3]{c_1s+c_2s^3}$ in (3.3), one can get the mean Landsberg curvature of F as follows

$$J_i:=\lambda\big(Ab_i+By_i+Cr_{i0}+Ds_i-Es_{i0}\big)\alpha^2, \tag{3.4}$$

where

$$\begin{aligned}
A:=\;&n\alpha^{13}b^8{c_1}^4s_0-12n\alpha^{11}b^8\beta^2{c_1}^3c_2s_0+54n\alpha^9b^8\beta^4{c_1}^2{c_2}^2s_0-108n\alpha^7b^8\beta^6c_1{c_2}^3s_0\\
&+81n\alpha^5b^8\beta^8{c_2}^4s_0-3\alpha^{13}b^8{c_1}^4s_0+24\alpha^{11}b^8\beta^2{c_1}^3c_2s_0-54\alpha^9b^8\beta^4{c_1}^2{c_2}^2s_0\\
&+81\alpha^5b^8\beta^8{c_2}^4s_0-15n\alpha^{11}b^6\beta^2{c_1}^4s_0+144n\alpha^9b^6\beta^4{c_1}^3c_2s_0-38n\alpha^7b^6\beta^6{c_1}^2{c_2}^2s_0\\
&+17\alpha^{11}b^6\beta^2{c_1}^4s_0+96\alpha^9b^6\beta^4{c_1}^3c_2r_0-192\alpha^9b^6\beta^4{c_1}^3c_2s_0+2n\alpha^9b^6\beta^3{c_1}^4r_{00}\\
&-144\alpha^7b^6\beta^6{c_1}^2{c_2}^2r_0-522\alpha^7b^6\beta^6{c_1}^2{c_2}^2s_0-30n\alpha^7b^6\beta^5{c_1}^3c_2r_{00}-6\alpha^9b^6\beta^3{c_1}^4r_{00}\\
&+216\alpha^5b^6\beta^8c_1{c_2}^3s_0+90n\alpha^5b^6\beta^7{c_1}^2{c_2}^2r_{00}+81\alpha^3b^6\beta^{10}{c_2}^4s_0+96n\alpha^9b^4\beta^4{c_1}^4s_0\\
&-54n\alpha^3b^6\beta^9c_1{c_2}^3r_{00}+162\alpha^7b^6\beta^5{c_1}^3c_2r_{00}-516n\alpha^7b^4\beta^6{c_1}^3c_2s_0+256\alpha^5\beta^8{c_1}^4r_0\\
&+18\alpha^5b^6\beta^7{c_1}^2{c_2}^2r_{00}+288n\alpha^5b^4\beta^8{c_1}^2{c_2}^2s_0-54\alpha^3b^6\beta^9c_1{c_2}^3r_{00}+256\alpha^5\beta^8{c_1}^4s_0\\
&+324n\alpha^3b^4\beta^{10}c_1{c_2}^3s_0+144\alpha^9b^4\beta^4{c_1}^4r_0-108\alpha^9b^4\beta^4{c_1}^4s_0-480\alpha^7b^4\beta^6{c_1}^3c_2r_0\\
&-648\alpha^7b^4\beta^6{c_1}^3c_2s_0-36n\alpha^7b^4\beta^5{c_1}^4r_{00}+144\alpha^5b^4\beta^8{c_1}^2{c_2}^2r_0-128n\alpha^3\beta^9{c_1}^4r_{00}\\
&-288n\alpha^3b^2\beta^9{c_1}^3c_2r_{00}-192\alpha^5b^2\beta^7{c_1}^4r_{00}+192n\alpha^5\beta^8{c_1}^4s_0-96\alpha^3b^2\beta^9{c_1}^3c_2r_{00}\\
&-216\alpha^3b^4\beta^9{c_1}^2{c_2}^2r_{00}+180\alpha^7b^4\beta^5{c_1}^4r_{00}-25n\alpha^7b^2\beta^6{c_1}^4s_0+192n\alpha^3\beta^{10}{c_1}^3c_2s_0\\
&+432n\alpha^3b^2\beta^{10}{c_1}^2{c_2}^2s_0-384\alpha^7b^2\beta^6{c_1}^4r_0-144\alpha^7b^2\beta^6{c_1}^4s_0+384\alpha^5b^2\beta^8{c_1}^3c_2r_0\\
&-156\alpha^5b^4\beta^7{c_1}^3c_2r_{00}+336n\alpha^5b^2\beta^8{c_1}^3c_2s_0-144\alpha^3b^4\beta^9{c_1}^2{c_2}^2\;r_{00}\\
&+828\alpha^5b^4\beta^8{c_1}^2{c_2}^2s_0+228n\alpha^5b^4\beta^7{c_1}^3c_2r_{00}+216\alpha^3b^4\beta^{10}c_1{c_2}^3s_0\\
&+960\alpha^5b^2\beta^8{c_1}^3c_2s_0+144n\alpha^5b^2\beta^7{c_1}^4r_{00}+144\alpha^3b^2\beta^{10}{c_1}^2{c_2}^2s_0\\
&+21n\alpha^5b^6\beta^8c_1{c_2}^3s_0+81n\alpha^3b^6\beta^{10}{c_2}^4s_0-16\alpha^{11}b^6\beta^2{c_1}^4r_0
\end{aligned}$$

