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ASSESSING THE GENERALIZED PROCESS CAPABILITY INDEX C_{pyk} FOR LOMAX DISTRIBUTION USING DIFFERENT ESTIMATION METHODS AND BOOTSTRAP CONFIDENCE INTERVAL

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Abstract. Process capability index is an important tool for assessing process performance, used mostly in industrial areas. Many process capability indices have been proposed in the literature. In this article, we consider different estimation methods to estimate the generalized process capability index, C_{pyk}, introduced by Maiti et al. [32] for the Lomax distribution. Maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramér-von Mises (CVM), Anderson Darling (AD), right-tail Anderson Darling (RAD) and maximum product of spacings (MPS) methods are used during the estimation process. Next, bootstrap confidence intervals, namely, standard bootstrap (SB), percentile bootstrap (PB), and bias-corrected percentile (BCPB) are considered to obtain 95% confidence intervals for the proposed estimators of C_{pyk}. The performances of proposed estimators are compared via a Monte-Carlo simulation study for different parameter settings. Furthermore, we perform a simulation study to compare the coverage probabilities (CP) and average lengths (AL) of bootstrap confidence intervals. Finally, two real data sets are analyzed for illustrative purposes.
Keywords: Process capability index, bootstrap confidence intervals, Lomax distribution, Estimation, Monte Carlo simulation.

1. Introduction

Assessing the process capability and performance is a very important topic in

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industrial areas. Process capability indices (PCIs) are widely used tools for measuring the capability and performance of a process within specifications known as lower specification limits (L), upper specification limits (U) and target value (T). For this purpose, the first PCI was introduced by Juran et al. [24]. Assuming the process distribution is normal with process mean μ and process standard deviation σ, several PCIs are developed, such as C_p, C_{pk}, C_{pmk} and C_{pm}; see Kane [25], Chan et al. [3], Pearn et al. [39]. Choi and Owen [7] compare these mentioned PCIs. However, many industrial processes follow non-normal distributions. For this reason, PCIs developed under normality assumption do not perform well, see Gunter [18]. Therefore, the generalization of PCIs for non-normal processes was suggested by Clements [9], Constable and Hobbs [10], Pearn et al. [40], Chen et al. [5], Mukherjee and Singh [33]. Recently, Maiti et al. [32] have defined a new generalized process capability index (C_{pyk}) that can be used for normal, non-normal, continuous and discrete quality characteristics. As stated in Maiti et al. [32], almost all the popular indices are directly or indirectly related to C_{pyk}. Practitioners can easily use this index, which is defined as:

$$\begin{aligned} C_{pyk} &= \min\left\{\frac{F(U)-F(\mu_e)}{\frac{1}{2}-\alpha_2}, \frac{F(\mu_e)-F(L)}{\frac{1}{2}-\alpha_1}\right\} \\ &= \min\left\{\frac{F(U-\frac{1}{2}}{\frac{1}{2}-\alpha_2}, \frac{\frac{1}{2}-F(L)}{\frac{1}{2}-\alpha_1}\right\}. \end{aligned} \tag{1.1}$$

where $F(.)$ and μ_e denote the cdf and median of the process distribution, respectively. Here, $F(\mu_e) = \frac{F(L)+F(U)}{2}$, i.e., $F(L) + F(U) = 1$, The lower α_1-th and the upper α_2-th percentage point of process distribution is determined as $\alpha_1 = P(X < LDL)$ and $\alpha_2 = P(X > UDL)$, respectively. LDL and UDL denote the lower and upper desired limits, respectively.

Point estimation and construction of confidence intervals (CIs) for PCIs are two common techniques used to assess process capability in the literature, see Chen et al. [5]. Researchers have developed various point estimation methods for PCIs, such as Pearn and Chen [38], Pearn et al. [41], Pearn and Chen [37]. Hsiang and Taguchi [**?**] introduced the construction of confidence limits for PCIs. Then, several authors have studied on construction of confidence intervals for non-normal process distributions, see Peng [42, 43], Leiva et al. [29], Chou et al. [8], Franklin and Wasserman [49], Kashif et al. [26], Kashif et al. [27], Weber et al. [50], Rao et al. [50], Dey et al. [12], Dey and Saha [11].

For estimating the parameters of a process distribution, the most frequently used estimation method is the well-known maximum likelihood (ML) methodology, see Mukherjee and Singh [33], Maiti et al. [32], Rao et al. [45], Dey et al. [12], Saha et al. [47]. However, ML methodology doesn't perform well in some cases, especially in the case of small samples and the presence of outliers. Therefore, researchers use other estimation methods to estimate the parameters of process distributions and propose PCIs by using estimated parameters, such as Dey and Saha [11], Saha et al. [47], Nooghabi [35], Gedik Balay [16].

