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CANAL HYPERSURFACES ACCORDING TO GENERALIZED BISHOP FRAMES IN 4-SPACE

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Abstract. In the present paper, we study canal hypersurfaces according to generalized Bishop frames of type B (parallel transport frame), type C and type D in Euclidean 4-space and obtain Gaussian, mean and principal curvatures of them in general form. We give some results for their flatness, minimality and we examine the Weingarten canal hypersurfaces according to these frames. Especially, we investigate the tubular hypersurfaces by taking the radius function is constant in these canal hypersurfaces.
Keywords: Canal Hypersurface, Tubular Hypersurface, Generalized Bishop Frames, Weingarten Hypersurface.

1. General Information and Basic Concepts

A canal surface given by the following parametric expression

$$\Omega(u,v) = \alpha(u) - \rho(u)\rho'(u)T(u) + \rho(u)\sqrt{1-\rho'^2(u)}\,(\cos vN(u) + \sin vB(u))$$

is formed by the envelope of the spheres whose centers lie on a curve and radius vary depending on this curve. Here, $\alpha(x)$ is a unit speed curve is called the spine curve or center curve, $\{T,N,B\}$ is the Frenet frame of $\alpha(x)$ and $\rho(x)$ is the radius function. If the radius function $\rho(x)$ is constant, then the canal surface is called tubular or pipe

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surface ([20], [21], [29]). Also, if the center curve of the canal surface is a straight line, then it becomes a revolution surface. Canal and tubular surfaces have been applied to many fields, such as the solid and the surface modeling for CAD/CAM, construction of blending surfaces, shape re-construction and so on. In this context, canal and tubular (hyper)surfaces have been studied by many geometers in different spaces (see [6], [15], [22]-[26], [29]-[31], [33], [37]-[40], [41] and etc).

Furthermore, although Frenet frame has been used in lots of studies about different differential geometric characterizations of curves and surfaces, sometimes geometers need alternative frames because of Frenet frame cannot be identified at the points where the curvature is zero. Hence, new alternative frames to the Frenet frame such as Bishop frame (parallel transport frame), generalized Bishop frames, Darboux frame or extended Darboux frame have been defined by geometers and the differential geometry of curves and surfaces started to be considered according to these alternative frames (see [1], [2], [7], [9]-[14], [27], [28], [30], [35], [36], and etc).

Here, let we recall some basic notions about Frenet frame, parallel transport frame, generalized Bishop frames and the curvatures of hypersurfaces in E^4.

Let $\{e_1, e_2, e_3, e_4\}$ be the standart basis of Euclidean 4-space E^4. If $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$ and $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ are three vectors in E^4, then the inner product is defined by $\langle \overrightarrow{x}, \overrightarrow{y} \rangle = \sum\limits_{i=1}^{4} x_i y_i$ and the vector product is defined by

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \det \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}.$$

If $\alpha : I \subset R \longrightarrow E^4$ is a unit speed curve in Euclidean 4-space and $\{T, N, B_1, B_2\}$ is the moving Frenet frame along α, then the Frenet formulas are given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{1.1}$$

where T, N, B_1 and B_2 denote the unit tangent, the principal normal, the first binormal and the second binormal vector fields, respectively; κ, τ and σ are the curvature functions according to Frenet frame of the curve α [16].

In [17], the authors have used the tangent vector $T(s)$ and three relatively parallel vector fields $M_1(s)$, $M_2(s)$ and $M_3(s)$ to construct an alternative frame which is called a parallel transport frame along the curve α in E^4. If $\{T, N, B_1, B_2\}$ is a Frenet frame along a unit speed curve $\alpha = \alpha(s) : I \to E^4$ and $\{T, M_1, M_2, M_3\}$ denotes the parallel transport frame of the curve α, then the relation may be expressed as

$$\begin{cases} T(s) = T(s), \\ N(s) = \cos\theta(s)\cos\psi(s)M_1(s) + (-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s))\, M_2(s) \\ \quad + (\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s))\, M_3(s), \\ B_1(s) = \cos\theta(s)\sin\psi(s)M_1(s) + (\cos\phi(s)\cos\psi(s) + \sin\phi(s)\sin\theta(s)\sin\psi(s))\, M_2(s) \\ \quad + (-\sin\phi(s)\cos\psi(s) + \cos\phi(s)\sin\theta(s)\sin\psi(s))\, M_3(s), \\ B_2(s) = -\sin\theta(s)M_1(s) + \sin\phi(s)\cos\theta(s)M_2(s) + \cos\phi(s)\cos\theta(s)M_3(s), \end{cases} \tag{1.2}$$

where $\theta(s)$, $\psi(s)$ and $\phi(s)$ are the Euler angles. Also, the alternative parallel transport frame (we'll call it as generalized Bishop frame of type B) equations are

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \tag{1.3}$$

where k_1, k_2 and k_3 are curvature functions according to parallel transport frame of the curve α and their expressions are as follows

$$\begin{cases} k_1(s) = \kappa(s)\cos\theta(s)\cos\psi(s), \\ k_2(s) = \kappa(s)\left(-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s)\right), \\ k_3(s) = \kappa(s)\left(\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s)\right). \end{cases} \tag{1.4}$$

On the other hand, if we regard an orthonormal frame on a regular curve $I \to E^n$ parametrized by arc-length parameters as a matrix valued function $\mathbb{Z} : I \to O(n)$ such that the frame consists of the row vectors of $\mathbb{Z}$, then for a frame on a regular curve, we will call the matrix valued function X such that $\mathbb{Z}' = X\mathbb{Z}$ the coefficient matrix of the frame. In this context, Nomoto and Nozawa have introduced 16 kinds of alternative frames called generalized Bishop frames on regular curves on E^4, excluding frames whose coefficient matrix has a zero column vector [35]. They have seen that these frames are classified into four types up to the action of the symmetric group $\mathfrak{S}_3$ of order 3 which swaps the second, third and fourth vector of the frame as follows:

$$\left.\begin{bmatrix} 0 & \blacksquare & \blacksquare & \blacksquare \\ -\blacksquare & 0 & 0 & 0 \\ -\blacksquare & 0 & 0 & 0 \\ -\blacksquare & 0 & 0 & 0 \end{bmatrix};\ \right\} \text{type B}$$

$$\left.\begin{matrix} \begin{bmatrix} 0 & \blacksquare & \blacksquare & 0 \\ -\blacksquare & 0 & 0 & \blacksquare \\ -\blacksquare & 0 & 0 & 0 \\ 0 & -\blacksquare & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & \blacksquare & 0 \\ -\blacksquare & 0 & 0 & 0 \\ -\blacksquare & 0 & 0 & \blacksquare \\ 0 & 0 & -\blacksquare & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & 0 & \blacksquare \\ -\blacksquare & 0 & 0 & 0 \\ 0 & 0 & 0 & \blacksquare \\ -\blacksquare & 0 & -\blacksquare & 0 \end{bmatrix}, \\ \begin{bmatrix} 0 & \blacksquare & 0 & \blacksquare \\ -\blacksquare & 0 & \blacksquare & 0 \\ 0 & -\blacksquare & 0 & 0 \\ -\blacksquare & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & 0 \\ -\blacksquare & -\blacksquare & 0 & 0 \\ -\blacksquare & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \blacksquare & \blacksquare \\ 0 & 0 & 0 & \blacksquare \\ -\blacksquare & 0 & 0 & 0 \\ -\blacksquare & -\blacksquare & 0 & 0 \end{bmatrix}; \end{matrix}\right\} typeC$$

