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ON RECURRENCE RELATIONS FOR BERNOULLI POLYNOMIALS AND NUMBERS

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Abstract. In this work, we connect Bernoulli numbers and polynomials to Mersenne numbers via recurrence relations. We find two explicit formulas of Bernoulli numbers by means of Mersenne numbers which are different from those given by F. Qi and X. Y. Chen et al. Finally, we explore additional interesting relationships, which serve as bridges between the Bernoulli polynomials and Mersenne numbers.
Keywords: Bernoulli numbers, Bernoulli polynomials, recurrence relations.

1. Introduction

Bernoulli polynomials $B_n(x)$ [3] are defined by the generating function

$$\frac{ze^{xz}}{e^z-1}=\sum_{n\geq 0}B_n(x)\frac{z^n}{n!}. \tag{1.1}$$

The numbers $B_n = B_n(0)$ are Bernoulli numbers and can be computed by the recursion formula

$$B_n=-\delta_{1,n}+\sum_{k=0}^{n}\binom{n}{k}B_k, \tag{1.2}$$

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where $\delta_{k,n}$ is the Kronecker symbol. We have $B_1=\frac{-1}{2},\ B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ B_6=\frac{1}{42}$ and $B_{2n+1}=0$ for $n\geq 1$. The identity (1.2) is extracted from the well-known relation (24.5.3) in [17]:

$$\sum_{k=0}^{n-1}\binom{n}{k}B_k=0,\ n=2,3,\cdots.$$

Many other properties can be found in [1]. It is well known that the sum

$$S_r(n)=\sum_{k=1}^{n-1}k^r$$

is closely connected to Bernoulli polynomials $B_n(x)$ via the relation

$$S_r(n)=\frac{1}{r+1}\left(B_{r+1}(n)-B_{r+1}\right). \tag{1.3}$$

Bernoulli polynomials have found numerous important applications, notably in number theory and asymptotic analysis. Recurrence relations were soon used as the most important tool [2, 10, 19] for computing Bernoulli polynomials and numbers. Several generalizations of Bernoulli polynomials are defined, the most recent being the degenerate Bernoulli polynomial, which has been studied in [12, 14]. Degenerate versions of special polynomials is an active area of research and has yielded many arithmetic and combinatorial results. For any positive integers n we consider the partition set

$$\pi(n)=\left\{(k_1,\cdots,k_n)\in\mathbb{N}^n:\ \sum_{r=1}^{n}rk_r=n\right\},$$

where $\mathbb{N}=\{0,1,2,\cdots\}$ is the set of nonnegative integers. For any positive integer k and complex number x, we denote by $B(n,k,x)$ the sum

$$B(n,k,x)=\sum_{\pi(n)}\binom{k}{k_1,k_2,\cdots,k_n}\prod_{r=1}^{n}\left(\frac{B_r(x)}{r!}\right)^{k_r},$$

where

$$\binom{k}{k_1,k_2,\cdots,k_n}=\begin{cases}\frac{k!}{k_1!\cdots k_n!}, & \text{if } k_1+\cdots+k_n=k,\\ 0, & \text{otherwise}\end{cases}$$

is the multinomial coefficient. We write $B(n,k)=B(n,k,0)$ and we remark that $B(n,k,x)=0$ for $n<k$.

The outline of this paper is as follows. In the first section, we give a recurrence relations for $B_n(x)$ and B_n involving Mersenne numbers and the sums $B(n,k,x)$ different from those in the literature which involve binomial coefficient. To end with two explicit formulas of B_n by means of Mersenne numbers. The second section is reserved for the proof of all the theorems in the first section. In the last section, we express Mersenne numbers as sums involving the numbers $S_n(3)$ and $S_n(4)$ to deduce a recurrence relation for $S_n(4)$.

2. Main results

To establish the link between Bernoulli polynomials and Mersenne numbers $M_n = 2^n - 1$, one needs relations more complicated than the recurrence relations [17, (24.5.1)], [1, (23.1.7), (23.1.6)] and others in the literature [9, 10, 8].

Theorem 2.1. *The connection of Bernoulli polynomials to Mersenne numbers is given by the following recurrence relation.*

$$B_n(x) = (x+1)^n - \sum_{k=0}^{n-1} \frac{n! M_{n-k+1}}{k!(n-k+1)!} B_k(x). \tag{2.1}$$

Relation (2.1) can be written under the form

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{M_{n-k+1}}{n-k+1} B_k(x) - x^k \right) = 0.$$

By setting $x = 0$ we get the corresponding recurrence relation for Bernoulli numbers.