In this paper, we propose $\hat{C}_{pyk}$ based on ML, least squares (LS), weighted least squares (WLS), Anderson Darling (AD) and Right-tail Anderson-Darling (RAD), Cramèr-von-Mises estimators (CVM) and maximum product of spacing (MPS) estimators for parameters of Lomax distribution. Then we compare their performances in terms of bias and mean squared error (MSE) criteria. Further, three bootstrap confidence intervals (BCIs), namely standard bootstrap (SB), percentile bootstrap (PB) and bias-corrected percentile bootstrap (BCPB) are considered for obtaining CIs of C_{pyk} based on proposed estimators. We also compare coverage probabilities (CP) and average lengths (AL) of them.

To the best of our knowledge, there is no study on C_{pyk} obtained using ML, LS, WLS, AD, RAD, CVM and MPS estimators and three bootstrap CIs based on them, when the process distribution is Lomax. The reason why we use Lomax distribution as the process distribution is that it can be used quite widely applied in the variety of contexts, see Giles et al. [17]. It was originally introduced by Lomax [28] for modelling business failure data. However, Lomax distribution has also been studied for reliability modelling and life testing, see Hassan and Al-Ghamdi [19], Hu and Gui [23], Mahmoud et al. [31]. It has also been used in economics, actuarial science, bioscience and engineering, see Holland et al. [21]. Bryson [2] suggested that Lomax distribution is a very good alternative to exponential distribution when the data exhibit heavy-tailed behaviour. As far as we know, although Lomax distribution gains more attention from researchers, the PCI C_{pyk} has not been examined when the underlying distribution is Lomax.

The probability density function (pdf) and the cdf of Lomax distribution are defined as follows:

$$f(x) = \frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}, \quad x>0, \quad \beta>0, \quad \lambda>0 \tag{1.2}$$

and

$$F(x) = 1-\left(1+\frac{x}{\lambda}\right)^{-\beta}, \quad x>0, \quad \beta>0, \quad \lambda>0, \tag{1.3}$$

respectively. The PCI C_{pyk} for Lomax distribution is

$$C_{pyk} = \min\left\{\frac{\frac{1}{2}-\left(1+\frac{U}{\lambda}\right)^{-\beta}}{\frac{1}{2}-\alpha_2}, \frac{\left(1+\frac{L}{\lambda}\right)^{-\beta}-\frac{1}{2}}{\frac{1}{2}-\alpha_1}\right\}. \tag{1.4}$$

The estimators of the parameters of Lomax distribution to be studied herein are defined in the following sections.

It should be stated that, to the best of our knowledge, this is the first study considering ML, LS, WLS, AD, RAD, CVM and MPS estimation of C_{pyk} for Lomax distribution and three BCIs based on them.

The rest of this paper is organized as follows: In Section 2., we develop estimators of C_{pyk} for Lomax distribution using different estimation methods. In Section 3., BCIs (SB, PB and BCPB) are discussed for $\hat{C}_{pyk}$. Monte Carlo simulation study is performed to compare the performance of the proposed estimators of C_{pyk} in terms of bias and MSE criteria in Section 4.. Further, we compare the performances of BCIs based on $\hat{C}_{pyk}$ obtained using the mentioned estimation methods via simulation study in Section 4.. In Section 5., we give two real data examples for illustrative purposes. Finally, concluding remarks are given in Section 6..

2. Different estimation methods of the C_{pyk}

In this section, we describe seven different estimators, namely, ML, LS, WLS, AD, RAD, CVM and MPS to obtain the estimators of parameters of Lomax distribution and the corresponding estimator of C_{pyk}.

2.1. Maximum likelihood estimator

In this section, we consider the ML estimate of C_{pyk}. First, the ML estimates of parameters of Lomax distribution are obtained. Let $y_1, y_2, ..., y_n$ be a random sample from Lomax distribution with β and λ parameters. Then the log-likelihood function is

$$\text{lnL} = \text{nlog}(\beta) - \text{nlog}(\lambda) - (1+\beta)\sum_{i=1}^{n}\log(1+y_i/\lambda). \tag{2.1}$$

By taking derivatives of (2.1) with respect to the parameters of interest, the likelihood equations are derived as

$$\frac{\partial \text{lnL}}{\partial \beta} = (n/\beta) - \sum_{i=1}^{n}(1+y_i/\lambda), \tag{2.2}$$

$$\frac{\partial \text{lnL}}{\partial \lambda} = -(n/\lambda) + [(1+\beta)/\lambda]\sum_{i=1}^{n}[y_i/(\lambda+y_i)]. \tag{2.3}$$

The likelihood equations above do not have closed-form solutions, so we need to use an iterative methods to get the ML estimates of β and λ. We use *optim()* function in the R software to solve nonlinear functions.
The ML-based C_{pyk} is obtained by substituting the ML estimators of parameters of Lomax distribution, $\hat{\beta}^{(1)}$ and $\hat{\lambda}^{(1)}$, in (1.4) as

$$\hat{C}_{pyk}^{(1)} = \min\left\{\frac{\frac{1}{2}-\left(1+\frac{U}{\hat{\lambda}^{(1)}}\right)^{-\hat{\beta}^{(1)}}}{\frac{1}{2}-\alpha_2}, \frac{\left(1+\frac{L}{\hat{\lambda}^{(1)}}\right)^{\hat{\beta}^{(1)}}-\frac{1}{2}}{\frac{1}{2}-\alpha_1}\right\}. \tag{2.4}$$

