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GENERALIZED CAUCHY-RIEMANN SCREEN PSEUDO-SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE SASAKIAN MANIFOLDS

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Abstract. This paper deals with the study of generalized Cauchy-Riemann (in short, GCR) screen pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds giving a characterization theorem with some non-trivial examples of such submanifolds. Integrability conditions of distributions D_1, D_2, D'_1 and D'_2 on GCR screen pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds have been obtained. Furthermore, we obtain necessary and sufficient conditions for foliations determined by the above distributions to be totally geodesic.
Keywords: Radical distribution, screen distribution, lightlike transversal vector bundle, screen transversal vector bundle, pseudo-slant lightlike submanifolds, Gauss and Weingarten formulae.

1. Introduction

Riemannian and semi-Riemannian geometries have been active as well as interesting areas of research work in differential geometry. In the present time, the geometry of manifolds and their submanifolds used by most of the branches of mathematics and physics. Indeed, lightlike geometry has its applications in general relativity as some smooth parts of event horizons of the Kruskal-Kerr black holes. As we know, the properties of a manifold depend on the metric which depened on it. We study manifolds with positive definite metric in Riemannian geometry.

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A lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is a submanifold on which the induced metric is degenerate.

Actually, the theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu in 1996 (see [13]). Moreover, Chen et al. defined bi-slant submanifolds in Kaehler manifolds (see [21]). In (see [11], [12]) the geometry of slant and semi-slant submanifolds of Kaehler manifolds was studied by Cabrerizo and his coauthors. A. Lotta coined the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold.

Furthermore, A. Carriazo introduced bi-slant submanifolds of almost Hermitian and almost contact metric manifolds and later the notion of pseudo-slant submanifolds was generalized by him. Recently in 1994, the notion of slant submanifold by semi-slant submanifolds of Kaehler manifolds was extended by N. Papaghuic (see [18]). In (see [2], [4]), Sahin studied the geometry of slant and screen-slant lightlike submanifolds of indefinite Hermitian manifolds. The theory of slant, CR lightlike submanifolds, SCR lightlike submanifolds of indefinite Kaehler manifolds and indefinite Sasakian manifolds has been studied in (see [13], [14]).

The main objective of the present paper is to introduce the notion of GCR screen pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds. The paper is arranged as follows : In section 2, we collected some useful basic results and formulaes. Section 3 focused on the study GCR screen pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold with example. Section 4 is dedicated to the study of foliations determined by distributions on GCR screen pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds.

2. Preliminaries

The notation and formulas used in this paper are followed by (see [12]). A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called a lightlike submanifold in which induced metric g from $\bar{g}$ is degenerate and the rank of radical distribution $Rad(TM)$ is r, where $1 \leq r \leq m$. $RadTM = TM \cap TM^{\perp}$, where

$$TM^{\perp} = \bigcup_{x \in M} \{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \ \forall v \in T_xM\}.$$

Now we consider the tangent bundle TM splits orthogonally into a non-degenerate distribution $S(TM)$, called screen distribution and a degenerate (radical) distribution $Rad(TM)$, called radical distribution, i.e.

$$TM = Rad(TM) \oplus_{orth} S(TM). \tag{2.1}$$

Let $S(TM^{\perp})$ be a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$, called screen transversal vector bundle. Since for any local basis $\{\xi_i\}$ of $Rad(TM)$ there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle

$ltr(TM)$ locally spanned by $\{N_i\}$. Now suppose that the transversal bundle $tr(TM)$ splits orthogonally into lightlike transversal vector bundle $ltr(TM)$ and screen transversal vector bundle $S(TM^{\perp})$,

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}). \tag{2.2}$$

Then, $tr(TM)$ is a complementary which is not orthogonal vector bundle to TM in $T\bar{M}|_M$, i.e.

$$T\bar{M}|_M = TM \oplus tr(TM). \tag{2.3}$$

and therefore

$$T\bar{M}|_M = S(TM) \oplus_{orth} [Rad(TM) \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}). \tag{2.4}$$

Although $S(TM)$ is not unique but it is canonically isomorphic to the factor vector bundle $TM/RadTM$ (see [13]).

Following result is important to this paper.

Proposition 2.1. [14] *The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM)$, $S(TM^{\perp})$ and $ltr(TM)$.*

Following this, we say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is
Case 1: r-lightlike if $r \leqslant min(m, n)$,
Case 2: co-isotropic if $r = n \leqslant m,\ S(TM^{\perp}) = \{0\}$,
Case 3: isotropic if $r = m \leqslant n,\ S(TM) = \{0\}$,
Case 4: totally lightlike if $r = m = n,\ S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V. \tag{2.6}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_V X\}$ belong to $\Gamma(TM)$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(tr(TM))$. Here, ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$ respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.5) and (2.6) we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall\ X, Y \in \Gamma(TM). \tag{2.7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \forall\ N \in \Gamma(ltr(TM)). \tag{2.8}$$

$$\bar{\nabla}_X W = -A_W X + D^l(X, W) + \nabla^s_X W, \quad \forall\ W \in \Gamma(S(TM^{\perp})). \tag{2.9}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l(X,W) = L(\nabla^t_X W)$, $D^s(X,N) = S(\nabla^t_X N)$, L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ respectively. Thus h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^\perp))$-valued lightlike second fundamental form and screen second fundamental form of M respectively. On the other hand, ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^\perp)$ called the lightlike connection and screen transversal connection on M respectively. Now by using (2.5), (2.7)-(2.9) and metric connection $\bar{\nabla}$, we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y,D^l(X,W)) = g(A_W X), \tag{2.10}$$

$$\bar{g}(D^s(X,N),W) = g(N,A_W X), \tag{2.11}$$

Suppose $\bar{P}$ is the projection of TM on $S(TM)$. Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X,\bar{P}Y), \quad \forall\ X,Y \in \Gamma(TM), \tag{2.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \quad \xi \in \Gamma(Rad(TM)), \tag{2.13}$$

where $\{\nabla^*_X \bar{P}Y, -A^*_\xi X\}$ and $\{h^*(X,\bar{P}Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively. It follows that ∇^* and ∇^{*t} are linear connections on $S(TM)$ and $Rad(TM)$ respectively. On the other hand, h^* and A^* are called the second fundamental forms of distributions $S(TM)$ and $Rad(TM)$ respectively, which are $\Gamma(Rad(TM))$-valued and $\Gamma(S(TM))$-valued $F(M)$-bilinear forms on $\Gamma(TM) \times \Gamma(S(TM))$ and $\Gamma(Rad(TM)) \times \Gamma(TM)$ respectively. Now by using the above equations, we obtain

$$\bar{g}(h^l(X,\bar{P}Y),\xi) = g(A^*_\xi X, \bar{P}Y), \tag{2.14}$$

$$\bar{g}(h^*(X,\bar{P}Y),N) = g(A_N X, \bar{P}Y), \tag{2.15}$$

$$\bar{g}(h^l(X,\xi),\xi) = 0, \ A^*_\xi \xi = 0. \tag{2.16}$$

Here the induced connection ∇ on M is not a metric connection in general. Since $\bar{\nabla}$ is a metric connection, by using (2.7) we get