Corollary 2.1. *The following relation holds true.*

$$B_n = 1 - \sum_{k=0}^{n-1} \frac{n! M_{n-k+1}}{k!(n-k+1)!} B_k. \tag{2.2}$$

Since $B_{2n+1} = 0$, then we can write

$$\sum_{k=0}^{2n} \frac{(2n+1)! M_{2n-k+2}}{k!(2n-k+2)!} B_k = 1,$$

and then

$$B_{2n} = \frac{2}{3(2n+1)} \left(1 - \sum_{k=0}^{2n-1} \frac{(2n+1)! M_{2n-k+2}}{k!(2n-k+2)!} B_k \right). \tag{2.3}$$

Also with relation (2.2), we compute the successive values of the Bernoulli numbers. The first are

$$B_1 = 1 - \frac{M_2}{2},$$

$$B_2 = \frac{6 - 6M_2 + 3M_2^2 - 2M_3}{6},$$

and

$$B_4 = \frac{90(4 - M_2)^2(M_2 - 3) + 5(6 - 6M_2 + 3M_2^2 - 2M_3)^2 - 6M_5}{30}.$$

The combination of the sums $B(n, k, x)$ and Mersenne numbers gives another formulation for Bernoulli polynomials. The result is established in the following theorem.

Theorem 2.2. *We have $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$ and for $n \geq 2$ we get*

$$\begin{aligned} B_n(x) &= (x+1)^n - \frac{M_{n+1}}{n+1} + n! \sum_{k=2}^{n} (-1)^k B(n,k,x) \\ &+ \sum_{j=1}^{n-1} \sum_{k=1}^{j} (-1)^k j! \binom{n}{j} B(j,k,x)(x+1)^{n-j}. \end{aligned}$$

Consequently the expression of Bernoulli numbers by means of numbers M_n and $B(j,k)$ is given by the following corollary.

Corollary 2.2. *For $n \geq 2$ we have*

$$\begin{aligned} B_n &= 1 - \frac{M_{n+1}}{n+1} + n! \sum_{k=2}^{n} (-1)^k B(n,k) \\ &+ \sum_{j=1}^{n-1} \sum_{k=1}^{j} (-1)^k j! \binom{n}{j} B(j,k). \end{aligned} \tag{2.4}$$

Identity (2.4) conducts to the following corollary.

Corollary 2.3. *We have*

$$\begin{aligned} B_{2n} &= 1 - \frac{2^{2n+1}-1}{2n+1} + (2n)! \sum_{k=2}^{2n} (-1)^k B(2n,k) \\ &+ \sum_{j=1}^{2n-1} \sum_{k=1}^{j} (-1)^k j! \binom{2n}{j} B(j,k) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \frac{M_{2n+2}-1}{2n+2} - 1 &= \sum_{j=1}^{2n} \sum_{k=1}^{j} (-1)^k j! \binom{2n+1}{j} B(j,k) \\ &+ (2n+1)! \sum_{k=2}^{2n+1} (-1)^k B(2n+1,k). \end{aligned} \tag{2.6}$$

In [18] (see identities (1.3) and (1.5)), F. Qi and R. J. Chapman provided two closed forms for Bernoulli numbers. The first is according to Stirling numbers of the second kind and the second is expressed as the determinant of a known matrix. Recently, X.Y. Chen et al. [4] derived two closed formulas for Bernoulli numbers in terms of central factorial numbers of the second kind. The partial Bell polynomials $B_{n,k} := B_{n,k}\left((x_r)_{r\geq 1}\right)$ [5] of an infinite sequence $(x_r)_{r\geq 1}$ are defined by the generating function

$$\frac{1}{k!} \left(\sum_{n\geq 1} x_n \frac{z^n}{n!} \right)^k = \sum_{n\geq 1} B_{n,k}\left(x_1, x_2, \cdots\right) \frac{z^n}{n!} \tag{2.7}$$

for which the explicit formula is

$$B_{n,k}\left((x_r)_{r\geq 1}\right)=\sum_{\substack{k_1+\cdots+k_n=k\\ k_1+2k_2+\cdots+nk_n=n}}\frac{n!}{k_1!\cdots k_n!}\prod_{r=1}^{n}\left(\frac{x_r}{r!}\right)^{k_r}. \tag{2.8}$$

The complete Bell polynomials $Y_n := Y_n\left((x_r)_{r\geq 1}\right)$ [5] are defined by

$$Y_n=\sum_{k=1}^{n}B_{n,k},\ Y_0=1.$$

For which the generating function is

$$\exp\left(\sum_{n\geq 1}x_n\frac{z^n}{n!}\right)=\sum_{n\geq 0}Y_n\frac{z^n}{n!}.$$

The degenerate version of these polynomials is studied in several works, we can refer to [11, 15, 16] and reference therein. In addition, researching this degenerate version by connecting it with degenerate Dowling lattice can produce very interesting results, as the case for Degenerate Whitney Numbers of First and Second Kind of Dowling Lattices [13]. According to these polynomials the following theorem gives two other closed formulas for Bernoulli numbers in terms of Mersenne numbers.