$$\begin{aligned}
B:=\;&-n\alpha^{11}b^8\beta{c_1}^4s_0+12n\alpha^9b^8\beta^3{c_1}^3c_2s_0-54n\alpha^7b^8\beta^5{c_1}^2{c_2}^2s_0+108n\alpha^5b^8\beta^7c_1{c_2}^3s_0\\
&-24\alpha^9b^8\beta^3{c_1}^3c_2s_0-6n\alpha^8b^8\beta^3{c_1}^3c_2s_0+54\alpha^7b^8\beta^5{c_1}^2{c_2}^2s_0+54n\alpha^4b^8\beta^7c_1{c_2}^3s_0\\
&+6\alpha^8b^8\beta^3{c_1}^3c_2s_0-2n\alpha^8b^8\beta^2{c_1}^4r_{00}-144n\alpha^7b^6\beta^5{c_1}^3c_2s_0+36\alpha^6b^8\beta^5{c_1}^2{c_2}^2s_0\\
&+18n\alpha^6b^8\beta^4{c_1}^3c_2r_{00}+378n\alpha^5b^6\beta^7{c_1}^2{c_2}^2s_0-54\alpha^4b^8\beta^7c_1{c_2}^3s_0-81\alpha^2b^8\beta^9{c_2}^4s_0\\
&+18\alpha^6b^6\beta^5{c_1}^3c_2s_0+26n\alpha^6b^6\beta^4{c_1}^4r_{00}-162\alpha^5b^6\beta^6{c_1}^3c_2r_{00}+516n\alpha^5b^4\beta^7{c_1}^3c_2s_0\\
&-216\alpha b^4\beta^{11}c_1{c_2}^3s_0+216n\alpha b^4\beta^{10}{c_1}^2{c_2}^2r_{00}-54b^6\beta^{10}c_1{c_2}^3r_{00}+32nb^4\beta^{11}c_1{c_2}^3s_0\\
&+54\alpha^2b^8\beta^8c_1{c_2}^3r_{00}-378n\alpha^2b^6\beta^9c_1{c_2}^3s_0-81\alpha b^6\beta^{11}{c_2}^4s_0+54n\alpha b^6\beta^{10}c_1{c_2}^3r_{00}\\
&+480\alpha^5b^4\beta^7{c_1}^3c_2r_0+648\alpha^5b^4\beta^7{c_1}^3c_2s_0+36n\alpha^5b^4\beta^6{c_1}^4r_{00}+126\alpha^4b^6\beta^6{c_1}^3c_2r_{00}\\
&-120n\alpha^4b^4\beta^6{c_1}^4r_{00}+16\alpha^3b^4\beta^8{c_1}^3c_2r_{00}-36\alpha^3b^2\beta^9{c_1}^3c_2s_0+72\alpha^2b^4\beta^9{c_1}^2{c_2}^2s_0\\
&+32n\alpha^2b^4\beta^8{c_1}^3c_2r_{00}+14\alpha b^4\beta^{10}{c_1}^2{c_2}^2r_{00}-432n\alpha\,b^2\beta^{11}{c_1}^2{c_2}^2s_0-12n\beta^{10}{c_1}^4r_{00}\\
&+216b^4\beta^{11}c_1{c_2}^3s_0-216nb^4\beta^{10}{}_1{}^2{c_2}^2r_{00}+384\alpha^5b^2\beta^7{c_1}^4r_0+144\alpha^5b^2\beta^7{c_1}^4s_0\\
&+144\alpha^4b^4\beta^6{c_1}^4r_{00}-112n\alpha^4b^2\beta^7{c_1}^4s_0-384\alpha^3b^2\beta^9{c_1}^3c_2r_0-960\alpha^3b^2\beta^9{c_1}^3c_2s_0\\
&-14n\alpha^3b^2\beta^8{c_1}^4r_{00}-12n\alpha^2b^2\beta^9{c_1}^3c_2s_0-14\alpha b^2\beta^{11}{c_1}^2{c_2}^2s_0+28\alpha b^2\beta^{10}{c_1}^3c_2r_{00}\\
&-144b^4\beta^{10}{c_1}^2{c_2}^2r_{00}+432nb^2\beta^{11}{c_1}^2{c_2}^2s_0+48\alpha^4b^2\beta^7{c_1}^4s_0+192\alpha^3b^2\beta^8{c_1}^4r_{00}
\end{aligned}$$