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y). \tag{2.17}$$

An odd dimensional semi-Riemannian manifold $(\bar{M},\bar{g})$ is called an ϵ-almost contact metric manifold if there exists a $(1,1)$ tensor field ϕ, a vector field V called characteristic vector field and a 1-form η, satisfying

$$\phi^2 X = -X + \eta(X)V, \ \eta(V) = \epsilon, \ \eta o \phi = 0, \ \phi V = 0, \tag{2.18}$$

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.19}$$

for all $X, Y \in \Gamma(T\bar{M})$, where $\epsilon = 1$ or -1. It follows that

$$\bar{g}(V, V) = \epsilon, \tag{2.20}$$

$$\bar{g}(X, V) = \eta(X), \tag{2.21}$$

$$\bar{g}(X, \phi Y) = \bar{g}(\phi X, Y), \ \forall X, Y \in \Gamma(T\bar{M}). \tag{2.22}$$

Then $(\phi, V, \eta, \bar{g})$ is called an ϵ-almost contact metric structure on $\bar{M}$. An ϵ-almost contact metric structure $(\phi, V, \eta, \bar{g})$ is called an indefinite Sasakian structure if and only if

$$(\bar{\nabla}_X \phi)Y = \bar{g}(X, Y)V - \epsilon\eta(Y)X, \tag{2.23}$$

for all $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ is Levi-Civita connection with respect to $\bar{g}$.

An indefinite Sasakian manifold is a semi-Riemannian manifold endowed with an indefinite Sasakian structure. From (2.23), for any $X \in \Gamma(T\bar{M})$, we get

$$\bar{\nabla}_X V = -\phi X. \tag{2.24}$$

Suppose $(\bar{M}, \bar{g}, \phi, \eta, V)$ be an ϵ-almost contact metric manifold. If $\epsilon = 1$, then $\bar{M}$ is said to be a spacelike ϵ-almost contact metric manifold and if $\epsilon = -1$, then $\bar{M}$ is called a timelike ϵ-almost contact metric manifold. In this paper, we consider indefinite Sasakian manifolds with spacelike characteristic vector field V.

3. Generalized Cauchy-Riemann Screen Pseudo-slant Lightlike Submanifolds

The notion of Generalized Cauchy-Riemann (GCR) screen pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds is introduced in this section. At first, we state the following lemma which was proved by Sahin ([14]). We shall use this lemma in defining the notion of GCR screen pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds.

Lemma 3.1. [14] *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Suppose that there exists a screen distribution $S(TM)$ such that $\phi Rad(TM) \subset S(TM)$ and $\phi ltr(TM) \subset S(TM)$. Then $\phi Rad(TM) \cap \phi ltr(TM) = \{0\}$ and any complementry distribution to $\phi Rad(TM) \oplus \phi ltr(TM)$ in $S(TM)$ is Riemannian.*

The proof of above lemma follows as in Lemma 4.1 of [4], so we omit it.

Definition 3.1. [14] Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$ such that $2q < dim(M)$. Then we say that M is a generalized Cauchy-Riemann screen pseudo-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:
(i) $\phi Rad(TM)$ is a distribution on M such that

$$Rad(TM) = D_1 \oplus D_2,$$

where $\phi D_1 \subset S(TM)$ and $\phi D_2 \subset S(TM^{\perp})$. Furthermore, we have $ltr(TM) = L_1 \oplus L_2$ where $\phi L_1 \subset S(TM)$ and $\phi L_2 \subset S(TM^{\perp})$,
(ii) there exist non-degenerate orthogonal distributions D_1' and D_2' on M such that

$$S(TM) = (\phi D_1 \oplus \phi L_1) \oplus_{orth} D_1' \oplus_{orth} D_2' \oplus_{orth} \{V\},$$

where L_1 is a distribution of $ltr(TM)$,
(iii) the distribution D_1' is anti-invariant, i,e. $\phi D_1' \subset S(TM^{\perp})$,
(v) the distribution D_2' is slant with angle θ, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2')_x$. This slant angle θ between ϕX and the vector subspace $(D_2')_x$ is a non-zero constant, which is not depend on the choice of $x \in M$ and $X \in (D_2')_x$.

A GCR screen pseudo-slant lightlike submanifold is said to be proper if $D_1' \neq \{0\}$; $D_2' \neq \{0\}$ and $\theta \neq \pi/2$.

From the above definition, we have the following decomposition:

$$TM = Rad(TM) \oplus_{orth} (\phi D_1 \oplus \phi L_1) \oplus_{orth} D_1' \oplus_{orth} D_2' \oplus_{orth} \{V\}.$$

Let $(\mathbb{R}^{2m+1}_{2q}, \bar{g}, \phi, \eta, V)$ denote the manifold $\mathbb{R}^{2m+1}_{2q}$ with its usual Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i \partial x^i), \quad V = 2\partial z,$$

$$\bar{g} = \eta \otimes \eta + \frac{1}{4}(-\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\phi(\sum_{i=1}^{m}(X_i dx_i + Y_i dy_i) + Z\partial z) = \sum_{i=1}^{m}(Y_i \partial x_i - X_i \partial y_i) + \sum_{i=1}^{m} Y_i y^i \partial z,$$

where (x^i, y^i, z) are the cartesian coordinates on $\mathbb{R}^{2m+1}_{2q}$. Now, we construct some examples of GCR screen pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold.

Example 3.1. Let $(\mathbb{R}^{17}_4, \bar{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-,-,+,+,+,+,+,+,-,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial y_1, \partial y_2, \partial y_3 . \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial y_8, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}^{17}_4$ given by $x^1 = -\cos u_2$, $y^1 = \sin u_2$,

$x^2 = u_1$, $y^2 = u_3 - \dfrac{u_4}{2}$, $x_3 = u_2$, $y_3 = 0$, $x^4 = u_1$, $y^4 = u_3 + \dfrac{u_4}{2}$, $x^5 = y^6 = u_5$, $y^5 = x^6 = u_6$, $x^7 = u_7$, $y^7 = u_8$, $x^8 = k \sin u_8$, $y^8 = k \cos u_8$, $z = u_9$ where k is a constant. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$, where

$$\begin{aligned}
Z_1 &= 2(\partial x_2 + \partial x_4 + y^2 \partial z + y^4 \partial z), \\
Z_2 &= 2(\sin u_2 \partial x_1 + \cos u_2 \partial y_1 + \partial x_3 + y^1 \sin u_2 \partial z + y^3 \partial z), \\
Z_3 &= 2(\partial y_2 + \partial y_4), \\
Z_4 &= (-\partial y_2 + \partial y_4), \\
Z_5 &= 2(\partial x_5 + \partial y_6 + y^5 \partial z), \\
Z_6 &= 2(\partial x_6 + \partial y_5 + y^6 \partial z), \\
Z_7 &= 2(\partial x_7 + y^7 \partial z), \\
Z_8 &= 2(\partial y_7 + k \cos u_8 \partial x_8 - k \sin u_8 \partial y_8 + y^8 k \cos u_8 \partial z), \\
Z_9 &= 2(\partial z) = V.
\end{aligned}$$