$$\left.\begin{bmatrix} 0 & \blacksquare & 0 & 0 \\ -\blacksquare & 0 & \blacksquare & \blacksquare \\ 0 & -\blacksquare & 0 & 0 \\ 0 & -\blacksquare & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare & 0 \\ -\blacksquare & -\blacksquare & 0 & \blacksquare \\ 0 & 0 & -\blacksquare & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \\ -\blacksquare & -\blacksquare & -\blacksquare & 0 \end{bmatrix};\right\} typeD$$

$$\left.\begin{matrix} \begin{bmatrix} 0 & \blacksquare & 0 & 0 \\ -\blacksquare & 0 & \blacksquare & 0 \\ 0 & -\blacksquare & 0 & \blacksquare \\ 0 & 0 & -\blacksquare & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \\ 0 & -\blacksquare & 0 & 0 \\ -\blacksquare & -\blacksquare & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \blacksquare \\ 0 & 0 & \blacksquare & 0 \\ 0 & -\blacksquare & 0 & \blacksquare \\ -\blacksquare & 0 & -\blacksquare & 0 \end{bmatrix}, \\ \begin{bmatrix} 0 & \blacksquare & 0 & 0 \\ -\blacksquare & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & \blacksquare \\ 0 & -\blacksquare & -\blacksquare & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare & \blacksquare \\ -\blacksquare & -\blacksquare & 0 & 0 \\ 0 & -\blacksquare & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & \blacksquare \\ -\blacksquare & 0 & 0 & \blacksquare \\ 0 & -\blacksquare & -\blacksquare & 0 \end{bmatrix}. \end{matrix}\right\} typeF$$

Here they have seen that, there are the following 4 equivalence classes of these 16 frames on curves up to the change of the order of vectors fixing the first one which is the tangent vector: If a frame has a coefficient matrix of the respective form for some functions x_1, x_2, x_3 up to the change of the order of vectors fixing the first one, we call it a generalized Bishop frame of type B, C, D and F, respectively:

$$\underset{type\ B}{\begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & 0 \\ -x_3 & 0 & 0 & 0 \end{bmatrix}}, \quad \underset{type\ C}{\begin{bmatrix} 0 & x_1 & x_2 & 0 \\ -x_1 & 0 & 0 & x_3 \\ -x_2 & 0 & 0 & 0 \\ 0 & -x_3 & 0 & 0 \end{bmatrix}},$$

$$\underset{type\ D}{\begin{bmatrix} 0 & x_1 & 0 & 0 \\ -x_1 & 0 & x_2 & x_3 \\ 0 & -x_2 & 0 & 0 \\ 0 & -x_3 & 0 & 0 \end{bmatrix}}, \quad \underset{type\ F}{\begin{bmatrix} 0 & x_1 & 0 & 0 \\ -x_1 & 0 & x_2 & 0 \\ 0 & -x_2 & 0 & x_3 \\ 0 & 0 & -x_3 & 0 \end{bmatrix}}.$$

So, if $\{T, M_1, M_2, M_3\}$ denotes the generalized Bishop frames of the curve α, then we can write

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \left\{ \begin{matrix} \underset{type\ B}{\begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ -b_1 & 0 & 0 & 0 \\ -b_2 & 0 & 0 & 0 \\ -b_3 & 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \quad \underset{type\ C}{\begin{bmatrix} 0 & c_1 & c_2 & 0 \\ -c_1 & 0 & 0 & c_3 \\ -c_2 & 0 & 0 & 0 \\ 0 & -c_3 & 0 & 0 \end{bmatrix}} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \\ \underset{type\ D}{\begin{bmatrix} 0 & d_1 & 0 & 0 \\ -d_1 & 0 & d_2 & d_3 \\ 0 & -d_2 & 0 & 0 \\ 0 & -d_3 & 0 & 0 \end{bmatrix}} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \quad \underset{type\ F}{\begin{bmatrix} 0 & f_1 & 0 & 0 \\ -f_1 & 0 & f_2 & 0 \\ 0 & -f_2 & 0 & f_3 \\ 0 & 0 & -f_3 & 0 \end{bmatrix}} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}. \end{matrix} \right. \tag{1.5}$$

It is obvious that, if we consider $b_i = k_i$ ($i = 1, 2, 3$ and k_i are the principal curvature functions according to generalized Bishop frame of type B of the curve α) in 1.5 of type B, then the frame becomes parallel transport frame (1.3) which has been stated above. Furthermore, the frame of type F is closely related to the Frenet frame; it is Frenet frame if both f_1 and f_2 are positive. By Bishop's theorem, every regular curve admits a Bishop frame, but there are well known examples of regular curves which do not admit the Frenet frame. Also, a curve is said to be 2-regular if both the tangent vector and its derivative are nowhere vanishing and in this context, the authors have given a main result as "Every 2-regular curve on E^4 admits a frame of type C and D, respectively" in [35]. In this study, we will construct our canal hypersurface with the aid of the generalized Bishop frames of type B, type C, type D of the center curve $\alpha(u)$ (which will be considered as regular for characterizations of type B and 2-regular for characterizations of type C and type D) and give some important geometric characterizations about them.

Furthermore, the differential geometry of different types of (hyper)surfaces in 4-dimensional spaces has recently become a popular topic studied by geometers ([3]-[5], [8], [18], [19], [34], and etc). If
$\Omega : U \subset E^3 \longrightarrow E^4, \Omega(x_1, x_2, x_3) = (\Omega_1(x_1, x_2, x_3), \Omega_2(x_1, x_2, x_3), \Omega(x_1, x_2, x_3), \Omega_4(x_1, x_2, x_3))$
is a hypersurface in E^4, then the unit normal vector field, the matrix forms of the first and second fundamental forms are

$$\mathcal{N}^\Omega = \frac{\Omega_{x_1} \times \Omega_{x_2} \times \Omega_{x_3}}{\|\Omega_{x_1} \times \Omega_{x_2} \times \Omega_{x_3}\|}, \tag{1.6}$$

$$[g_{ij}^\Omega] = \begin{bmatrix} g_{11}^\Omega & g_{12}^\Omega & g_{13}^\Omega \\ g_{21}^\Omega & g_{22}^\Omega & g_{23}^\Omega \\ g_{31}^\Omega & g_{32}^\Omega & g_{33}^\Omega \end{bmatrix} \tag{1.7}$$

and

$$[h_{ij}^\Omega] = \begin{bmatrix} h_{11}^\Omega & h_{12}^\Omega & h_{13}^\Omega \\ h_{21}^\Omega & h_{22}^\Omega & h_{23}^\Omega \\ h_{31}^\Omega & h_{32}^\Omega & h_{33}^\Omega \end{bmatrix}, \tag{1.8}$$

respectively. Here $g_{ij}^\Omega = \left\langle \Omega_{x_i}, \Omega_{x_j} \right\rangle$, $h_{ij}^\Omega = \left\langle \Omega_{x_i x_j}, \mathcal{N}^\Omega \right\rangle$, $\Omega_{x_i} = \frac{\partial \Omega(x_1,x_2,x_3)}{\partial x_i}$, $\Omega_{x_i x_j} = \frac{\partial^2 \Omega(x_1,x_2,x_3)}{\partial x_i x_j}$, $i, j \in \{1, 2, 3\}$. Also, the shape operator of the hypersurface Ω is

$$S^\Omega = [a_{ij}^\Omega] = [g_{ij}^\Omega]^{-1}.[h_{ij}^\Omega], \tag{1.9}$$

where $[g_{ij}^\Omega]^{-1}$ is the inverse matrix of $[g_{ij}^\Omega]$.

With the aid of (1.6)-(1.9), the Gaussian and mean curvatures of a hypersurface in E^4 are given by

$$K^\Omega = \det(S^\Omega) = \frac{\det[h_{ij}^\Omega]}{\det[g_{ij}^\Omega]} \tag{1.10}$$

and

$$3H^\Omega = tr(S^\Omega), \tag{1.11}$$

respectively [32]. We say that a hypersurface is flat or minimal, if it has zero Gaussian curvature or zero mean curvature, respectively.