$$
\begin{aligned}
&-192n\alpha^3\beta^9{c_1}^4s_0+192\alpha^2b^2\beta^9{c_1}^3c_2s_0+224n\alpha^2b^2\beta^8{c_1}^4r_{00}+96\alpha b^2\beta^{10}{c_1}^3c_2r_{00}\\
&-192n\alpha\beta^{11}{c_1}^3c_2s_0+144b^2\beta^{11}{c_1}^2{c_2}^2s_0-288nb^2\beta^{10}{c_1}^3c_2r_{00}-256\alpha^3\beta^9{c_1}^4r_0\\
&-72\alpha^6b^4\beta^5{c_1}^4s_0-180\alpha^5b^4\beta^6{c_1}^4r_{00}+256n\alpha^5b^2\beta^7{c_1}^4s_0-216\alpha^4b^4\beta^7{c_1}^3c_2s_0\\
&-13n\alpha^8b^6\beta^3{c_1}^4s_0-96\alpha^7b^6\beta^5{c_1}^3c_2r_0+192\alpha^7b^6\beta^5{c_1}^3c_2s_0-2n\alpha^7b^6\beta^4{c_1}^4r_{00}\\
&+54\alpha b^6\beta^{10}c_1{c_2}^3r_{00}-324n\alpha b^4\beta^{11}c_1{c_2}^3s_0+81b^6\beta^{11}{c_2}^4s_0-54nb^6\beta^{10}c_1{c_2}^3r_{00}\\
&-144\alpha^7b^4\beta^5{c_1}^4r_0+108\alpha^7b^4\beta^5{c_1}^4s_0-54\alpha^6b^6\beta^4{c_1}^4r_{00}+60n\alpha^6b^4\beta^5{c_1}^4s_0\\
&-228n\alpha^3b^4\beta^8{c_1}^3c_2r_{00}+126\alpha^2b^6\beta^8{c_1}^2{c_2}^2r_{00}-50n\alpha^2b^4\beta^9{c_1}^2{c_2}^2s_0\\
&-256\alpha^3\beta^9{c_1}^4s_0-96\alpha^2b^2\beta^8{c_1}^4r_{00}+64n\alpha^2\beta^9{c_1}^4s_0+128n\alpha\beta^{10}{c_1}^4r_{00}\\
&-81\alpha^3b^8\beta^9{c_2}^4s_0-81n\alpha^2b^8\beta^9{c_2}^4s_0-3\alpha^{10}b^8\beta{c_1}^4s_0+15n\alpha^9b^6\beta^3{c_1}^4s_0\\
&-54n\alpha^4b^8\beta^6{c_1}^2{c_2}^2r_{00}-216n\alpha^3b^6\beta^9c_1{c_2}^3s_0+54n\alpha^2b^8\beta^8c_1{c_2}^3r_{00}\\
&-81n\alpha b^6\beta^{11}{c_2}^4s_0+16\alpha^9b^6\beta^3{c_1}^4r_0-17\alpha^9b^6\beta^3{c_1}^4s_0+6\alpha^8b^8\beta^2{c_1}^4r_{00}\\
&-30\alpha^6b^8\beta^4{c_1}^3c_2r_{00}+42n\alpha^6b^6\beta^5{c_1}^3c_2s_0+144\alpha^5b^6\beta^7{c_1}^2{c_2}^2r_0\\
&+522\alpha^5b^6\beta^7{c_1}^2{c_2}^2s_0+30n\alpha^5b^6\beta^6{c_1}^3c_2r_{00}+18\alpha^4b^8\beta^6{c_1}^2{c_2}^2r_{00}\\
&+108n\alpha^4b^6\beta^7{c_1}^2{c_2}^2s_0-216\alpha^3b^6\beta^9c_1{c_2}^3s_0-90n\alpha^3b^6\beta^8{c_1}^2{c_2}^2r_{00}\\
&+81nb^6\beta^{11}{c_2}^4s_0+27\alpha^8b^6\beta^3{c_1}^4s_0+6\alpha^7b^6\beta^4{c_1}^4r_{00}-96n\alpha^7b^4\beta^5{c_1}^4s_0\\
&-252\alpha^4b^6\beta^7{c_1}^2{c_2}^2s_0-162n\alpha^4b^6\beta^6{c_1}^3c_2r_{00}-18\alpha^3b^6\beta^8{c_1}^2{c_2}^2r_{00}\\
&-288n\alpha^3b^4\beta^9{c_1}^2{c_2}^2s_0-162\alpha^2b^6\beta^9c_1{c_2}^3s_0+270n\alpha^2b^6\beta^8{c_1}^2{c_2}^2r_{00}\\
&-36n\alpha^4b^4\beta^7{c_1}^3c_2s_0-144\alpha^3b^4\beta^9{c_1}^2{c_2}^2r_0-828\alpha^3b^4\beta^9{c_1}^2{c_2}^2s_0\\
&-96b^2\beta^{10}{c_1}^3c_2r_{00}+192n\beta^{11}{c_1}^3c_2s_0-81n\alpha^3b^8\beta^9{c_2}^4s_0\\
&+3\alpha^{11}b^8\beta{c_1}^4s_0+n\alpha^{10}b^8\beta{c_1}^4s_0
\end{aligned}
$$