Hence $Rad(TM) = span\{Z_1, Z_2\}$. Also it is easy to see that $D_1 = span\{Z_1\}$ and $D_2 = span\{Z_2\}$, where $\phi Z_1 = -Z_3 \in \Gamma(S(TM))$ and $\phi Z_2 = W_2 \in \Gamma(S(TM^\perp))$. Moreover $S(TM) = span\{Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$, where we can see that $D_1' = span\{Z_5, Z_6\}$ such that $\phi Z_5 = W_3$, $\phi Z_6 = W_4$, which implies that D_1' is anti-invariant with respect to ϕ. Also $D_2' = span\{Z_7, Z_8\}$ is slant distribution with slant angle $\theta = \cos^{-1}(1/\sqrt{1+k^2})$. On the other hand the lightlike transversal bundle $ltr(TM)$ is spanned by

$$\begin{aligned}
N_1 &= (-\partial x_2 + \partial x_4 - y^2 \partial z + y^4 \partial z), \\
N_2 &= (-\sin u_2 \partial x_1 - \cos u_2 \partial y_1 + \partial x_3 - y^1 \sin u_2 \partial z + y^3 \partial z).
\end{aligned}$$

From this we have $ltr(TM) = span\{N_1, N_2\}$, where $L_1 = span\{N_1\}$ and $L_2 = span\{N_2\}$. Here $\phi N_1 = -Z_4 \in \Gamma(S(TM))$ and $\phi N_2 = W_1 \in \Gamma(S(TM^\perp))$. Also $S(TM^\perp)$ is spanned by

$$\begin{aligned}
W_1 &= 2(\cos u_2 \partial x_1 - \sin u_2 \partial y_1 - \partial y_3 + y^1 \cos u_2 \partial z), \\
W_2 &= 2(-\cos u_2 \partial x_1 + \sin u_2 \partial y_1 - \partial y_3 - y^1 \cos u_2 \partial z), \\
W_3 &= 2(\partial x_6 - \partial y_5 + y^6 \partial z), \\
W_4 &= 2(\partial x_5 - \partial y_6 + y^5 \partial z), \\
W_5 &= 2(u_7 \cos u_8 \partial x_8 + u_7 \sin u_8 \partial y_8 + y^8 u_7 \cos u_8 \partial z), \\
W_6 &= 2(k^2 \partial y_7 + k \cos u_8 \partial x_8 - k \sin u_8 \partial y_8 + y^8 k \cos u_8 \partial z).
\end{aligned}$$

Hence M is a proper GCR screen pseudo-slant 2-lightlike submanifold of $\mathbb{R}^{17}_4$.

Example 3.2. Let $(\mathbb{R}^{17}_4, \bar{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-,-,+,+,+,+,+,+,-,+,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial y_8, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}^{17}_4$ given by $x^1 = \sin u_2$, $y^1 = -\cos u_2$, $x^2 = u_1$, $y^2 = u_3 - \dfrac{u_4}{2}$, $x_3 = u_2$, $y_3 = 0$, $x^4 = u_1$, $y^4 = u_3 + \dfrac{u_4}{2}$, $x^5 = y^6 = u_5$, $y^5 = x^6 = u_6$, $x^7 = u_7$, $y^7 = u_8$, $x^8 = k \cos u_8$, $y^8 = k \sin u_8$, where k is a constant. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9\}$, where

$$\begin{aligned}
Z_1 &= 2(\partial x_2 + \partial x_4 + y^2\partial z + y^4\partial z),\\
Z_2 &= 2(\cos u_2\partial x_1 + \sin u_2\partial y_1 + \partial x_3 + y^1\cos u_2\partial z + y^3\partial z),\\
Z_3 &= 2(\partial y_2 + \partial y_4),\\
Z_4 &= (-\partial y_2 + \partial y_4),\\
Z_5 &= 2(\partial x_5 + \partial y_6 + y^5\partial z),\\
Z_6 &= 2(\partial x_6 + \partial y_5 + y^6\partial z),\\
Z_7 &= 2(\partial x_7 + y^7\partial z),\\
Z_8 &= 2(\partial y_7 - k\sin u_8\partial x_8 + k\cos u_8\partial y_8 - y^8 k\sin u_8\partial z),\\
Z_9 &= 2(\partial z) = V.
\end{aligned}$$

Hence $Rad(TM) = span\{Z_1, Z_2\}$. Also it is easy to see that $D_1 = span\{Z_1\}$ and $D_2 = span\{Z_2\}$, where $\phi Z_1 = -Z_3 \in \Gamma(S(TM))$ and $\phi Z_2 = W_2 \in \Gamma(S(TM^\perp))$. Moreover $S(TM) = span\{Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where we can see that $D_1' = span\{Z_5, Z_6\}$ such that $\phi Z_5 = W_3$, $\phi Z_6 = W_4$, which implies that D_1' is anti-invariant with respect to ϕ. Also $D_2' = span\{Z_7, Z_8\}$ is slant distribution with slant angle $\theta = \cos^{-1}(1/\sqrt{1+k^2})$. On the other hand the lightlike transversal bundle $ltr(TM)$ is spanned by

$$\begin{aligned}
N_1 &= (-\partial x_2 + \partial x_4 - y^2\partial z + y^4\partial z),\\
N_2 &= (-\cos u_2\partial x_1 - \sin u_2\partial y_1 + \partial x_3 - y^1\cos u_2\partial z + y^3\partial z).
\end{aligned}$$

From this we have $ltr(TM) = span\{N_1, N_2\}$, where $L_1 = span\{N_1\}$ and $L_2 = span\{N_2\}$. Here $\phi N_1 = -Z_4 \in \Gamma(S(TM))$ and $\phi N_2 = W_1 \in \Gamma(S(TM^\perp))$. Also $S(TM^\perp)$ is spanned by

$$\begin{aligned}
W_1 &= 2(-\sin u_2\partial x_1 + \cos u_2\partial y_1 - \partial y_3 - y^1\sin u_2\partial z),\\
W_2 &= 2(\sin u_2\partial x_1 - \cos u_2\partial y_1 - \partial y_3) + y^1\sin u_2\partial z,\\
W_3 &= 2(\partial x_6 - \partial y_5 + y^6\partial z),\\
W_4 &= 2(\partial x_5 - \partial y_6 + y^5\partial z),\\
W_5 &= 2(u_7\cos u_8\partial x_8 + u_7\sin u_8\partial y_8 + y^8 u_7\cos u_8\partial z),\\
W_6 &= 2(k^2\partial y_7 - k\sin u_8\partial x_8 + k\cos u_8\partial y_8 - y^8 k\sin u_8\partial z).
\end{aligned}$$

Hence M is a proper GCR screen pseudo-slant 2-lightlike submanifold of $\mathbb{R}^{17}_4$.