After recalling some basic notions about generalized Bishop frames and the Gaussian and mean curvatures of hypersurfaces in E^4 in this section, we study on canal hypersurfaces according to generalized Bishop frames in E^4 in the second section. We obtain the

Gaussian, mean and principal curvatures with the aid of first and second derivatives of canal hypersurface according to generalized Bishop frames and give some results about their flatness and minimality. Also we give a characterization about Weingarten canal hypersurfaces according to generalized Bishop frames. In the third section of this study, we give some characterizations, which have been given for canal hypersurfaces in the second section, for tubular hypersurfaces according to generalized Bishop frames in E^4.

2. Canal Hypersurfaces according to Generalized Bishop Frames in E^4

In this section, we study the canal hypersurfaces according to generalized Bishop frames in Euclidean 4-space E^4. In this context, firstly we obtain Gaussian, mean and principal curvatures of a canal hypersurface $\mathfrak{C}$ according to generalized Bishop frames with the aid of its first and second derivatives. Also, we give some results for flat, minimal and Weingarten canal hypersurfaces.

Let us consider the canal hypersurface $\mathfrak{C}$ according to generalized Bishop frames of type B, type C and type D in E^4 given by

$$\begin{aligned} &\mathfrak{C}(u,v,t) = \alpha(u) - \left(\rho(u)\rho'(u)\right) T(u) \\ &\pm \rho(u)\sqrt{1-\rho'(u)^2}\left[(\cos v \cos t)\, M_1(u) + (\sin v \cos t)\, M_2(u) + (\sin t)\, M_3(u)\right], \end{aligned} \tag{2.1}$$

where $u \in [0, l]$ and $v, t \in [0, 2\pi)$. We must note that, from now on we state $\alpha = \alpha(u)$, $\rho = \rho(u)$, $\rho' = \frac{d\rho(u)}{du}$, $T = T(u)$, $M_1 = M_1(u)$, $M_2 = M_2(u)$, $M_3 = M_3(u)$; we will consider the "$\pm$" in equation (2.1) as "$+$" and we will give our results for "$+$". One can obtain similar results by taking the sign as "$-$". Also throughout this study, the center curve $\alpha(u)$ will be consider as a regular curve when we investigate the generalized Bishop frames of type B and $\alpha(u)$ will be consider as a 2-regular curve when we investigate the generalized Bishop frames of type C and D.
In all of the following calculations and results,

- if one takes α is regular and $a = b = 1$, $c = d = 0$, $x_i = b_i = k_i$, then the results belong to generalized Bishop frame of type B (parallel transport frame);
- if one takes α is 2-regular and $a = d = 1$, $b = c = 0$, $x_i = c_i$, then the results belong to generalized Bishop frame of type C;
- if one takes α is 2-regular and $a = b = 0$, $c = d = 1$, $x_i = d_i$, then the results belong to generalized Bishop frame of type D,

where $i \in \{1, 2, 3\}$.

First, from (1.5) and (2.1), the first derivatives of the canal hypersurface (2.1) according to generalized Bishop frames of type B, type C and type D are obtained as

$$\begin{cases} \mathfrak{C}_u = \left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)T \\ \quad + \left(\rho'\sqrt{1-\rho'^2}\cos v\cos t-\rho\left(\rho' x_1+\frac{(cx_2\sin v\cos t+dx_3\sin t)\left(1-\rho'^2\right)+\rho\rho''\cos v\cos t}{\sqrt{1-\rho'^2}}\right)\right)M_1 \\ \quad + \left(\rho'\sqrt{1-\rho'^2}\sin v\cos t+\rho\left(x_2\left(c\sqrt{1-\rho'^2}\cos v\cos t-a\rho'\right)-\frac{\rho'\rho''\sin v\cos t}{\sqrt{1-\rho'^2}}\right)\right)M_2 \\ \quad + \left(\rho'\sqrt{1-\rho'^2}\sin t+\rho\left(x_3\left(-b\rho'+d\sqrt{1-\rho'^2}\cos v\cos t\right)-\frac{\rho'\rho''\sin t}{\sqrt{1-\rho'^2}}\right)\right)M_3, \\ \mathfrak{C}_v = -\rho\sqrt{1-\rho'^2}\left((\sin v\cos t)M_1-(\cos v\cos t)M_2\right), \\ \mathfrak{C}_t = -\rho\sqrt{1-\rho'^2}\left((\cos v\sin t)M_1+(\sin v\sin t)M_2-(\cos t)M_3\right), \end{cases} \tag{2.2}$$

where

$$\mathcal{W} = x_1\cos v\cos t+ax_2\sin v\cos t+bx_3\sin t. \tag{2.3}$$

From (1.6) and (2.2), the unit normal vector field of $\mathfrak{C}$ in E^4 is

$$\mathcal{N}^{\mathfrak{C}} = -\rho' T+\sqrt{1-\rho'^2}\left((\cos v\cos t)M_1+(\sin v\cos t)M_2+(\sin t)M_3\right). \tag{2.4}$$

Also, the coefficients of the first fundamental form are given by

$$\begin{cases} g_{11}^{\mathfrak{C}} = \frac{1}{1-\rho'^2}\begin{pmatrix} \left(1-\rho'^2\right)\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)^2 \\ +\begin{pmatrix} \rho\left(cx_2\sin v\cos t+dx_3\sin t\right)\left(1-\rho'^2\right) \\ \rho\rho'\sqrt{1-\rho'^2}x_1-\rho'\left(1-\rho'^2-\rho\rho''\right)\cos v\cos t \end{pmatrix}^2 \\ +\begin{pmatrix} a\rho\rho'\sqrt{1-\rho'^2}x_2-c\rho\left(1-\rho'^2\right)x_2\cos v\cos t \\ -\rho'\left(1-\rho'^2-\rho\rho''\right)\sin v\cos t \end{pmatrix}^2 \\ +\begin{pmatrix} b\rho\rho'\sqrt{1-\rho'^2}x_3-d\rho\left(1-\rho'^2\right)x_3\cos v\cos t \\ -\rho'\left(1-\rho'^2-\rho\rho''\right)\sin t \end{pmatrix}^2 \end{pmatrix}, \\ g_{12}^{\mathfrak{C}} = g_{21}^{\mathfrak{C}} = \rho^2\begin{pmatrix} -x_2\left(a\rho'\sqrt{1-\rho'^2}\cos v-c\left(1-\rho'^2\right)\cos t\right) \\ +\sin v\left(x_1\rho'\sqrt{1-\rho'^2}+d\left(1-\rho'^2\right)x_3\sin t\right) \end{pmatrix}\cos t, \\ g_{13}^{\mathfrak{C}} = g_{31}^{\mathfrak{C}} = \rho^2\begin{pmatrix} \rho'\sqrt{1-\rho'^2}\left(x_1\cos v+ax_2\sin v\right)\sin t \\ -x_3\left(b\rho'\sqrt{1-\rho'^2}\cos t-d\left(1-\rho'^2\right)\cos v\right) \end{pmatrix}, \\ g_{22}^{\mathfrak{C}} = \rho^2(1-\rho'^2)\cos^2 t, \quad g_{23}^{\mathfrak{C}} = g_{32}^{\mathfrak{C}} = 0, \quad g_{33}^{\mathfrak{C}} = \rho^2(1-\rho'^2) \end{cases} \tag{2.5}$$

and it follows that

$$\det[g_{ij}^{\mathfrak{C}}] = \rho^4(1-\rho'^2)\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)^2\cos^2 t. \tag{2.6}$$