Theorem 2.3. *For $n\geq 1$ we have*

$$B_n=1+\sum_{m=1}^{n}\binom{n}{m}\sum_{k=1}^{m}(-1)^k k!B_{m,k}\left(\left(\frac{M_{r+1}}{r+1}\right)_{r\geq 1}\right), \tag{2.9}$$

and

$$B_n=-n-\delta_{1,n}+(-1)^n\sum_{k=1}^{n}(-1)^k k!B_{n,k}\left(\left(\frac{M_{r+1}}{r+1}\right)_{r\geq 1}\right) \tag{2.10}$$

3. Proof of the main results

First we recall the tools necessary for the demonstration, in this case Mersenne numbers and the inverse of a generating function. The sequence of Mersenne numbers $M_n = 2^n - 1$ is defined by the ordinary generating function (see [6, Identity 4])

$$\frac{z}{1-3z+2z^2}=\sum_{n\geq 0}M_n z^n, \tag{3.1}$$

and the exponential generating function (see [6, Identity 7])

$$e^{2z}-e^{z}=2e^{\frac{3}{2}z}\sinh\left(\frac{z}{2}\right)=\sum_{n\geq 0}M_n\frac{z^n}{n!}. \tag{3.2}$$

Bell polynomials appears among others in the series expansion of the inverse of any invertible generating function. Let $f(z) = 1 + \sum_{n\geq 1} a_n z^n$; the series expansion of $f^{-1}(z)$ by means of $B_{n,k}$ is given as follows [7]

$$f^{-1}(z) = 1 + \sum_{n\geq 1}\sum_{k=1}^{n}(-1)^k k! B_{n,k}\left((r!a_r)_{r\geq 1}\right)\frac{z^n}{n!}. \tag{3.3}$$

3.1. Proof of Theorem 2.1

We have

$$\left(\sum_{n\geq 0} B_n(x)\frac{z^n}{n!}\right)\left(\sum_{n\geq 0}\frac{M_{n+1}}{n+1}\frac{z^n}{n!}\right) = e^{(x+1)z}$$

and then

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{M_{n-k+1}}{n-k+1}B_k(x)\right)\frac{z^n}{n!} = \sum_{n\geq 0}(x+1)^n\frac{z^n}{n!}.$$

Finally

$$\sum_{k=0}^{n}\binom{n}{k}\frac{M_{n-k+1}}{n-k+1}B_k(x) = (x+1)^n$$

and

$$B_n(x) = (x+1)^n - \sum_{k=0}^{n-1}\binom{n}{k}\frac{M_{n-k+1}}{n-k+1}B_k(x).$$

3.2. Proof of Theorem 2.2

To prove Theorem 2.2 we need the following lemma.

Lemma 3.1. *The connection of Mersenne numbers to partial Bell polynomials is given by the following relation.*

$$\begin{aligned}\frac{M_{n+1}}{n+1} &= \sum_{j=1}^{n}\sum_{k=1}^{j}\binom{n}{j}(x+1)^{n-j}(-1)^k k! B_{j,k}\left((B_r(x))_{r\geq 1}\right)\\ &+ (x+1)^n.\end{aligned} \tag{3.4}$$

Proof. From (3.3) we have

$$\frac{e^z-1}{ze^{xz}} = 1 + \sum_{n\geq 1}\sum_{k=1}^{n}(-1)^k k! B_{n,k}\left((B_r(x))_{r\geq 1}\right)\frac{z^n}{n!}.$$

Then

$$\sum_{n\geq 0}\frac{M_{n+1}}{n+1}\frac{z^n}{n!} = \left(\sum_{n\geq 0}\frac{(x+1)^n z^n}{n!}\right)\left(1 + \sum_{n\geq 1}\sum_{k=1}^{n}(-1)^k k! B_{n,k}\left((B_r(x))_{r\geq 1}\right)\frac{z^n}{n!}\right).$$