Now, for any vector field X tangent to M, we put

$$\phi X = PX + FX, \tag{3.1}$$

where PX and FX are the tangential and transversal parts of ϕX respectively. We denote the projections on D_1, D_2, ϕD_1, ϕL_1, D_1' and D_2' in TM by P_1, P_2, P_3, P_4, P_5 and P_6 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ by Q and R respectively. Thus, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + P_4X + P_5X + P_6X + \eta(X)V, \tag{3.2}$$

Now applying ϕ to (3.2), we have

$$\phi X = \phi P_1X + \phi P_2X + \phi P_3X + \phi P_4X + \phi P_5X + \phi P_6X, \tag{3.3}$$

which gives

$$\phi X = \phi P_1X + \phi P_2X + \phi P_3X + \phi P_4X + \phi P_5X + fP_6X + FP_6X, \tag{3.4}$$

where fP_6X and FP_6X denotes the tangential and transversal component of ϕP_6X. Thus we get $\phi P_1X \in \Gamma(S(TM))$, $\phi P_2X \in \Gamma(\phi D_2) \subset \Gamma(S(TM^{\perp}))$, $\phi P_3X \in \Gamma(D_1)$, $\phi P_4X \in \Gamma(L_1) \subset \Gamma(ltr(TM))$, $\phi P_5X \in \Gamma(D_1')$, $fP_6X \in \Gamma(D_2')$, $FP_6X \in \Gamma(CR_4W) \subset \Gamma(S(TM^{\perp})$. Also, for any $W \in \Gamma(tr(TM))$, we have

$$W = QW + RW, \tag{3.5}$$

Applying ϕ to (3.5), we obtain

$$\phi W = \phi QW + \phi RW, \tag{3.6}$$

which gives

$$\phi W = \phi Q_1W + \phi Q_2W + \phi R_1W + \phi R_2W + \phi R_3W + BR_4W + CR_4W, \tag{3.7}$$

where, BR_3W (resp. CR_3W) denotes the tangential (resp. transversal) component of ϕR_3W. Thus we get $\phi Q_1W \in \Gamma(S(TM))$, $\phi Q_2W \in \Gamma(S(TM^{\perp}))$, $\phi R_1W \in \Gamma(D_2)$, $\phi R_2W \in \Gamma(L_2)$, $\phi R_3W \in \Gamma(D_1')$ $BR_4W \in \Gamma(D_2')$, $CR_4W \in \Gamma(S(TM^{\perp}))$. Now, by using (2.23), (3.4), (3.7) and (2.7)-(2.9) and identifying the components on D_1, D_2, ϕD_1, ϕL_1, D_1', D_2', $ltr(TM)$ and $S(TM^{\perp})$, we obtain

(3.8)
$$P_1(\nabla_X\phi P_1Y) + P_1(\nabla_X\phi P_3Y) + P_1(\nabla_X fP_6Y) = P_1(A_{\phi P_2Y}X) + P_1(A_{\phi P_4Y}X) + P_1(A_{\phi P_5Y}X) + P_1(A_{FP_6Y}X) + \phi P_3\nabla_XY - \eta(Y)P_1X,$$

(3.9)
$$P_2(\nabla_X\phi P_1Y) + P_2(\nabla_X\phi P_3Y) + P_2(\nabla_X fP_6Y) = P_2(A_{\phi P_2Y}X) + P_2(A_{\phi P_4Y}X) + P_2(A_{\phi P_5Y}X) + P_2(A_{FP_6Y}X) + \phi R_1h^s(X,Y) - \eta(Y)P_2X,$$

(3.10)
$$P_3(\nabla_X\phi P_1Y) + P_3(\nabla_X\phi P_3Y) + P_3(\nabla_X fP_6Y) = P_3(A_{\phi P_2Y}X) + P_3(A_{\phi P_4Y}X) + P_3(A_{\phi P_5Y}X) + P_3(A_{FP_6Y}X) + \phi P_1\nabla_XY - \eta(Y)P_3X,$$

(3.11)
$$P_4(\nabla_X\phi P_1Y) + P_4(\nabla_X\phi P_3Y) + P_4(\nabla_X fP_6Y) = P_4(A_{\phi P_2Y}X) + P_4(A_{\phi P_4Y}X) + P_4(A_{\phi P_5Y}X) + P_4(A_{FP_6Y}X) + \phi Q_1h^l(X,Y) - \eta(Y)P_4X,$$

(3.12)
$$P_5(\nabla_X\phi P_1Y) + P_5(\nabla_X\phi P_3Y) + P_5(\nabla_X fP_6Y) = fP_6\nabla_XY + P_5(A_{\phi P_2Y}X) + P_5(A_{\phi P_4Y}X) + P_5(A_{\phi P_5Y}X) + P_5(A_{FP_6Y}X) + \phi R_3h^s(X.Y) - \eta(Y)P_5X,$$

(3.13)
$$P_6(\nabla_X \phi P_1 Y) + P_6(\nabla_X \phi P_3 Y) + P_6(\nabla_X f P_6 Y) = P_6(A_{\phi P_2 Y} X) + P_6(A_{\phi P_4 Y} X) + P_6(A_{\phi P_5 Y} X) + P_6(A_{F P_6 Y} X) + B R_4 h^s(X, Y) - \eta(Y) P_6 X,$$

(3.14)
$$Q_1 h^l(X, \phi P_1 Y) + Q_1 D^l(X, \phi P_2 Y) + Q_1 h^l(X, \phi P_3 Y) + Q_1 h^l(X, f P_6 Y) = \phi P_4 \nabla_X Y + \phi P_4 h^l(X, Y) - Q_1 \nabla^l_X \phi P_4 Y - Q_1 D^l(X, \phi P_5 Y) - Q_1 D^l(X, F P_6 Y),$$

(3.15)
$$Q_2 h^l(X, \phi P_1 Y) + Q_2 D^l(X, \phi P_2 Y) + Q_2 h^l(X, \phi P_3 Y) + Q_2 h^l(X, f P_6 Y) = Q_2 \nabla^l_X \phi P_4 Y - Q_2 D^l(X, \phi P_5 Y) - Q_2 D^l(X, F P_6 Y),$$

(3.16)
$$R_1 h^s(X, \bar{J} P_1 Y) + R_1 h^s(X, \bar{J} P_3 Y) + R_1 D^s(X, \bar{J} P_4 Y) + R_1 h^s(X, f P_6 Y) = \bar{J} P_2 \nabla_X Y - R_1 \nabla^s_X \bar{J} P_2 Y - R_1 \nabla^s_X F P_6 Y - R_1 \nabla^s_X \bar{J} P_5 Y,$$