$$
\begin{aligned}
C:=\ &2n\alpha^{10}b^8\beta^2{c_1}^4-18n\alpha^8b^8\beta^4{c_1}^3c_2+54n\alpha^6b^8\beta^6{c_1}^2{c_2}^2-54n\alpha^4b^8\beta^8c_1{c_2}^3\\
&-6\alpha^{10}b^8\beta^2{c_1}^4+30\alpha^8b^8\beta^4{c_1}^3c_2-18\alpha^6b^8\beta^6{c_1}^2{c_2}^2-54\alpha^4b^8\beta^8c_1{c_2}^3\\
&-26n\alpha^8b^6\beta^4{c_1}^4+162n\alpha^6b^6\beta^6{c_1}^3c_2-270n\alpha^4b^6\beta^8{c_1}^2{c_2}^2+54n\alpha^2b^6\beta^{10}c_1{c_2}^3\\
&+54\alpha^8b^6\beta^4{c_1}^4-126\alpha^6b^6\beta^6{c_1}^3c_2-126\alpha^4b^6\beta^8{c_1}^2{c_2}^2+54\alpha^2b^6\beta^{10}c_1{c_2}^3\\
&+120n\alpha^6b^4\beta^6{c_1}^4-432n\alpha^4b^4\beta^8{c_1}^3c_2+216n\alpha^2b^4\beta^{10}{c_1}^2{c_2}^2-144\alpha^6b^4\beta^6{c_1}^4\\
&+144\alpha^2b^4\beta^{10}{c_1}^2{c_2}^2-224n\alpha^4b^2\beta^8{c_1}^4+288n\alpha^2b^2\beta^{10}{c_1}^3c_2+96\alpha^4b^2\beta^8{c_1}^4\\
&+96\alpha^2b^2\beta^{10}{c_1}^3c_2+128n\alpha^2\beta^{10}{c_1}^4,
\end{aligned}
$$

$$
\begin{aligned}
D:=\ &6n\alpha^{10}b^8\beta^3{c_1}^3c_2-n\alpha^{12}b^8\beta{c_1}^4-54n\alpha^6b^8\beta^7c_1{c_2}^3+81n\alpha^4b^8\beta^9{c_2}^4-192n\alpha^2\beta^{11}{c_1}^3c_2\\
&-81n\alpha^2b^6\beta^{11}{c_2}^4-27\alpha^{10}b^6\beta^3{c_1}^4-18\alpha^8b^6\beta^5{c_1}^3c_2+252\alpha^6b^6\beta^7{c_1}^2{c_2}^2-64n\alpha^4\beta^9{c_1}^4\\
&+3\alpha^{12}b^8\beta{c_1}^4-6\alpha^{10}b^8\beta^3{c_1}^3c_2-36\alpha^8b^8\beta^5{c_1}^2{c_2}^2+54\alpha^6b^8\beta^7c_1{c_2}^3+81\alpha^4b^8\beta^9{c_2}^4\\
&+13n\alpha^{10}b^6\beta^3{c_1}^4-42n\alpha^8b^6\beta^5{c_1}^3c_2-108n\alpha^6b^6\beta^7{c_1}^2{c_2}^2+378n\alpha^4b^6\beta^9c_1{c_2}^3\\
&+540n\alpha^4b^4\beta^9{c_1}^2{c_2}^2-324n\alpha^2b^4\beta^{11}c_1{c_2}^3+72\alpha^8b^4\beta^5{c_1}^4+216\alpha^6b^4\beta^7{c_1}^3c_2\\
&-72\alpha^4b^4\beta^9{c_1}^2{c_2}^2-216\alpha^2b^4\beta^{11}c_1{c_2}^3+112n\alpha^6b^2\beta^7{c_1}^4+192n\alpha^4b^2\beta^9{c_1}^3c_2\\
&-432n\alpha^2b^2\beta^{11}{c_1}^2{c_2}^2-48\alpha^6b^2\beta^7{c_1}^4-192\alpha^4b^2\beta^9{c_1}^3c_2-144\alpha^2b^2\beta^{11}{c_1}^2{c_2}^2\\
&+162\alpha^4b^6\beta^9c_1{c_2}^3-81\alpha^2b^6\beta^{11}{c_2}^4-60n\alpha^8b^4\beta^5{c_1}^4+36n\alpha^6b^4\beta^7{c_1}^3c_2
\end{aligned}
$$