According to Cauchy product of series, the desired result follows. □

To get Theorem 2.2, we remark that

$$k!B_{j,k}\left((B_r(x))_{r\geq 1}\right) = j!B(j,k,x)$$

and then

$$\frac{M_{n+1}}{n+1} = (x+1)^n + \sum_{j=1}^{n}\sum_{k=1}^{j}\binom{n}{j}(x+1)^{n-j}(-1)^k j!B(j,k,x).$$

Since $B(n,1,x) = B_n(x)$, then

$$\begin{aligned}\frac{M_{n+1}}{n+1} &= (x+1)^n - B_n(x) + \sum_{k=2}^{n}(-1)^k n!B(n,k,x)\\ &+ \sum_{j=1}^{n-1}\sum_{k=1}^{j}\binom{n}{j}(x+1)^{n-j}(-1)^k j!B(j,k,x)\end{aligned}$$

and

$$\begin{aligned}B_n(x) &= (x+1)^n - \frac{M_{n+1}}{n+1} + \sum_{k=2}^{n}(-1)^k n!B(n,k,x)\\ &+ \sum_{j=1}^{n-1}\sum_{k=1}^{j}\binom{n}{j}(x+1)^{n-j}(-1)^k j!B(j,k,x).\end{aligned}$$

3.3. Proof of Theorem 2.3

We have

$$\sum_{n\geq 0} B_n\frac{z^n}{n!} = e^z\left(1+\sum_{n\geq 1}\frac{M_{n+1}}{n+1}\frac{z^n}{n!}\right)^{-1}.$$

But

$$\left(1+\sum_{n\geq 1}\frac{M_{n+1}}{n+1}\frac{z^n}{n!}\right)^{-1} = 1+\sum_{n\geq 1}\sum_{k=1}^{n}(-1)^k k!B_{n,k}\left(\left(\frac{M_{r+1}}{r+1}\right)_{r\geq 1}\right)\frac{z^n}{n!}.$$

We consider the sequence a_n defined by

$$a_0 = 1 \text{ and } a_n = \sum_{k=1}^{n}(-1)^k k!B_{n,k}\left(\left(\frac{M_{r+1}}{r+1}\right)_{r\geq 1}\right).$$

Then we get

$$\sum_{n\geq 0} B_n \frac{z^n}{n!} = 1 + \sum_{n\geq 1} \left(1 + \sum_{m=1}^{n} \binom{n}{m} a_m\right) \frac{z^n}{n!},$$

and the first identity (2.9) follows. For the second identity we have

$$\left(1 + \sum_{n\geq 1} \frac{M_{n+1}}{n+1} \frac{z^n}{n!}\right)^{-1} = e^{-z} \sum_{n\geq 0} B_n \frac{z^n}{n!},$$

and

$$e^{-z} \sum_{n\geq 0} B_n \frac{z^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} B_k \frac{z^n}{n!}.$$

But

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} B_k = (-1)^n \sum_{k=0}^{n} \binom{n}{k} B_k + (-1)^n n.$$

Then

$$e^{-z} \sum_{n\geq 0} B_n \frac{z^n}{n!} = \sum_{n\geq 0} (-1)^n \left(B_n + n + \delta_{1,n}\right) \frac{z^n}{n!}.$$

which leads to the desired result (2.10).

4. Further identities

Using the roots of the polynomial $1 - 3z + 2z^2$ we write the ordinary generating function of M_{n+1} under the form $f_2(z) = \frac{1}{(1-z)(1-2z)}$. Then we obtain

$$f_2(z) = \exp\left[-\log(1-z) - \log(1-2z)\right].$$

Since we have

$$\log(1-z) = -\sum_{n\geq 1} \frac{1}{n} z^n \text{ and } \log(1-2z) = -\sum_{n\geq 1} \frac{2^n}{n} z^n.$$

Then we can write

$$f_2(z) = \exp\left(\sum_{n\geq 1} \frac{1+2^n}{n} z^n\right).$$

One remarks that $1 + 2^n = S_n(3) = \frac{1}{n+1}\left(B_{n+1}(3) - B_{n+1}\right)$ and

$$n! M_{n+1} = Y_n \left(\left(\frac{r!\left(B_{r+1}(3) - B_{r+1}\right)}{r(r+1)}\right)_{r\geq 1}\right).$$

So the following theorem holds true.