Now, for obtaining the coefficients of the second fundamental form, let we give the second derivatives $\mathfrak{C}_{x_i x_j}=\frac{\partial^2\mathfrak{C}}{\partial x_i x_j}$ of the canal hypersurface (2.1):

$$(2.7)\left\{\begin{array}{l}
\mathfrak{C}_{uu}=\mathfrak{C}_{uu}^{1}T+\mathfrak{C}_{uu}^{2}M_1+\mathfrak{C}_{uu}^{3}M_2+\mathfrak{C}_{uu}^{4}M_3,\\[2mm]
\mathfrak{C}_{uv}=\mathfrak{C}_{vu}=\left(\sqrt{1-\rho'^2}\rho\left(x_1\sin v-ax_2\cos v\right)\cos t\right)T\\
\quad+\frac{\cos t}{\sqrt{1-\rho'^2}}\left(\rho\rho'\rho''\sin v-\left(1-\rho'^2\right)\left(c\rho x_2\cos v+\rho'\sin v\right)\right)M_1\\
\quad-\frac{\cos t}{\sqrt{1-\rho'^2}}\left(\left(c\rho x_2\sin v-\rho'\cos v\right)\left(1-\rho'^2\right)+\rho\rho'\rho''\cos v\right)M_2\\
\quad-(d\rho\sqrt{1-\rho'^2}x_3\sin v\cos t)M_3,\\[2mm]
\mathfrak{C}_{ut}=\mathfrak{C}_{tu}=\sqrt{1-\rho'^2}\rho\left(x_1\cos v\sin t+ax_2\sin v\sin t-bx_3\cos t\right)T\\
\quad+\frac{1}{\sqrt{1-\rho'^2}}\left(\begin{array}{l}\left(\left(\rho\left(cx_2\sin v\sin t-dx_3\cos t\right)-\rho'\cos v\sin t\right)\left(1-\rho'^2\right)\right)\\+\rho\rho'\rho''\cos v\sin t\end{array}\right)M_1\\
\quad+\frac{\sin t}{\sqrt{1-\rho'^2}}\left(\rho\rho'\rho''\sin v-\left(c\rho x_2\cos v+\rho'\sin v\right)\left(1-\rho'^2\right)\right)M_2\\
\quad+\frac{1}{\sqrt{1-\rho'^2}}\left(\rho'\left(1-\rho'^2-\rho\rho''\right)\cos t-d\rho(1-\rho'^2)x_3\cos v\sin t\right)M_3,\\[2mm]
\mathfrak{C}_{vv}=-\rho\sqrt{1-\rho'^2}\left(\left(\cos v\cos t\right)M_1+\left(\sin v\cos t\right)M_2\right),\\[2mm]
\mathfrak{C}_{vt}=\mathfrak{C}_{tv}=\rho\sqrt{1-\rho'^2}\left(\left(\sin v\sin t\right)M_1-\left(\cos v\sin t\right)M_2\right),\\[2mm]
\mathfrak{C}_{tt}=-\rho\sqrt{1-\rho'^2}\left(\left(\cos v\cos t\right)M_2+\left(\sin v\cos t\right)M_3+\left(\sin t\right)M_4\right),
\end{array}\right.$$

where

$$\begin{array}{l}
\mathrm{C}_{uu}^{1}=\rho\rho'\left(x_1^2+a^2x_2^2+b^2x_3^2\right)-2\rho'\sqrt{1-\rho'^2}\mathcal{W}-3\rho'\rho''-\rho\rho'''+\frac{2\rho\rho'\rho''W}{\sqrt{1-\rho'^2}}\\
\quad+\rho\sqrt{1-\rho'^2}\left(\begin{array}{l}x_1\left(cx_2\sin v\cos t+dx_3\sin t\right)\\-\left(acx_2^2+bdx_3^2+x_1'\right)\cos v\cos t-ax_2'\sin v\cos t-bx_3'\sin t\end{array}\right),
\end{array}$$

$$\begin{array}{l}
\mathrm{C}_{uu}^{2}=x_1-2\rho'^2x_1-2\rho'\sqrt{1-\rho'^2}\left(cx_2\sin v\cos t+dx_3\sin t\right)-2\rho\rho''x_1\\
\quad-\rho\sqrt{1-\rho'^2}\left(\begin{array}{l}\left(x_1^2+c^2x_2^2+d^2x_3^2\right)\cos v\cos t\\+\left(ax_1x_2+cx_2'\right)\sin v\cos t+\left(bx_1x_3+dx_3'\right)\sin t\end{array}\right)\\
\quad+\frac{2\rho\rho'\rho''\left(cx_2\sin v\cos t+dx_3\sin t\right)-2\rho'^2\rho''\cos v\cos t-\rho\rho''^2\cos v\cos t-\rho\rho'\rho'''\cos v\cos t}{\sqrt{1-\rho'^2}}\\
\quad-\frac{\rho\rho'^2\rho''^2\cos v\cos t}{\left(1-\rho'^2\right)^{3/2}}+\sqrt{1-\rho'^2}\rho''\cos v\cos t+\rho\rho'\left(acx_2^2+bdx_3^2-x_1'\right),
\end{array}$$

$$\begin{array}{l}
\mathrm{C}_{uu}^{3}=-\left(\left(a^2+c^2\right)\rho\sqrt{1-\rho'^2}x_2^2\sin v\cos t\right)\\
\quad+x_2\left(\begin{array}{l}a-c\rho\rho'x_1-2a\rho'^2-\left(ab+cd\right)\rho\sqrt{1-\rho'^2}x_3\sin t-2a\rho\rho''\\+\frac{\left(\left(2c\rho'-a\rho x_1\right)\left(1-\rho'^2\right)-2c\rho\rho'\rho''\right)}{\sqrt{1-\rho'^2}}\cos v\cos t\end{array}\right)\\
\quad-\frac{1}{\left(1-\rho'^2\right)^{3/2}}\left(\begin{array}{l}\left(4\rho'^2-3\rho'^4-1\right)\rho''\sin v\cos t\\+\rho\left(\begin{array}{l}\left(\left(1-\rho'^2\right)\left(\begin{array}{l}a\rho'\sqrt{1-\rho'^2}\\-c\left(1-\rho'^2\right)\cos v\cos t\end{array}\right)x_2'\right)\\+\left(\rho''^2+\rho'\left(1-\rho'^2\right)\rho'''\right)\sin v\cos t\end{array}\right)\end{array}\right),
\end{array}$$

$$
\begin{aligned}
C^4_{uu} = \frac{1}{\left(1-\rho'^2\right)^{3/2}} & \left(\begin{array}{l} \rho \left(\begin{array}{l} \left(1-\rho'^2\right) \left(\begin{array}{l} \left(\begin{array}{l} d\left(1-\rho'^2\right)\cos v \cos t \\ -b\rho'\sqrt{1-\rho'^2} \end{array} \right) x_3' \\ -\rho'\rho''' \sin t \end{array} \right) \\ -\rho''^2 \sin t \end{array} \right) \\ +\left(1-4\rho'^2+3\rho'^4\right)\rho'' \sin t \end{array} \right) \\
& - \left(\left(b^2+d^2\right)\rho\sqrt{1-\rho'^2}x_3^2 \sin t \right) \\
& + x_3 \left(\begin{array}{l} b - d\rho\rho' x_1 - 2b\rho'^2 - (ab+cd)\rho\sqrt{1-\rho'^2}x_2 \sin v \cos t - 2b\rho\rho'' \\ -\frac{\left(\left(b\rho x_1 - 2d\rho'\right)\left(1-\rho'^2\right)+2d\rho\rho'\rho''\right)}{\sqrt{1-\rho'^2}} \cos v \cos t \end{array} \right).
\end{aligned}
$$