$$
\begin{aligned}
E := {} & n\alpha^{12}b^{10}{c_1}^4 - 12n\alpha^{10}b^{10}\beta^2{c_1}^3c_2 + 54n\alpha^8b^{10}\beta^4{c_1}^2{c_2}^2 - 192\alpha^4b^2\beta^8{c_1}^4 \\
& -108n\alpha^6b^{10}\beta^6c_1{c_2}^3 + 81n\alpha^4b^{10}\beta^8{c_2}^4 - 3\alpha^{12}b^{10}{c_1}^4 + 24\alpha^{10}b^{10}\beta^2{c_1}^3c_2 \\
& -54\alpha^8b^{10}\beta^4{c_1}^2{c_2}^2 + 81\alpha^4b^{10}\beta^8{c_2}^4 - 17n\alpha^{10}b^8\beta^2{c_1}^4 + 156n\alpha^8b^8\beta^4{c_1}^3c_2 \\
& -486n\alpha^6b^8\beta^6{c_1}^2{c_2}^2 + 540n\alpha^4b^8\beta^8c_1{c_2}^3 - 81n\alpha^2b^8\beta^{10}{c_2}^4 + 39\alpha^{10}b^8\beta^2{c_1}^4 \\
& -204\alpha^8b^8\beta^4{c_1}^3c_2 + 162\alpha^6b^8\beta^6{c_1}^2{c_2}^2 + 324\alpha^4b^8\beta^8c_1{c_2}^3 - 81\alpha^2b^8\beta^{10}{c_2}^4 \\
& -180\alpha^8b^6\beta^4{c_1}^4 + 468\alpha^6b^6\beta^6{c_1}^3c_2 + 324\alpha^4b^6\beta^8{c_1}^2{c_2}^2 - 324\alpha^2b^6\beta^{10}c_1{c_2}^3 \\
& -352n\alpha^6b^4\beta^6{c_1}^4 + 1344n\alpha^4b^4\beta^8{c_1}^{c_2} - 864n\alpha^2b^4\beta^{10}{c_1}^2{c_2}^2 + 336\alpha^6b^4\beta^6{c_1}^4 \\
& -96\alpha^4b^4\beta^8{c_1}^3c_2 - 432\alpha^2b^4\beta^{10}{c_1}^2{c_2}^2 + 512n\alpha^4b^2\beta^8{c_1}^4 - 768n\alpha^2b^2\beta^{10}{c_1}^3c_2 \\
& -720n\alpha^6b^6\beta^6{c_1}^3c_2 + 1296n\alpha^4b^6\beta^8{c_1}^2{c_2}^2 - 432n\alpha^2b^6\beta^{10}c_1{c_2}^3 \\
& -192\alpha^2b^2\beta^{10}{c_1}^3c_2 - 256n\alpha^2\beta^{10}{c_1}^4 + 112n\alpha^8b^6\beta^4{c_1}^4.
\end{aligned}
$$

By assumption, we have $J_i = 0$. Contracting $J_i = 0$ with b^i and using (3.4) imply that

$$
d_5\alpha^5 + d_4\alpha^4 + d_3\alpha^3 + d_2\alpha^2 + d_1\alpha + d_0 = 0, \tag{3.5}
$$