(3.17)
$$R_2 h^s(X, \bar{J} P_1 Y) + R_2 h^s(X, \bar{J} P_3 Y) + R_2 D^s(X, \bar{J} P_4 Y) + R_2 h^s(X, f P_6 Y) = \bar{J} Q_2 h^l(X, Y) - R_2 \nabla^s_X \bar{J} P_2 Y - R_2 \nabla^s_X F P_6 Y - R_2 \nabla^s_X \bar{J} P_5 Y,$$

(3.18)
$$R_3 h^s(X, \phi P_1 Y) + R_3 h^s(X, \phi P_3 Y) + R_3 D^s(X, \phi P_4 Y) + R_3 h^s(X, f P_6 Y) = \phi P_5 \nabla_X Y - R_3 \nabla^s_X \phi P_2 Y - R_3 \nabla^s_X F P_6 Y - R_3 \nabla^s_X \phi P_5 Y,$$

(3.19)
$$R_4 h^s(X, \phi P_1 Y) + R_4 h^s(X, \phi P_3 Y) + R_4 D^s(X, \phi P_4 Y) + R_4 h^s(X, f P_6 Y) = F P_6 \nabla_X Y - R_4 \nabla^s_X \phi P_2 Y - R_4 \nabla^s_X F P_6 Y - R_4 \nabla^s_X \phi P_5 Y,$$

(3.20)
$$\eta(\nabla_X \phi P_1 Y) + \eta(\nabla_X \phi P_3 Y) + \eta(\nabla_X f P_6 Y) = \eta(A_{\phi P_2 Y} X) + \eta(A_{\phi P_4 Y} X) + \eta(A_{\phi P_5 Y} X) + \eta(A_{F P_6 Y} X) + \bar{g}(X, Y) V.$$

Theorem 3.1. *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Then M is a GCR screen pseudo-slant lightlike submanifold if and only if*
(i) there exists degenerate orthogonal distributions L_1 and L_2 such that $ltr(TM) = L_1 \oplus L_2$ where $\phi L_1 \subset S(TM)$ and $\phi L_2 \subset S(TM^\perp)$,
(ii) the distribution D'_1 is anti-invariant, i.e. $\phi D'_1 \subset S(TM^\perp)$,
(iii) there exists a constant $\lambda \in [0, 1)$ such that $P^2 X = -\lambda X$, for all $X \in \Gamma(D'_2)$, where $\lambda = cos^2\theta$ and θ is the slant angle of D'_2.

Proof. Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the distribution D'_1 is anti-invariant with respect to ϕ and $Rad(TM)$ is a distribution on M such that $Rad(TM) = D_1 \oplus D_2$, where

$\phi D_1 \subset S(TM)$ and $\phi D_2 \subset S(TM^{\perp})$. Thus $ltr(TM) = L_1 \oplus L_2$, where $\phi L_1 \subset S(TM)$ and $\phi L_2 \subset S(TM^{\perp})$. Therefore for any $X \in \Gamma(L_1)$, $\phi X \in \Gamma S(TM)$. Hence $\phi(\phi X) \in \Gamma(L_1)$, which implies $-X \in \Gamma(L_1)$, which proves (i) and (ii).

Now, for any $X \in \Gamma(D'_1)$ we have $|PX| = |\phi X| \cos\theta$, i.e.

$$\cos\theta = \frac{|PX|}{|\phi X|}. \tag{3.21}$$

In view of (3.21), we get $\cos^2\theta = \dfrac{|PX|^2}{|\phi X|^2} = \dfrac{g(PX, PX)}{g(\phi X, \phi X)} = \dfrac{g(X, P^2X)}{g(X, \phi^2 X)}$, which gives

$$g(X, P^2X) = \cos^2\theta g(X, \phi^2 X). \tag{3.22}$$

Since M is a GCR screen pseudo-slant lightlike submanifold, $\cos^2\theta = \lambda$(constant) $\in [0, 1)$ and therefore from (3.22) we get $g(X, P^2X) = \lambda g(X, \phi^2X) = g(X, \lambda\phi^2X)$, for all $X \in \Gamma(D'_1)$, which implies

$$g(X, (P^2 - \lambda\phi^2)X) = 0 \tag{3.23}$$

Since $(P^2 - \lambda\phi^2)X \in \Gamma(D'_1)$ and the induced metric $g = g|_{D'_1 \times D'_1}$ is non-degenerate (positive definite). From (3.23) we have $(P^2 - \lambda\phi^2)X = 0$, which implies

$$P^2X = \lambda\phi^2X = -\lambda X, \ \forall X \in \Gamma(D'_1). \tag{3.24}$$

This proves (iii).

Conversely, suppose that conditions (i), (ii) and (iii) are satisfied. From (iii), we have $P^2X = \lambda\phi^2X, \ \forall X \in \Gamma(D'_1)$, where $\lambda \in [0, 1)$.

Now $\cos\theta = \dfrac{g(\phi X, PX)}{|\phi X||PX|} = -\dfrac{g(X, \phi PX)}{|\phi X||PX|} = -\dfrac{g(X, P^2X)}{|\phi X||PX|} = -\lambda\dfrac{g(X, \phi^2X)}{|\phi X||PX|} = \lambda\dfrac{g(\phi X, \phi X)}{|\phi X||PX|}$.

From the above equation, we obtain

$$\cos\theta = \lambda\frac{|\phi X|}{|PX|}. \tag{3.25}$$

Therefore (3.21) and (3.25) give $\cos^2\theta = \lambda$(constant). □

Theorem 3.2. *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Then M is a GCR screen pseudo-slant lightlike submanifold if and only if*
(i) there exist degenerate orthogonal distributions L_1 and L_2 such that $ltr(TM) = L_1 \oplus L_2$ where $\phi L_1 \subset S(TM)$ and $\phi L_2 \subset S(TM^{\perp})$,

(ii) the distribution D_1' is anti-invariant, i.e. $\phi D_1' \subset S(TM^\perp)$,
(iii) there exists a constant $\lambda \in (0,1]$ such that $P^2X = -\lambda X$.

Moreover, there exists a constant $\mu \in [0,1)$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2')$, where D_1' and D_2' are non-degenerate orthogonal distributions on M such that and $\lambda = \cos^2\theta$, θ is slant angle of D_2'.