Thus, from (1.8), (2.4) and (2.7), the coefficients of the second fundamental form are given by

$$
(2.8)\quad \begin{cases}
h^{\mathfrak{c}}_{11} = \rho'' + \dfrac{\left(\begin{array}{l} -4W(1-\rho'^2)^{3/2} \\ +\rho \left(\begin{array}{l} 4x_1^2(1-\rho'^2)\left(\cos^2 v \cos^2 t + \rho'^2(1-\cos^2 v\cos^2 t)\right) \\ -x_2^2(1-\rho'^2) \left(\begin{array}{l} -4(c^2+a^2\sin^2 v)\cos^2 t \\ +\rho' \left(\begin{array}{l} 4c^2\rho'\cos^2 t \\ -a^2\rho' \left(\begin{array}{l} 3+\cos(2v) \\ -2\cos(2t)\sin^2 v \end{array} \right) \\ +8ac\sqrt{1-\rho'^2}\cos v\cos t \end{array} \right) \end{array} \right) \\ -4x_2 \left(\begin{array}{l} (ab+cd)(1-\rho'^2)^2 x_3 \sin(2t) \\ -2a\sqrt{1-\rho'^2}\rho''\cos t \end{array} \right) \sin v \\ +4x_1 \left(\begin{array}{l} x_2(1-\rho'^2)\left(\begin{array}{l} 2c\rho'\sqrt{1-\rho'^2}\sin v \\ +a(1-\rho'^2)\cos t\sin(2v) \end{array} \right)\cos t \\ -x_3(1-\rho'^2)\left(\begin{array}{l} -2d\rho'\sqrt{1-\rho'^2}\sin t \\ -b(1-\rho'^2)\cos v\sin(2t) \end{array} \right) \\ +2\sqrt{1-\rho'^2}\rho''\cos v\cos t \end{array} \right) \\ -4 \left(\begin{array}{l} x_3^2(1-\rho'^2)\left(\begin{array}{l} 2bd\rho'\sqrt{1-\rho'^2}\cos v\cos t \\ -d^2(1-\rho'^2)\left(\begin{array}{l} \cos^2 v\cos^2 t \\ +\sin^2 t \end{array} \right) \\ -b^2\left(\sin^2 t + \rho'^2\cos^2 t\right) \end{array} \right) \\ -2b\sqrt{1-\rho'^2}\rho'' x_3 \sin t + \rho''^2 \end{array} \right) \end{array} \right) \end{array} \right)}{4(-1+\rho'^2)} \\
h^{\mathfrak{c}}_{12} = h^{\mathfrak{c}}_{21} = \rho \left(\begin{array}{l} x_2\left(a\rho'\sqrt{1-\rho'^2}\cos v - c(1-\rho'^2)\cos t\right) \\ -\left(d(1-\rho'^2)x_3\sin t + \rho'\sqrt{1-\rho'^2}x_1\right)\sin v \end{array} \right)\cos t, \\
h^{\mathfrak{c}}_{13} = h^{\mathfrak{c}}_{31} = \rho \left(\begin{array}{l} -\rho'\sqrt{1-\rho'^2}\left(x_1\cos v + a x_2 \sin v\right)\sin t \\ +x_3\left(b\rho'\sqrt{1-\rho'^2}\cos t - d(1-\rho'^2)\cos v\right) \end{array} \right), \\
h^{\mathfrak{c}}_{22} = -\rho(1-\rho'^2)\cos^2 t, \quad h^{\mathfrak{c}}_{23} = h^{\mathfrak{c}}_{32} = 0, \quad h^{\mathfrak{c}}_{33} = -\rho(1-\rho'^2)
\end{cases}
$$

and it implies

$$
(2.9)\quad \det[h^{\mathfrak{c}}_{ij}] = \rho^2\left(1-\rho'^2\right)\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)\cos^2 t.
$$

So, from (1.10), (2.6) and (2.9), we have

Proposition 2.1. *The Gaussian curvature of the canal hypersurfaces (2.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in*

Euclidean 4-space is

$$K^{\mathfrak{C}} = \frac{\sqrt{1-\rho'^2}\mathcal{W}+\rho''}{\rho^2\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)}. \tag{2.10}$$

Corollary 2.1. *The canal hypersurfaces (2.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in Euclidean 4-space cannot be flat when $x_1 \neq 0$.*

Proof. Let we suppose that canal hypersurfaces (2.1) is flat; i.e. $K^{\mathfrak{C}} = 0$. From (2.3) and (2.10), we get

$$(x_1 \cos v + ax_2 \sin v)\cos t + bx_3 \sin t + \frac{\rho''}{\sqrt{1-\rho'^2}} = 0. \tag{2.11}$$

Since the set $\{\cos t, \sin t, 1\}$ is linear independent, we have

$$x_1 \cos v + ax_2 \sin v = bx_3 = \frac{\rho''}{\sqrt{1-\rho'^2}} = 0. \tag{2.12}$$

Also, since the set $\{\cos v, \sin v\}$ is linear independent in the first part of (2.12), it must be $x_1 = ax_2 = 0$ and this is a contradiction. □

Corollary 2.2. *Let $\mathfrak{C}$ be a canal hypersurface according to generalized Bishop frame of type B (parallel transport frame) given by (2.1) in E^4. When α is a straight line, the canal hypersurface $\mathfrak{C}$ is flat if and only if $\rho(u) = \lambda u + \mu$, $(\lambda, \mu \in \mathbb{R}, \lambda \neq \pm 1)$.*

Proof. If α is a straight line, then all of the curvature functions k_i, $i \in \{1, 2, 3\}$, according to generalized Bishop frame of type B (parallel transport frame) of α vanish. So from (2.10), the Gaussian curvature becomes (for $a = b = 1$, $c = d = 0$)

$$K^{\mathfrak{C}} = \frac{\rho''}{\rho^2\left(1-\rho'^2-\rho\rho''\right)} \tag{2.13}$$

and so, the proof completes from (2.13). □

Also, after finding the inverse of the matrix of the first fundamental form and using this and (2.8) in (1.9), the shape operator of the canal hypersurface (2.1) is obtained by

$$S^{\mathfrak{C}} = \begin{bmatrix} S^{\mathfrak{C}}_{11} & S^{\mathfrak{C}}_{12} & S^{\mathfrak{C}}_{13} \\ S^{\mathfrak{C}}_{21} & S^{\mathfrak{C}}_{22} & S^{\mathfrak{C}}_{23} \\ S^{\mathfrak{C}}_{31} & S^{\mathfrak{C}}_{32} & S^{\mathfrak{C}}_{33} \end{bmatrix}, \tag{2.14}$$

where the nonzero components of this matrix are

$$\begin{aligned}
S^{\mathfrak{C}}_{11} &= \frac{\sqrt{1-\rho'^2}\mathcal{W}+\rho''}{1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)}, \\
S^{\mathfrak{C}}_{21} &= \frac{\rho'\sqrt{1-\rho'^2}(x_1 \sin v - ax_2 \cos v)\sec^2 t + \left(1-\rho'^2\right)(cx_2 + dx_3 \sin v \tan t)\sec t}{\rho\left(-\sec t + \rho\sqrt{1-\rho'^2}(x_1 \cos v + ax_2 \sin v + bx_3 \tan t) + \left(\rho'^2+\rho\rho''\right)\sec t\right)}, \\
S^{\mathfrak{C}}_{31} &= -\frac{\rho'\sqrt{1-\rho'^2}(x_1 \cos v \sin t + ax_2 \sin v \sin t - bx_3 \cos t) + d\left(1-\rho'^2\right)x_3 \cos v}{\rho\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)}, \\
S^{\mathfrak{C}}_{22} &= S^{\mathfrak{C}}_{33} = -\frac{1}{\rho}.
\end{aligned}$$