where

$$
\begin{aligned}
d_0 &:= \beta^9\left(3c_2b^2 + 4nc_1\right)^2\left(3nc_2b^2 + 3nb^2c_2 + 4c_1\right)\left(3c_2\beta s_0 - 2c_1r_{00}\right), \\
d_1 &:= -\beta^9(3c_2b^2 + 4nc_1)^2(3nc_2b^2 + 3nc_2b^2 + 4c_1)(3c_2\beta s_0 - 2c_1r_{00}), \\
d_2 &:= \beta^7\left(3c_2b^2 + 4c_1\right)\Big(27nb^6\beta{c_2}^3s_0 + 27b^6\beta{c_2}^3s_0 + 18b^4\beta c_1{c_2}^2r_0 + 90b^4\beta c_1{c_2}^2s_0 \\
&\quad +132nb^2\beta{c_1}^2c_2s_0 + 18nb^4{c_1}^2c_2r_{00} - 6b^4{c_1}^2c_2r_{00} + 48nb^2\beta{c_1}^2c_2r_0 \\
&\quad +90nb^4\beta c_1{c_2}^2s_0 + 24b^2\beta{c_1}^2c_2r_0 + 96b^2\beta{c_1}^2c_2s_0 + 24nb^2{c_1}^3r_{00} \\
&\quad +18b^4\beta c_1{c_2}^2r_0 - 24b^2{c_1}^3r_{00} + 80n\beta{c_1}^3s_0 + 32n\beta{c_1}^3r_0\Big), \\
d_3 &:= -\beta^7\Big(81n{c_2}^4b^8\beta s_0 + 81b^8\beta{c_2}^4s_0 + 216nb^6\beta c_1{c_2}^3s_0 + 216b^6\beta c_1{c_2}^3s_0 \\
&\quad +18b^6{c_1}^2{c_2}^2r_{00} + 288nb^4\beta{c_1}^2{c_2}^2s_0 + 144b^4\beta{c_1}^2{c_2}^2r_0 + 828b^4\beta{c_1}^2{c_2}^2s_0 \\
&\quad -156b^4{c_1}^3c_2r_{00} + 336nb^2\beta{c_1}^3c_2s_0 + 384b^2\beta{c_1}^3c_2r_0 + 960b^2\beta{c_1}^3c_2s_0 \\
&\quad -192b^2{c_1}^4r_{00} + 192n\beta{c_1}^4s_0 + 256\beta{c_1}^4r_0 + 256\beta{c_1}^4s_0 \\
&\quad +90nb^6{c_1}^2{c_2}^2r_{00} + 228nb^4{c_1}^3c_2r_{00} + 144nb^2{c_1}^4r_{00}\Big), \\
d_4 &:= -2c_1b^2\beta^5\Big(54nb^6\beta{c_2}^3s_0 + 27nb^4\beta c_1{c_2}^2r_0 + 216nb^4\beta c_1{c_2}^2s_0 \\
&\quad -36b^4\beta c_1{c_2}^2s_0 + 9nb^4{c_1}^2c_2r_{00} - 15b^4{c_1}^2c_2r_{00} + 72nb^2\beta{c_1}^2c_2r_0 \\
&\quad -48b^2\beta{c_1}^2c_2r_0 - 132b^2\beta{c_1}^2c_2s_0 + 12nb^2{c_1}^3r_{00} - 24b^2{c_1}^3r_{00} \\
&\quad +152n\beta{c_1}^3s_0 - 48\beta{c_1}^3r_0 - 120\beta{c_1}^3s_0 - 9b^4\beta c_1{c_2}^2r_0 \\
&\quad +36nb^2\beta{c_1}^2c_2s_0 + 48n\beta{c_1}^3r_0\Big), \\
d_5 &:= 2b^2\beta^5c_1\Big(54nb^6\beta{c_2}^3s_0 + 189nb^4\beta c_1{c_2}^2s_0 + 72b^4\beta c_1{c_2}^2r_0 \\
&\quad +15nb^4{c_1}^2c_2r_{00} - 81b^4{c_1}^2c_2r_{00} + 258nb^2\beta{c_1}^2c_2s_0 + 72\beta{c_1}^3s_0 \\
&\quad +324b^2\beta{c_1}^2c_2s_0 + 18nb^2{c_1}^3r_{00} - 90b^2{c_1}^3r_{00} + 128n\beta{c_1}^3s_0
\end{aligned}
$$

$$+240b^2\beta {c_1}^2c_2r_0 + 261b^4\beta c_1{c_2}^2s_0 + 192\beta {c_1}^3r_0\Big).$$

By (3.5), we get

$$d_5\alpha^4 + d_3\alpha^2 + d_1 = 0, \tag{3.6}$$

$$d_4\alpha^4 + d_2\alpha^2 + d_0 = 0. \tag{3.7}$$

(3.6) implies that there exists a non-zero function $f = f(x, y)$ such that

$$r_{00} = c\beta s_0 + f\alpha^2, \tag{3.8}$$

where $c := 3c_2/(2c_3)$ is a real constant. Similarly, (3.7) implies that there exists a non-zero function $g = g(x, y)$ such that the following holds

$$r_{00} = c\beta s_0 + g\alpha^2. \tag{3.9}$$

Since $f \neq g$ and also f is not a multiplication of g, then by (3.8) and (3.9) we get