Proof. Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the distribution D_1' is anti-invariant with respect to ϕ and $Rad(TM)$ is a distribution on M such that $Rad(TM) = D_1 \oplus D_2$, where $\phi D_1 \subset S(TM)$ and $\phi D_2 \subset S(TM^\perp)$. Thus $ltr(TM) = L_1 \oplus L_2$, where $\phi L_1 \subset S(TM)$ and $\phi L_2 \subset S(TM^\perp)$. Therefore for any $X \in \Gamma(L_1)$, $\phi X \in \Gamma S(TM)$. Hence $\phi(\phi X) \in \Gamma(L_1)$, which implies $-X \in \Gamma(L_1)$, which proves (i) and (ii).
Now, for any vector field $X \in \Gamma(D_1')$, we have

$$\phi X = PX + FX, \tag{3.26}$$

where PX and FX are the tangential and transversal parts of ϕX respectively. Applying ϕ to (3.24) and taking the tangential component, we get

$$-X = P^2X + BFX, \ \forall X \in \Gamma(D_1'). \tag{3.27}$$

Since M is a GCR screen pseudo-slant lightlike submanifold, $P^2X = -\lambda X$, $\forall X \in \Gamma(D_1')$, where $\lambda \in (0,1]$ and therefore from (3.25) we get

$$BFX = -\mu X, \ \forall X \in \Gamma(D_1'), \tag{3.28}$$

where $1-\lambda = \mu(\text{constant}) \in [0,1)$. Now, in view of Theorem 3.1, we have $\lambda = \cos^2\theta$. This proves (iii).
Conversely, assume that conditions (i), (ii) and (iii) are satisfied. From (3.24) we get

$$-X = P^2X - \mu_1 X, \ \forall X \in \Gamma(D_1'), \tag{3.29}$$

which implies

$$P^2X = -\lambda_1 X, \ \forall X \in \Gamma(D_1') \tag{3.30}$$

where $1-\mu_1 = \lambda_1(\text{constant}) \in (0,1]$. Therefore, M is a GCR screen pseudo-slant lightlike submanifold. □

Corollary 3.1. *Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then for any slant distribution D of M with slant angle θ, we have*

$$\begin{aligned} g(PX,PY) &= \cos^2\theta g((X,Y) - \eta(X)\eta(Y)), \\ g(FX,FY) &= \sin^2\theta g((X,Y) - \eta(X)\eta(Y)), \end{aligned}$$

for all $X, Y \in \Gamma(D)$.

The proof of the above corollary follows by using similar steps as in the proof of Corollary 3.1 of [4].

Theorem 3.3. *Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the distribution $D_1 \subset Rad(TM)$ is integrable if and only if*
(i) $P_1(\nabla_X \phi P_1 Y) = P_1(\nabla_Y \phi P_1 X)$ *and* $P_6(\nabla_X \phi P_1 Y) = P_6(\nabla_Y \phi P_1 X)$,
(ii) $Q_1 h^l(X, \phi P_1 Y) = Q_1 h^l(Y, \phi P_1 X)$ *and* $R_1 h^s(Y, \phi P_1 X) = R_1 h^s(X, \phi P_1 Y)$,
(iii) $R_3 h^s(Y, \phi P_1 X) = R_3 h^s(X, \phi P_1 Y)$ *and* $R_4 h^s(Y, \phi P_1 X) = R_4 h^s(X, \phi P_1 Y)$,
for all $X, Y \in \Gamma(D_1)$.

Proof. Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Suppose $X, Y \in \Gamma(D_1)$. From (3.8), we have $P_1(\nabla_X \phi P_1 Y) = \phi P_3 \nabla_X Y$, which implies $P_1(\nabla_X \phi P_1 Y) - P_1(\nabla_Y \phi P_1 X) = \phi P_3[X, Y]$. From (3.13), we have $P_6(\nabla_X \phi P_1 Y) = f P_6 \nabla_X Y + B R_4 h^s(X, Y)$, which gives $P_6(\nabla_X \phi P_1 Y) - P_6(\nabla_Y \phi P_1 X) = f P_6[X, Y]$. From (3.14), we have $Q_1 h^l(X, \phi P_1 Y) = \phi P_4 \nabla_X Y$, which gives $Q_1 h^l(X, \phi P_1 Y) - Q_1 h^l(Y, \phi P_1 X) = \phi P_4[X, Y]$.
From (3.16), we have $R_1 h^s(X, \phi P_1 Y) = \phi P_2 \nabla_X Y$, which implies $R_1 h^s(X, \phi P_1 Y) - R_1 h^s(Y, \phi P_1 X) = \phi P_2[X, Y]$. From (3.18), we have $R_3 h^s(X, \phi P_1 Y) = \phi P_5 \nabla_X Y$, which implies $R_3 h^s(X, \phi P_1 Y) - R_3 h^s(Y, \phi P_1 X) = \phi P_5[X, Y]$. From (3.19), we have $R_4 h^s(X, \phi P_1 Y) = F P_6 \nabla_X Y + C R_4 h^s(X, Y)$, which gives $R_4 h^s(X, \phi P_1 Y) - R_4 h^s(Y, \phi P_1 X) = F P_6[X, Y]$, which completes the proof. □

Theorem 3.4. *Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the distribution $D_2 \subset Rad(TM)$ is integrable if and only if*
(i) $P_1(A_{\phi P_2 Y} X) = P_1(A_{\phi P_2 X} Y)$ *and* $P_3(A_{\phi P_2 Y} X) = P_3(A_{\phi P_2 X} Y)$,
(ii) $P_6(\nabla_X \phi P_2 Y) = P_6(\nabla_Y \phi P_2 X)$ *and* $Q_1 D^l(X, \phi P_2 Y) = Q_1 D^l(Y, \phi P_2 X)$,
(iii) $R_3 \nabla^s_X \phi P_2 Y = R_3 \nabla^s_Y \phi P_2 X$ *and* $R_4 \nabla^s_X \phi P_2 Y = R_4 \nabla^s_Y \phi P_2 X$,
for all $X, Y \in \Gamma(D_2)$.

Proof. Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Suppose $X, Y \in \Gamma(D_1)$. From (3.8), we have $P_1(A_{\phi P_2 Y} X) = -\phi P_3 \nabla_X Y$, which implies $P_1(A_{\phi P_2 Y} X) - P_1(A_{\phi P_2 X} Y) = -\phi P_3[X, Y]$. From (3.10), we have $P_3(A_{\phi P_2 Y} X) = -\phi P_1 \nabla_X Y$, which gives $P_3(A_{\phi P_2 Y} X) - P_3(A_{\phi P_2 X} Y) = -\phi P_1[X, Y]$. From (3.13), we have $P_6(\nabla_X \phi P_2 Y) = f P_6 \nabla_X Y + B R_4 h^s(X, Y)$, which gives $P_6(\nabla_X \phi P_2 Y) - P_6(\nabla_Y \phi P_2 X) = f P_6[X, Y]$.
From (3.14), we have $Q_1 D^l(X, \phi P_2 Y) = -\phi P_4 \nabla_X Y + \phi P_4 h^l(X, Y)$, which implies $Q_1 D^l(X, \phi P_2 Y) - Q_1 D^l(Y, \phi P_2 X) = -\phi P_4[X, Y]$. From (3.16), we have $R_3 \nabla^s_X \phi P_2 Y = \phi P_5 \nabla_X Y$, which gives $R_3 \nabla^s_X \phi P_2 Y - R_3 \nabla^s_Y \phi P_2 X = \phi P_5[X, Y]$. From (3.17), we have $R_4 \nabla^s_X \phi P_2 Y = F P_6 \nabla_X Y$, which implies $R_4 \nabla^s_X \phi P_2 Y - R_4 \nabla^s_Y \phi P_2 X = F P_6[X, Y]$, which completes the proof. □