Hence from (1.11) and (2.14), we get

Proposition 2.2. *The mean curvature of the canal hypersurfaces (2.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in Euclidean 4-space is*

$$H^{\mathfrak{C}} = -\frac{2\left(1-\rho'^2\right) - 3\rho\left(\sqrt{1-\rho'^2}\mathcal{W} + \rho''\right)}{3\rho\left(1-\rho'^2 - \rho\left(\sqrt{1-\rho'^2}\mathcal{W} + \rho''\right)\right)}. \tag{2.15}$$

Corollary 2.3. *The canal hypersurfaces (2.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in Euclidean 4-space cannot be minimal when $x_1 \neq 0$.*

Proof. Let we suppose that canal hypersurfaces (2.1) is minimal; i.e. $H^{\mathfrak{C}} = 0$. From (2.3) and (2.15), we get

$$(x_1 \cos v + a x_2 \sin v)\cos t + b x_3 \sin t - \frac{2\left(1-\rho'^2\right) - 3\rho\rho''}{3\rho\sqrt{1-\rho'^2}} = 0. \tag{2.16}$$

With similar procedure in the proof of Corollary 2.1, the proof completes. □

Corollary 2.4. *Let $\mathfrak{C}$ be a canal hypersurface according to generalized Bishop frame of type B (parallel transport frame) given by (2.1) in E^4. When α is a straight line, the canal hypersurface $\mathfrak{C}$ is minimal if and only the differential equation*

$$2 - 2\rho'^2(u) - 3\rho(u)\rho''(u) = 0 \tag{2.17}$$

holds.

Proof. If α is a straight line, then from (2.10), the mean curvature becomes (for $a = b = 1$, $c = d = 0$)

$$H^{\mathfrak{C}} = -\frac{2 - 2\rho'^2(u) - 3\rho(u)\rho''(u)}{3\rho\left(1-\rho'^2 - \rho\rho''\right)}. \tag{2.18}$$

Thus from (2.18), the proof is obvious. □

By solving the equation (2.17) (see [26]), we get the following corollary:

Corollary 2.5. *Let α be a straight line. Then, the canal hypersurface (2.1) according to generalized Bishop frame of type B (parallel transport frame) in E^4 is minimal if and only if the radius function $\rho(x)$ is given by $\int \frac{d\rho}{\sqrt{1-\left(\frac{\lambda}{\rho}\right)^{\frac{4}{3}}}} = \pm x + \mu$, $(\lambda, \mu \in \mathbb{R})$.*

Here, from (2.10) and (2.15), we can state the following theorem which gives an important relation between Gaussian and mean curvatures of the canal hypersurfaces (2.1) according to generalized Bishop frames of type B, type C and type D:

Corollary 2.6. *The Gaussian curvature K and the mean curvature H of the canal hypersurfaces (2.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in E^4 satisfy*

$$H^{\mathfrak{C}} = \frac{1}{3}(K^{\mathfrak{C}}\rho^2 - \frac{2}{\rho}). \tag{2.19}$$

Now, if $H_uK_v - H_vK_u = 0$, $H_uK_t - H_tK_u = 0$ or $H_vK_t - H_tK_v = 0$ hold on a hypersurface, then we call the hypersurface as $(H,K)_{\{u,v\}}$-Weingarten, $(H,K)_{\{u,t\}}$-Weingarten or $(H,K)_{\{v,t\}}$-Weingarten hypersurface, respectively.

From (2.10) and (2.15), we have

$$\tag{2.20} \begin{cases} K_u^{\mathfrak{C}} = \dfrac{\begin{pmatrix} 3\rho\rho'(1-\rho'^2)^{3/2}\mathcal{W}^2 + (2\rho'^3\rho'' + \rho'\rho''(-2+5\rho\rho''))\sqrt{1-\rho'^2} \\ +\rho'(1-\rho'^2)\left(-2+2\rho'^2+7\rho\rho''\right)\mathcal{W} + \rho\left(\sqrt{1-\rho'^2}\rho''' + \mathcal{W}_u\right) \\ -\rho\rho'^2\left(\sqrt{1-\rho'^2}\rho''' + 2\mathcal{W}_u\right) + \rho\rho'^4\mathcal{W}_u \end{pmatrix}}{\rho^3\sqrt{1-\rho'^2}\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)^2}, \\ K_v^{\mathfrak{C}} = \dfrac{(1-\rho'^2)^{3/2}\mathcal{W}_v}{\rho^2\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)^2}, \\ K_t^{\mathfrak{C}} = \dfrac{(1-\rho'^2)^{3/2}\mathcal{W}_t}{\rho^2\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)^2} \end{cases}$$

and

$$\tag{2.21} \begin{cases} H_u^{\mathfrak{C}} = \dfrac{\begin{pmatrix} 2\rho'(1-\rho'^2)^{5/2} - 4\rho\rho'(1-\rho'^2)\left((1-\rho'^2)\mathcal{W} + \sqrt{1-\rho'^2}\rho''\right) \\ +\rho^2\begin{pmatrix} 3\rho'(1-\rho'^2)^{3/2}\mathcal{W}^2 + 7\rho'(1-\rho'^2)\rho''\mathcal{W} + 5\rho'\rho''^2\sqrt{1-\rho'^2} \\ +\sqrt{1-\rho'^2}\rho''' - \rho'^2\left(\sqrt{1-\rho'^2}\rho''' + 2\mathcal{W}_u\right) + \mathcal{W}_u + \rho'^4\mathcal{W}_u \end{pmatrix} \end{pmatrix}}{3\rho^2\sqrt{1-\rho'^2}\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)^2}, \\ H_v^{\mathfrak{C}} = \dfrac{(1-\rho'^2)^{3/2}\mathcal{W}_v}{3\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)^2}, \\ H_t^{\mathfrak{C}} = \dfrac{(1-\rho'^2)^{3/2}\mathcal{W}_t}{3\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)^2}, \end{cases}$$

where

$$\tag{2.22} \begin{cases} W_u = x_1'\cos v\cos t + ax_2'\sin v\cos t + bx_3'\sin t, \\ W_v = \cos t\left(-x_1\sin v + ax_2\cos v\right), \\ W_t = -\sin t\left(x_1\cos v + ax_2\sin v\right) + bx_3\cos t. \end{cases}$$

So (2.20)-(2.22), we have

Proposition 2.3. *The canal hypersurface (2.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in E^4 is $(H^{\mathfrak{C}}, K^{\mathfrak{C}})_{\{v,t\}}$-Weingarten hypersurface.*

Proposition 2.4. *The canal hypersurface (2.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in E^4 cannot be $(H^{\mathfrak{C}}, K^{\mathfrak{C}})_{\{u,v\}}$ and $(H^{\mathfrak{C}}, K^{\mathfrak{C}})_{\{u,t\}}$-Weingarten hypersurface when $x_1 \neq 0$.*

Proof. Using (2.20)-(2.22), we get

$$\tag{2.23} H_u^{\mathfrak{C}}K_v^{\mathfrak{C}} - H_v^{\mathfrak{C}}K_u^{\mathfrak{C}} = \frac{2\rho'(1-\rho'^2)^2\begin{pmatrix} \rho\left((1-\rho'^2)\mathcal{W} + \sqrt{1-\rho'^2}\rho''\right) \\ -(1-\rho'^2)^{3/2} \end{pmatrix}(x_1\sin v - ax_2\cos v)\cos t}{3\rho'^4\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)^4}.$$