$$r_{00} = c\beta s_0 \tag{3.10}$$

or equivalently

$$r_{ij} = \frac{c}{2}\big(b_is_j + b_js_i\big). \tag{3.11}$$

(3.11) implies that

$$r_0 = \frac{c}{2}b^2s_0. \tag{3.12}$$

Putting (3.10) and (3.12) in (3.7) yields

$$F(x)s_0 = 0, \tag{3.13}$$

where

$$\begin{aligned} F(x) \;=\; & 27nb^8c{c_1}^2{c_2}^2\alpha^4 - 27nb^8cc_1{c_2}^3\beta^2\alpha^2 - b^8c{c_1}^2{c_2}^29\alpha^4 + 108\ nb^8c_1{c_2}^3\ \alpha^4 \\ & -81nb^8{c_2}^4\beta^2\alpha^2 + 90nb^6c{c_1}^3c_2\alpha^4 - 81b^8{c_2}^4\beta^2\alpha^2 - 162nb^6c{c_1}^2{c_2}^2\beta^2\alpha^2 \\ & +48b^4c{c_1}^3c_2\beta^2\alpha^2 - 756nb^4{c_1}^2{c_2}^2\beta^2\alpha^2 + 144b^4c{c_1}^2{c_2}^2\beta^4 - 34b^4c_1{c_2}^3\beta^4 \\ & -264b^4{c_1}^3c_2\alpha^4 - 648b^4{c_1}^2{c_2}^2\beta^2\alpha^2 - 160nb^2c{c_1}^4\beta^2\alpha^2 + 28nb^2c{c_1}^3c_2\beta^4 \\ & -54b^6c{c_1}^2{c_2}^2\beta^2\alpha^2 - 378nb^6c_1{c_2}^3\beta^2\alpha^2 + 54b^6cc_1{c_2}^3\beta^4 - 81nb^6{c_2}^4\beta^4 \\ & -72b^6{c_1}^2{c_2}^2\alpha^4 + 72nb^4{c_1}^4\alpha^4 - 378b^6c_1{c_2}^3\beta^2\alpha^2 - 288nb^4c{c_1}^3c_2\beta^2\alpha^2 \\ & +304nb^2{c_1}^4\alpha^4 + 96b^2c{c_1}^4\beta^2\alpha^2 - 768nb^2{c_1}^3c_2\beta^2\alpha^2 + 96b^2c{c_1}^3c_2\beta^4 \\ & +216nb^4c{c_1}^2{c_2}^2\beta^4 - 96b^4c{c_1}^4\alpha^4 + 612nb^4{c_1}^3c_2\alpha^4 - 216b^4c_1{c_2}^3\beta^4 \\ & -78b^6c{c_1}^3c_2\alpha^4 + 432nb^6{c_1}^2{c_2}^2\alpha^4 + 128nc{c_1}^4\beta^4 - 320n{c_1}^4\beta^2\alpha^2 \\ & -43b^2{c_1}^2{c_2}^2\beta^4 - 240b^2{c_1}^4\alpha^4 - 38b^2{c_1}^3c_2\beta^2\alpha^2 - 144b^2{c_1}^2{c_2}^2\beta^4 \\ & -27b^8cc_1{c_2}^3\beta^2\alpha^2 - 81b^6{c_2}^4\beta^4 + 54nb^6cc_1{c_2}^3\beta^4 - 192n{c_1}^3c_2\beta^4 \end{aligned}$$

Let $F(x) = 0$. Then we get

$$f_1\alpha^4 + f_2\beta^2\alpha^2 + f_3\beta^2 = 0, \tag{3.14}$$

where $f_1 = f_1(x)$, $f_2 = f_2(x)$ and $f_3 = f_3(x)$ are scalar functions on M. (3.14) implies that

$$\alpha^2 = \Big(\frac{-g \pm \sqrt{g^2 - 4fh}}{2f}\Big)\beta^2. \tag{3.15}$$

This contradicts with the positive-definiteness of α. Thus $F \neq 0$. Therefore, by (3.13) we get $s_i = 0$. Putting it in (3.10) yields $r_{ij} = 0$. On the other hand, the S-curvature of an (α,β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, on an n-dimensional manifold M is given by

$$\mathbf{S} = \Big[2\Psi - \frac{f'(b)}{bf(b)}\Big](r_0 + s_0) - \frac{\Phi}{2\alpha\Delta^2}(r_{00} - 2\alpha Q s_0), \tag{3.16}$$

where

$$\begin{aligned} f(b) &:= \frac{\int_0^\pi \sin^{n-2} t\ T(b\cos t)dt}{\int_0^\pi \sin^{n-2} tdt}, \\ T(s) &:= \phi(\phi - s\phi')^{n-2}\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]. \end{aligned}$$

The relation (3.16) is independent of dimension of manifold. By putting $s_i = r_{ij} = 0$ in (3.16), we get $\mathbf{S} = 0$. □

Proof of Theorem 1.1: By Proposition 3.1, every weakly Landsberg cubic metric on a manifold M of dimension $n \geq 3$ satisfies $\mathbf{S} = 0$. In [3], Cheng-Shen proved that an (α,β)-metric $F := \alpha\phi(s)$, $s = \beta/\alpha$, satisfies $\mathbf{S} = 0$ if and only if $r_{ij} = s_i = 0$. In [7], Li-Shen considered weakly Landsberg (α,β)-metric and proved that a weakly Landsberg (α,β)-metric satisfies $s_{ij} = 0$. Then β is parallel with respect to α and F is reduced to a Berwald metric. Thus Theorem 1.1 is proved for the $dim(M) = n \geq 3$.