Theorem 4.1. *For $n \geq 1$ we have*

$$M_{n+1} = \sum_{k=1}^{n} \sum_{\pi(n)} \frac{1}{k!} \binom{k}{k_1, \cdots, k_n} \prod_{r=1}^{n} \left(\frac{B_{r+1}(3) - B_{r+1}}{r(r+1)} \right)^{k_r}. \tag{4.1}$$

Now we consider the generating function

$$f_3(z) = \frac{1}{(1-z)(1-2z)(1-3z)} = \sum_{n \geq 0} A_n(3) z^n.$$

The explicit formula of $A_3(x)$ involves Mersenne numbers as it is explained in the following lemma.

Lemma 4.1. *For $n \geq 0$ we have*

$$A_n(3) = \frac{3^{n+2} - M_{n+3}}{2}. \tag{4.2}$$

Proof. We have

$$f_3(z) = \left(\sum_{n \geq 0} 3^n z^n \right) \left(\sum_{n \geq 0} \left(2^{n+1} - 1\right) z^n \right)$$

According to Cauchy product of generating functions we deduce that

$$A_n(3) = \sum_{k=0}^{n} 3^{n-k} \left(2^{k+1} - 1\right) = 3^n \left(2 \sum_{k=0}^{n} (2/3)^k - \sum_{k=0}^{n} (1/3)^k \right).$$

Since

$$2 \cdot 3^n \sum_{k=0}^{n} (2/3)^k = 2 \left(3^{n+1} - 2^{n+1}\right)$$

and

$$3^n \sum_{k=0}^{n} (1/3)^k = \frac{3^{n+1} - 1}{2}.$$

Then

$$A_n(3) = \frac{3^{n+2} - M_{n+3}}{2}.$$

□

It should be noted that the function f_3 allows us to build a recurrence relation for M_n. By noticing that

$$A_n(3) = \sum_{k=0}^{n} 3^{n-k} M_{k+1}$$

to obtain

$$M_{n+3} = 3^{n+2} - 2\sum_{k=0}^{n} 3^{n-k} M_{k+1}.$$

M_{n+3} is useful for constructing a recurrence relation for computing $B_n(4)$ as shown in the following theorem.

Theorem 4.2. *For $n \geq 1$ we have*

$$(4.3)\; M_{n+3} = 3^{n+2} - 2\sum_{k=1}^{n}\sum_{\pi(n)} \frac{1}{k!}\binom{k}{k_1, \cdots, k_n}\prod_{r=1}^{n}\left(\frac{B_{r+1}(4) - B_{r+1}}{r(r+1)}\right)^{k_r}.$$

Proof. On the one hand we have

$$f_3(z) = \sum_{n\geq 0} A_n(3) z^n.$$

On the other hand it is obvious to remark that

$$f_3(z) = \exp\left(\sum_{n\geq 1}\frac{B_{n+1}(4) - B_{n+1}}{n(n+1)} z^n\right)$$

and

$$f_3(z) = 1 + \sum_{n\geq 1} Y_n\left(\left(r!\frac{B_{r+1}(4) - B_{r+1}}{r(r+1)}\right)_{r\geq 1}\right)\frac{z^n}{n!}.$$

The comparison between the two identities makes it possible to deduce that

$$A_n(3) = \frac{1}{n!} Y_n\left(\left(r!\frac{B_{r+1}(4) - B_{r+1}}{r(r+1)}\right)_{r\geq 1}\right).$$

The desired result follows from the Lemma 4.1 and the expression of Y_n for the sequence $\left(r!\frac{B_{r+1}(4)-B_{r+1}}{r(r+1)}\right)_{r\geq 1}$. $\square$

So to be precise we get this recurrence relation for $S_n(4)$.

$$\frac{B_{n+1}(4) - B_{n+1}}{n(n+1)} = \frac{3^{n+2} - M_{n+3}}{2} - \sum_{k=2}^{n}\sum_{\pi(n)}\frac{1}{k!}\binom{k}{k_1, \cdots, k_n}\prod_{r=1}^{n}\left(\frac{B_{r+1}(4) - B_{r+1}}{r(r+1)}\right)^{k_r}.$$

As it is also possible to construct a recurrence relation for $S_n(3)$ with the Theorem 4.1.

5. Conclusion

In this work, we have extended some recurrence relations satisfied by Bernoulli numbers and polynomials to those involving Mersenne numbers. This allows us to give two closed formulas of Bernoulli numbers by means of Mersenne numbers. Moreover, we highlighted the link of Mersenne numbers with the sums $S_n(r)$ to give a recurrence relation for the numbers $S_n(4)$. In our approach we combined the generating functions of the Mersenne numbers and Bernoulli polynomials to build the necessary bridges using Bell polynomials.

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