Let the canal hypersurface (2.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in E^4 is $(H^{\mathfrak{C}}, K^{\mathfrak{C}})_{\{u,v\}}$-Weingarten and $x_1 \neq 0$. Then from (2.23), we get

$$2\rho'(1-\rho'^2)^2\left(-(1-\rho'^2)^{3/2}+\rho\left((1-\rho'^2)\mathcal{W}+\sqrt{1-\rho'^2}\rho''\right)\right)(x_1\sin v - ax_2\cos v)\cos t = 0$$

and here, we have

$$\left(-(1-\rho'^2)^{3/2}+\rho\left((1-\rho'^2)\mathcal{W}+\sqrt{1-\rho'^2}\rho''\right)\right)(x_1\sin v - ax_2\cos v) = 0. \tag{2.24}$$

Because of the set $\{\sin v, \cos v\}$ is linear independent and $x_1 \neq 0$, the second part of (2.24) cannot be zero. Hence the first part of (2.24) must be zero, i.e.

$$-(1-\rho'^2)^{3/2}+\rho\left((1-\rho'^2)\mathcal{W}+\sqrt{1-\rho'^2}\rho''\right) = 0. \tag{2.25}$$

Using (2.3) in (2.25), we get

$$\cos t\,(x_1\cos v + ax_2\sin v) + bx_3\sin t - \frac{1-\rho'^2-\rho\rho''}{\rho\sqrt{1-\rho'^2}} = 0.$$

Since the set $\{\cos t, \sin t, 1\}$ is linear independent, we have

$$x_1\cos v + ax_2\sin v = bx_3 = \frac{1-\rho'^2-\rho\rho''}{\rho\sqrt{1-\rho'^2}} = 0. \tag{2.26}$$

So, from the first part of (2.26), it must be $x_1 = 0$ and this is a contradiction.

Similarly, from (2.20)-(2.22), we get

$$H_u^{\mathfrak{C}}K_t^{\mathfrak{C}} - H_t^{\mathfrak{C}}K_u^{\mathfrak{C}} = \frac{2\rho'(1-\rho'^2)^2\begin{pmatrix}\mathcal{W}\rho(1-\rho'^2)+\sqrt{1-\rho'^2}\rho\rho''\\ -(1-\rho'^2)^{3/2}\end{pmatrix}\begin{pmatrix}x_1\cos v\sin t\\ +ax_2\sin v\sin t\\ -bx_3\cos t\end{pmatrix}}{3\rho'^4\left(1-\rho'^2-\rho\left(\mathcal{W}\sqrt{1-\rho'^2}+\rho''\right)\right)^4} \tag{2.27}$$

and if the canal hypersurface (2.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in E^4 is $(H^{\mathfrak{C}}, K^{\mathfrak{C}})_{\{u,t\}}$, then from (2.27), we reach a similar contradiction with above. So, the proof completes. □

Proposition 2.5. *The canal hypersurface (2.1) according to generalized Bishop frame of type B (parallel transport frame) is $(H^{\mathfrak{C}}, K^{\mathfrak{C}})_{\{u,v\}}$-Weingarten and $(H^{\mathfrak{C}}, K^{\mathfrak{C}})_{\{u,t\}}$-Weingarten hypersurface when α is a straight line.*

Also, from (2.14) we have

$$\det(S^{\mathfrak{C}}-\lambda I_3) = \frac{(1+\lambda\rho)^2\left(-\lambda+\lambda\rho'^2+\sqrt{1-\rho'^2}\mathcal{W}+\rho''+\lambda\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)}{\rho^2\left(1-\rho'^2-\rho\left(\sqrt{1-\rho'^2}\mathcal{W}+\rho''\right)\right)}. \tag{2.28}$$

By solving the equation $\det(S^{\mathfrak{C}}-\lambda I_3) = 0$ from (2.28), we obtain the principal curvatures of the canal hypersurfaces (2.1) according to generalized Bishop frames of type B, type C and type D in E^4 as follows:

Proposition 2.6. *The principal curvatures of the canal hypersurfaces (2.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in E^4 are*

$$\lambda_1^{\mathfrak{C}} = \lambda_2^{\mathfrak{C}} = -\frac{1}{\rho},\ \lambda_3^{\mathfrak{C}} = \frac{\sqrt{1-\rho'^2}\mathcal{W} + \rho''}{1 - \rho'^2 - \rho\left(\sqrt{1-\rho'^2}\mathcal{W} + \rho''\right)}. \tag{2.29}$$

3. Tubular Hypersurfaces according to Generalized Bishop Frames in E^4

In this section, we study the tubular hypersurfaces according to generalized Bishop frames in E^4. By taking $\rho(u) = \rho$ =constant in (2.1), we get the tubular hypersurface $\mathcal{T}$ according to generalized Bishop frames in E^4 as

$$\mathcal{T}(u,v,t) = \alpha(u) \pm \rho\left[(\cos v \cos t)\, M_1(u) + (\sin v \cos t)\, M_2(u) + (\sin t)\, M_3(u)\right], \tag{3.1}$$

where $u \in [0,l]$ and $v,t \in [0,2\pi)$. Considering "$\pm$" as "$+$" in (3.1), we obtain the following results:

Firstly, from (1.5) and (3.1), the first derivatives of the tubular hypersurface (3.1) are obtained as

$$\begin{cases} \mathcal{T}_u = (1-\rho\mathcal{W})\, T - \rho\,(cx_2 \cos t \sin v + dx_3 \sin t)\, M_1 \\ \qquad + (c\rho x_2 \cos v \cos t)\, M_2 + (d\rho x_3 \cos v \cos t)\, M_3, \\ \mathcal{T}_v = -\rho\left((\sin v \cos t)\, M_1 - (\cos v \cos t)\, M_2\right), \\ \mathcal{T}_t = -\rho\left((\cos v \sin t)\, M_1 + (\sin v \sin t)\, M_2 - (\cos t)\, M_3\right). \end{cases} \tag{3.2}$$

From (1.6) and (3.2), the unit normal vector field of $\mathcal{T}$ in E^4 is

$$\mathcal{N}^{\mathcal{T}} = (\cos v \cos t)\, M_1 + (\sin v \cos t)\, M_2 + (\sin t)\, M_3. \tag{3.3}$$

Also, the nonzero coefficients of the first fundamental form are given by

$$\begin{cases} g_{11}^{\mathcal{T}} = (\rho \cos t \cos v)^2 \left(c^2 x_2^2 + d^2 x_3^2\right) + \rho^2 \left(cx_2 \cos t \sin v + dx_3 \sin t\right)^2 + (1-\rho\mathcal{W})^2, \\ g_{12}^{\mathcal{T}} = \rho^2 \left(cx_2 \cos t + dx_3 \sin v \sin t\right) \cos t,\ g_{13}^{\mathcal{T}} = d\rho^2 x_3 \cos v,\ g_{22}^{\mathcal{T}} = \rho^2 \cos^2 t,\ g_{33}^{\mathcal{T}} = \rho^2 \end{cases} \tag{3.4}$$

and it follows that

$$\det[g_{ij}^{\mathcal{T}}] = \rho^4 (1-\rho\mathcal{W})^2 \cos^2 t. \tag{3.5}$$