Theorem 3.5. *Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the distribution D'_1 is integrable if and only*

if
(i) $P_1(A_{\phi P_5 Y}X) = P_1(A_{\phi P_5 X}Y)$ *and* $P_3(A_{\phi P_5 Y}X) = P_3(A_{\phi P_5 X}Y)$,
(ii) $P_6(\nabla_X \phi P_5 Y) = P_6(\nabla_Y \phi P_5 X)$ *and* $Q_1 D^l(X, \phi P_5 Y) = Q_1 D^l(Y, \phi P_5 X)$,
(iii) $R_1 h^s(Y, \phi P_5 X) = R_1 h^s(X, \phi P_5 Y)$ *and* $R_4 \nabla^s_X \phi P_5 Y = R_4 \nabla^s_Y \phi P_5 X$,
for all $X, Y \in \Gamma(D'_1)$.

Proof. Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Suppose $X, Y \in \Gamma(D_1)$. From (3.8), we have $P_1(A_{\phi P_5 Y}X) = -\phi P_3 \nabla_X Y$, which implies $P_1(A_{\phi P_5 Y}X) - P_1(A_{\phi P_5 X}Y) = -\phi P_3[X, Y]$. From (3.10), we have $P_3(A_{\phi P_5 Y}X) = -\phi P_1 \nabla_X Y$, which gives $P_3(A_{\phi P_5 Y}X) - P_3(A_{\phi P_5 X}Y) = -\phi P_1[X, Y]$. From (3.13), we have $P_6(\nabla_X \phi P_5 Y) = f P_6 \nabla_X Y + B R_4 h^s(X, Y)$, which gives $P_6(\nabla_X \phi P_5 Y) - P_6(\nabla_Y \phi P_5 X) = f P_6[X, Y]$.
From (3.14), we have $Q_1 D^l(X, \phi P_5 Y) = \phi P_4 \nabla_X Y + \phi P_4 h^l(Y, X)$, which implies $Q_1 D^l(X, \phi P_5 Y) - Q_1 D^l(Y, \phi P_5 X) = \phi P_4[X, Y]$. From (3.16), $R_1 h^s(X, \phi P_5 Y) = \phi P_2 \nabla_X Y$, which implies $R_1 h^s(X, \phi P_5 Y) - R_1 h^s(Y, \phi P_5 X) = \phi P_2[X, Y]$. From (3.17), we have $R_4 \nabla^s_X \phi P_5 Y = F P_6 \nabla_X Y$, which gives $R_4 \nabla^s_X \phi P_5 Y - R_4 \nabla^s_Y \phi P_5 X = F P_6[X, Y]$, which completes the proof. □

Theorem 3.6. *Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the distribution D'_2 is integrable if and only if*
(i) $P_1(\nabla_X f P_6 Y) - P_1(\nabla_Y f P_6 X) = P_1(A_{F P_6 Y}X) - P_1(A_{F P_6 X}Y)$ *and* $P_3(\nabla_X f P_6 Y) - P_3(\nabla_Y f P_6 X) = P_3(A_{F P_6 Y}X) - P_3(A_{F P_6 X}Y)$,
(ii) $Q_1 D^l(X, f P_6 Y) - Q_1 D^l(Y, f P_6 X) = Q_1 D^l(Y, F P_6 X) - Q_1 D^l(X, F P_6 Y)$,
(iii) $R_3 h^s(X, f P_6 Y) - R_3 h^s(Y, f P_6 X) = R_3 \nabla^s_Y F P_6 X - R_3 \nabla^s_X F P_6 Y$ *and*
$R_4 h^s(X, f P_6 Y) - R_4 h^s(Y, f P_6 X) = R_4 \nabla^s_Y F P_6 X - R_4 \nabla^s_X F P_6 Y$,
for all $X, Y \in \Gamma(D'_2)$.

Proof. Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Suppose $X, Y \in \Gamma(D_1)$. From (3.8), we have $P_1(\nabla_X f P_6 Y) = P_1(A_{F P_6 Y}X) + \phi P_3 \nabla_X Y$, which gives $P_1(\nabla_X f P_6 Y) - P_1(\nabla_Y f P_6 X) = P_1(A_{F P_6 Y}X) - P_1(A_{F P_6 X}Y) + \phi P_3[X, Y]$. From (3.10), we have $P_3(\nabla_X f P_6 Y) = P_3(A_{F P_6 Y}X) + \phi P_1 \nabla_X Y$, which gives $P_3(\nabla_X f P_6 Y) - P_3(\nabla_Y f P_6 X) = P_3(A_{F P_6 Y}X) - P_3(A_{F P_6 X}Y) + \phi P_1[X, Y]$.

From (3.14), $Q_1 D^l(X, f P_6 Y) = \phi P_4 \nabla_X Y - Q_1 D^l(X, F P_6 Y) + \phi P_4 h^l(X, Y)$, which gives $Q_1 D^l(X, f P_6 Y) - Q_1 D^l(Y, f P_6 X) = \phi P_4[X, Y] - Q_1 D^l(X, F P_6 Y) + Q_1 D^l(Y, F P_6 X)$. From (3.16), we have $R_3 h^s(X, f P_6 Y) = \phi P_5 \nabla_X Y - R_3 \nabla^s_X F P_6 Y$, which implies $R_3 h^s(X, f P_6 Y) - R_3 h^s(Y, f P_6 X) = \phi P_5[X, Y] - R_3 \nabla^s_X F P_6 Y + R_3 \nabla^s_Y F P_6 X$. From (3.17), we have $R_4 h^s(X, f P_6 Y) = F P_6 \nabla_X Y - R_4 \nabla^s_X F P_6 Y$, which implies $R_4 h^s(X, f P_6 Y) - R_4 h^s(Y, f P_6 X) = F P_6[X, Y] - R_4 \nabla^s_X F P_6 Y + R_4 \nabla^s_Y F P_6 X$, which completes the proof. □

4. Foliations Determined By Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold to be totally geodesic.

Definition 4.1. A GCR screen pseudo-slant lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(X,Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is a mixed geodesic GCR screen pseudo-slant lightlike submanifold if $h^l(X,Y) = 0$ and $h^s(X,Y) = 0$, $\forall\ X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

Theorem 4.1. *Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $D_1 \subset Rad(TM)$ defines a totally geodesic foliation if and only if*

$$\bar{g}(\nabla_X \phi P_3 Z + \nabla_X f P_6 Z, \phi P_1 Y) = \bar{g}(A_{\phi P_4 Z} X + A_{\phi P_5 Z} X + A_{F P_6 Z} X, \phi P_1 Y),$$

for all $X \in \Gamma(D_1)$ and $Z \in \Gamma(S(TM))$.