Now, we consider the class of 2-dimensional cubic (α,β)-metrics. Every 2-dimensional Finsler manifold is C-reducible

$$C_{ijk} = \frac{1}{3}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\}. \tag{3.17}$$

Taking a horizontal derivation of (3.17) yields

$$L_{ijk} = \frac{1}{3}\Big\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\Big\}. \tag{3.18}$$

Putting $\mathbf{J} = 0$ in (3.18) implies that $\mathbf{L} = 0$. On the other hand, the Berwald curvature of 2-dimensional Finsler manifold can be written as follows

$$B^i{}_{jkl} = -\frac{2}{F}L_{jkl}\ell^i + \frac{2}{3}\Big\{E_{jk}h^i_l + E_{kl}h^i_j + E_{jl}h^i_k\Big\}. \tag{3.19}$$

For more details see formula (3.15) in [22]. By Putting $\mathbf{L} = 0$ and $\mathbf{E} = 0$ in (3.19) it follows that F is a Berwald metric. This completes the proof. □

Proof of Corollary 1.1: In [5], Kim-Park proved that the cubic metric $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$ is a Berwald metric if and only if (1.1) holds. Also, in [8] Matsumoto showed that every cubic Finsler metric with vanishing Landsberg curvature is a Berwald metric. Then by Theorem 1.1, we get the proof. □

Proof of Corollary 1.2: Let $F = \sqrt[3]{A}$ be a cubic metric on M, where A is given by

$$A := a_{ijk}(x)y^iy^jy^k \tag{3.20}$$

with a_{ijk} symmetric in all its indices. Put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^k} = \frac{\partial A}{\partial x^k}, \quad A_0 = A_{x^k}y^k, \quad A_{0j} = A_{x^ky^j}y^k.$$

Suppose that (A_{ij}) is a positive definite tensor and (A^{ij}) denotes its inverse. Then the following hold

$$g_{ij} = \frac{1}{9}A^{-\frac{4}{3}}\mathbb{A}_{ij}, \quad g^{ij} = A^{-\frac{2}{3}}\mathbb{A}^{ij}, \quad y_i = \frac{1}{3}A^{-\frac{1}{3}}A_i,$$

where

$$\mathbb{A}_{ij} := 3AA_{ij} - A_iA_j, \qquad \mathbb{A}^{ij} := 3AA^{ij} + \frac{1}{2}y^iy^j.$$

The Cartan tensor of F is given by the following

$$C_{ijk} = \frac{1}{3}A^{-\frac{7}{3}}\mathbb{C}_{ijk}, \tag{3.21}$$

where

$$\mathbb{C}_{ijk} := A^2A_{ijk} + \frac{4}{9}A_iA_jA_k - \frac{1}{3}A\Big\{A_iA_{jk} + A_jA_{ki} + A_kA_{ij}\Big\}.$$

Thus the mean Cartan torsion is as follows

$$I_k = g^{ij}C_{ijk} = \frac{1}{3}A^{-3}\mathbb{A}^{ij}\mathbb{C}_{ijk}. \tag{3.22}$$

In [28], Yu-You found that the spray coefficients of F are given by

$$G^i = \frac{1}{2}(A_{0j} - A_{x^j})A^{ij}. \tag{3.23}$$

It is easy to see that G^i are rational functions in y. Since $L_{ijk} = -\frac{1}{2}y_sG^s{}_{y^iy^jy^k}$, then we have

$$L_{ijk} = -\frac{1}{6}A^{-\frac{1}{3}}A_sG^s{}_{y^iy^jy^k}.$$

Therefore, we have

$$J_k = g^{ij}L_{ijk} = -\frac{1}{6}A^{-1}\mathbb{A}^{ij}A_sG^s{}_{y^iy^jy^k}. \tag{3.24}$$

Since F has relatively isotropic mean Landsberg curvature $\mathbf{J} = cF\mathbf{I}$, then by (3.22), (3.24) and $F = \sqrt[3]{A}$ we get

$$A^2 A_s G^s{}_{y^i y^j y^k} = -2c\sqrt[3]{A}\mathbb{C}_{ijk}. \tag{3.25}$$

The left hand side of (3.25) is a rational function in y, while its right hand side is an irrational function in y. Thus $c = 0$ and F is reduced to a weakly Landsberg metric. By Theorem 1.1, we get the proof. □

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