Now, for obtaining the coefficients of the second fundamental form, let we give the second derivatives $\mathcal{T}_{x_i x_j} = \frac{\partial^2 \mathcal{T}}{\partial x_i x_j}$ of the tubular hypersurface (3.1) as follows

$$\left\{ \begin{array}{l} \mathcal{T}_{uu} = \left(\rho \left(\begin{array}{l} x_1 \left(c x_2 \cos t \sin v + d x_3 \sin t \right) \\ -\left(a c x_2^2 + b d x_3^2 + x_1' \right) \cos v \cos t \\ -a x_2' \sin v \cos t - b x_3' \sin t \end{array} \right) - \rho \rho''' \right) T \\ \quad + \left(x_1 - \rho \left(\begin{array}{l} \left(x_1^2 + c^2 x_2^2 + d^2 x_3^2 \right) \cos v \cos t \\ + \left(a x_1 x_2 + c k_1' \right) \sin v \cos t + \left(b x_1 x_3 + d x_1' \right) \sin t \end{array} \right) \right) M_1 \\ \quad + \left(\begin{array}{l} -\left(a^2 + c^2 \right) \rho x_2^2 \sin v \cos t \\ + x_2 \left(a - a \rho x_1 \cos v \cos t - (ab + cd) \rho x_3 \sin t \right) + c \rho x_2' \cos v \cos t \end{array} \right) M_2 \\ \quad + \left(\begin{array}{l} x_3 \left(b - b \rho x_1 \cos v \cos t - (ab + cd) \rho x_2 \sin v \cos t \right) \\ -\left(b^2 + d^2 \right) \rho x_3^2 \sin t + d \rho x_3' \cos v \cos t \end{array} \right) M_3, \\ \mathcal{T}_{uv} = \mathcal{T}_{vu} = \left(\rho \left(x_1 \sin v - a x_2 \cos v \right) \cos t \right) T - \left(c \rho x_2 \cos v \cos t \right) M_1 \\ \quad - \left(c \rho x_2 \sin v \cos t \right) M_2 - \left(d \rho x_3 \sin v \cos t \right) M_3, \\ \mathcal{T}_{ut} = \mathcal{T}_{tu} = \rho \left(x_1 \cos v \sin t + a x_2 \sin t \sin v - b x_3 \cos t \right) T \\ \quad + \rho \left(c x_2 \sin v \sin t - d x_3 \cos t \right) M_1 - \left(c \rho x_2 \sin t \cos v \right) M_2 - \left(d \rho x_3 \sin t \cos v \right) M_3, \\ \mathcal{T}_{vv} = -\rho \left(\left(\cos v \cos t \right) M_1 + \left(\sin v \cos t \right) M_2 \right), \\ \mathcal{T}_{vt} = \mathcal{T}_{tv} = \rho \left(\left(\sin v \sin t \right) M_1 - \left(\sin t \cos v \right) M_2 \right), \\ \mathcal{T}_{tt} = -\rho \left(\left(\cos v \cos t \right) M_1 + \left(\sin v \cos t \right) M_2 + \left(\sin t \right) M_3 \right). \end{array} \right. \tag{3.6}$$

Thus, from (1.8), (3.3) and (3.6), the nonzero coefficients of the second fundamental form are given by

$$\left\{ \begin{array}{l} h_{11}^{\mathcal{T}} = -\rho x_1^2 \cos^2 v \cos^2 t + \frac{\rho}{2} \left(-a^2 - 2c^2 + a^2 \cos(2v) \right) x_2^2 \cos^2 t \\ \quad + x_2 \left(a - 2(ab + cd) \rho \sin t x_3 \right) \cos t \sin v \\ \quad + x_3 \left(b \sin t - \rho \left(d^2 \cos^2 v \cos^2 t + (b^2 + d^2) \sin^2 t \right) x_3 \right) \\ \quad + x_1 \left(1 - 2\rho \left(a x_2 \cos t \sin v + b x_3 \sin t \right) \right) \cos v \cos t, \\ h_{12}^{\mathcal{T}} = -\rho \left(c x_2 \cos t + d x_3 \sin t \sin v \right) \cos t, \\ h_{13}^{\mathcal{T}} = -d \rho x_3 \cos v, h_{22}^{\mathcal{T}} = -\rho \cos^2 t, h_{33}^{\mathcal{T}} = -\rho \end{array} \right. \tag{3.7}$$

and it implies

$$\det[h_{ij}^{\mathcal{T}}] = \rho^2 \mathcal{W} \left(1 - \rho \mathcal{W} \right) \cos^2 t. \tag{3.8}$$

So from (1.10), (3.5) and (3.8), we get

Proposition 3.1. *The Gaussian curvature of the tubular hypersurfaces (3.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in E^4 is*

$$K^{\mathcal{T}} = \frac{\mathcal{W}}{\rho^2 (1 - \rho \mathcal{W})}. \tag{3.9}$$

Thus from (3.9), we have

Corollary 3.1. *Let $\mathcal{T}$ be a tubular hypersurface according to generalized Bishop frame of type B (parallel transport frame) given by (3.1) in E^4. When α is a straight line, the tubular hypersurface $\mathcal{T}$ is flat.*

Also, after finding the inverse of the matrix of the first fundamental form and using this and (3.7) in (1.9), the shape operator of the tubular hypersurface (3.1) is obtained by

$$S^{\mathcal{T}} = \left[\begin{array}{ccc} S_{11}^{\mathcal{T}} & S_{12}^{\mathcal{T}} & S_{13}^{\mathcal{T}} \\ S_{21}^{\mathcal{T}} & S_{22}^{\mathcal{T}} & S_{23}^{\mathcal{T}} \\ S_{31}^{\mathcal{T}} & S_{32}^{\mathcal{T}} & S_{33}^{\mathcal{T}} \end{array} \right], \tag{3.10}$$

where the nonzero components of this matrix are

$$S_{11}^{\mathcal{T}} = \frac{\mathcal{W}}{1-\rho\mathcal{W}},$$
$$S_{21}^{\mathcal{T}} = \frac{(cx_2+dx_3\sin v\tan t)\sec t}{\rho(-\sec t+\rho x_1\cos v+a\rho x_2\sin v+b\rho x_3\tan t)},$$
$$S_{31}^{\mathcal{T}} = -\frac{dx_3\cos v}{\rho(1-\rho\mathcal{W})},$$
$$S_{22}^{\mathcal{T}} = S_{33}^{\mathcal{T}} = -\frac{1}{\rho}.$$

Hence from (1.11) and (3.10), we reach that

Proposition 3.2. *The mean curvature of the tubular hypersurface (3.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in E^4 is*

$$H^{\mathcal{T}} = \frac{-2+3\rho\mathcal{W}}{3\rho(1-\rho\mathcal{W})}. \tag{3.11}$$

Thus from (3.11), we get

Corollary 3.2. *Let $\mathcal{T}$ be a tubular hypersurface according to generalized Bishop frame of type B (parallel transport frame) given by (3.1) in E^4. When α is a straight line, the tubular hypersurface $\mathcal{T}$ is not minimal and it has negative mean curvature $\frac{-2}{3\rho}$.*

Furthermore, from (3.9) and (3.11), we have

Proposition 3.3. *The tubular hypersurface (3.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in E^4 is $(H^{\mathcal{T}}, K^{\mathcal{T}})_{\{u,v\}}$, $(H^{\mathcal{T}}, K^{\mathcal{T}})_{\{u,t\}}$ and $(H^{\mathcal{T}}, K^{\mathcal{T}})_{\{v,t\}}$-Weingarten hypersurface.*

Also, from (3.10) we have

$$\det(S^{\mathcal{T}} - \lambda I_3) = \frac{(1+\lambda\rho)^2\left(-1-\lambda\rho+\frac{1}{1-\rho\mathcal{W}}\right)}{\rho^3}. \tag{3.12}$$

By solving the equation $\det(S^{\mathcal{T}} - \lambda I_3) = 0$ from (3.12), we obtain the principal curvatures of the tubular hypersurfaces (3.1) in E^4 as follows:

Proposition 3.4. *The principal curvatures of the tubular hypersurfaces (3.1) according to generalized Bishop frames of type B (parallel transport frame), type C and type D in E^4 are*

$$\lambda_1^{\mathcal{T}} = \lambda_2^{\mathcal{T}} = -\frac{1}{\rho},\ \lambda_3^{\mathcal{T}} = \frac{\mathcal{W}}{1-\rho\mathcal{W}}. \tag{3.13}$$

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