Proof. Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. To prove that $D_1 \subset Rad(TM)$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in D_1$, for all $X, Y \in \Gamma(D_1)$. Since $\bar{\nabla}$ is a metric connection, using (2.7) and (2.19), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(S(TM))$, we get

$$\bar{g}(\nabla_X Y, Z) = \bar{g}((\bar{\nabla}_X \phi) Z - \bar{\nabla}_X \phi Z, \phi Y). \tag{4.1}$$

Now from (2.20), (3.4) and (4.1) we get

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X(\phi P_3 Z + \phi P_4 Z + \phi P_5 Z + \phi P_6 Z), \phi P_1 Y). \tag{4.2}$$

In view of (2.7)-(2.9) and (4.2), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(S(TM))$ we obtain (4.3)

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X \phi P_3 Z - A_{\phi P_4 Z} X - A_{\phi P_5 Z} X + \nabla_X f P_6 Z - A_{F P_6 Z} X, \phi P_1 Y),$$

which completes the proof. □

Theorem 4.2. *Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $D_2 \subset Rad(TM)$ defines a totally geodesic foliation if and only if*

$$\begin{aligned} & \bar{g}(h^s(X, \phi P_3 Z) + h^s(X, \phi P_5 Z) + h^s(X, f P_6 Z) + h^s(X, f P_7 Z), \phi P_2 Y) \\ = \; & -\bar{g}(D^s(X, \phi P_4 Z) + \nabla^s_X F P_6 Z + \nabla^s_X F P_7 Z, \phi P_2 Y), \end{aligned}$$

for all $X \in \Gamma(D_2)$ and $Z \in \Gamma(S(TM))$.

Proof. Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. To prove that $D_2 \subset Rad(TM)$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in D_2$, for all $X, Y \in \Gamma(D_2)$. Since $\bar{\nabla}$ is a metric connection, using (2.7) and (2.19), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(S(TM))$, we get

$$\bar{g}(\nabla_X Y, Z) = \bar{g}((\bar{\nabla}_X \phi) Z - \bar{\nabla}_X \phi Z, \phi Y). \tag{4.4}$$

Now from (2.20), (3.4) and (4.1) we get

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X(\phi P_3 Z + \phi P_4 Z + \phi P_5 Z + \phi P_6 Z), \phi P_2 Y). \tag{4.5}$$

In view of (2.7)-(2.9) and (4.2), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(S(TM))$ we obtain

$$\begin{aligned}\bar{g}(\nabla_X Y, Z) = -\bar{g}(h^s(X, \phi P_3 Z) + D^s(X, \phi P_4 Z) + \nabla^s_X \phi P_5 Z \\ + h^s(X, f P_6 Z) + \nabla^s_X F P_6 Z, \phi P_2 Y)\end{aligned} \tag{4.6}$$

which completes the proof. □

Theorem 4.3. *Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then D'_2 defines a totally geodesic foliation if and only if*
(i) $\bar{g}(-A_X \phi Z, fY) = -\bar{g}(\nabla^s_X \phi Z, FY)$,
(ii) $\bar{g}(A_{\phi Q_2 N} X - \nabla_X \phi Q_1 N, fY) = \bar{g}(h^s(X, \phi Q_1 N) + \nabla^s_X \phi Q_2 N, FY)$,
(iii) $\bar{g}(A_{\phi W} X, fY) = \bar{g}(D^s(X, \phi W), FY)$
for all $X, Y \in \Gamma(D'_2)$, $Z \in \Gamma(D'_1)$, $W \in \Gamma(\phi ltr(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a GCR screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold. The distribution D'_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D'_2)$, $\forall\ X, Y \in \Gamma(D'_2)$. Since $\bar{\nabla}$ is a metric connection for any $X, Y \in \Gamma(D'_2)$ and $Z \in \Gamma(D'_1)$, we get

$$\bar{g}(\nabla_X Y, Z) = \bar{g}(\bar{\nabla}_X \phi Y, \phi Z) = -\bar{g}(\bar{\nabla}_X \phi Z, \phi Y). \tag{4.7}$$

From (2.7), (3.1) and (4.7) we get

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(-A_X \phi Z + \nabla^s_X \phi Z, fY + FY). \tag{4.8}$$

In view of (2.8) and (4.8) we obtain

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(-A_X \phi Z, fY) - \bar{g}(\nabla^s_X \phi Z, FY). \tag{4.9}$$

Now by (4.9) we get the required result

$$\bar{g}(-A_X \phi Z, fY) = -\bar{g}(\nabla^s_X \phi Z, FY). \tag{4.10}$$

Now for any $X, Y \in \Gamma(D'_2)$ and $N \in \Gamma(ltr(TM))$ we have

$$\bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X \phi Y, \phi N) = -\bar{g}(\bar{\nabla}_X \phi N, \phi Y). \tag{4.11}$$

From (2.7), (3.1) and (4.11) we get

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(\phi Q_1 N + \phi Q_2 N, fY + FY). \tag{4.12}$$

In view of (4.12) we obtain

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(\nabla_X \phi Q_1 N - A_{\phi Q_2 N} X, fY) - \bar{g}(h^s(X, \phi Q_1 N) + \nabla^s_X \phi Q_2 N, FY). \tag{4.13}$$

Now from (4.13) we get the required result

$$\bar{g}(A_{\phi Q_2 N}X - \nabla_X \phi Q_1 N, fY) = \bar{g}(h^s(X, \phi Q_1 N) + \nabla^s_X \phi Q_2 N, FY). \tag{4.14}$$

Now for any $X, Y \in \Gamma(D'_2)$ and $W \in \Gamma(\phi ltr(TM))$ we have

$$\bar{g}(\nabla_X Y, W) = \bar{g}(\bar{\nabla}_X \phi Y, \phi W) = -\bar{g}(\bar{\nabla}_X \phi W, \phi Y). \tag{4.15}$$

From (2.9), (3.1) and (4.15) we get

$$\bar{g}(\nabla_X Y, W) = -\bar{g}(-A_{\phi W}X + D^s(X, \phi W), fY + FY). \tag{4.16}$$

In view of (4.16) we obtain

$$\bar{g}(\nabla_X Y, W) = \bar{g}(A_{\phi W}X, fY) - \bar{g}(FY, D^s(X, \phi W)). \tag{4.17}$$

Now from (4.17) we get the required result

$$\bar{g}(A_{\phi W}X, fY) = \bar{g}(D^s(X, \phi W), FY), \tag{4.18}$$

which completes the proof. □

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