2.2. Least squares estimator

Let $x_{(1)} < x_{(2)} < ... < x_{(n)}$ be the order statistics of a random sample from Lomax distribution. The LS estimators of the parameters β and λ are obtained by minimizing the following function with respect to the parameters of interest, see Swain et al. [48].

$$S = \sum_{i=1}^{n} \left(F(x_{(i)}) - \frac{i}{n+1} \right)^2 . \tag{2.5}$$

Here $F(.)$ is the cdf of Lomax given in (1.3). LS estimators of β and λ can be also obtained by solving

$$\begin{aligned} \frac{\partial S}{\partial \beta} &= \sum_{i=1}^{n} \left(F(x_{(i)}; \beta, \lambda) - \frac{i}{n+1} \right) \Lambda_1(x_{(i)}; \beta, \lambda) = 0, \\ \frac{\partial S}{\partial \lambda} &= \sum_{i=1}^{n} \left(F(x_{(i)}; \beta, \lambda) - \frac{i}{n+1} \right) \Lambda_2(x_{(i)}; \beta, \lambda) = 0, \end{aligned} \tag{2.6}$$

where,

$$\Lambda_1(x_{(i)}; \beta, \lambda) = \left(1 + \frac{x}{\lambda} \right)^{-\beta} \ln \left(1 + \frac{x}{\lambda} \right) \tag{2.7}$$

and

$$\Lambda_2(x_{(i)}; \beta, \lambda) = \beta \left(1 + \frac{x}{\lambda} \right)^{-\beta - 1} \left(- \frac{x}{\lambda^2} \right). \tag{2.8}$$

It is obvious that, since equations given in (2.6) include nonlinear functions, numerical methods should be performed to obtain LS estimators of β and λ. Substituting the LS estimators of β and λ, say $\hat{\beta}^{(2)}$ and $\hat{\lambda}^{(2)}$ in equation (1.4), we can get the estimator of C_{pyk} as

$$\hat{C}_{pyk}^{(2)} = \min \left\{ \frac{\frac{1}{2} - \left(1 + \frac{U}{\hat{\lambda}^{(2)}} \right)^{-\hat{\beta}^{(2)}}}{\frac{1}{2} - \alpha_2}, \frac{\left(1 + \frac{L}{\hat{\lambda}^{(2)}} \right)^{\hat{\beta}^{(2)}} - \frac{1}{2}}{\frac{1}{2} - \alpha_1} \right\}. \tag{2.9}$$

2.3. Weighted least squares estimator

The WLS estimators of the parameters β and λ are obtained by minimizing the following function:

$$S_w = \sum_{i=1}^{n} w_i \left(F(x_{(i)}) - \frac{i}{n+1} \right)^2 \tag{2.10}$$

where w_i denotes the weight and is computed by

$$w_i = \frac{1}{\mathrm{Var}(F(X_{(i)}))} = \frac{(n+1)^2(n+2)}{i(n-i-1)}, \quad i = 1, 2, ..., n.$$

The WLS estimators of β and λ are obtained by solving the following nonlinear equations:

$$\begin{aligned} \frac{\partial S_w}{\partial \beta} &= \sum_{i=1}^{n} w_i \left(F(x_{(i)}; \beta, \lambda) - \frac{i}{n+1} \right) \Lambda_1(x_{(i)}; \beta, \lambda) = 0, \\ \frac{\partial S_w}{\partial \lambda} &= \sum_{i=1}^{n} w_i \left(F(x_{(i)}; \beta, \lambda) - \frac{i}{n+1} \right) \Lambda_2(x_{(i)}; \beta, \lambda) = 0, \end{aligned} \tag{2.11}$$

respectively. Here Λ_1 and Λ_2 are given in (2.7) and (2.8), respectively. It is clear that WLS estimators should also be obtained using numerical methods, since equations given in (2.11) cannot be solved explicitly. Substituting the WLS estimators of β and λ, say $\hat{\beta}^{(3)}$ and $\hat{\lambda}^{(3)}$ in equation (1.4), we can get the estimator of C_{pyk} as

$$\hat{C}_{pyk}^{(3)} = \min \left\{ \frac{\frac{1}{2} - \left(1 + \frac{U}{\hat{\lambda}^{(3)}}\right)^{-\hat{\beta}^{(3)}}}{\frac{1}{2} - \alpha_2}, \frac{\left(1 + \frac{L}{\hat{\lambda}^{(3)}}\right)^{\hat{\beta}^{(3)}} - \frac{1}{2}}{\frac{1}{2} - \alpha_1} \right\}. \tag{2.12}$$

2.4. Anderson-Darling estimators

The AD estimators $\hat{\beta}^{(4)}$ and $\hat{\lambda}^{(4)}$ of β and λ are obtained by minimizing the following equation with respect to the parameters of interest, see Anderson and Darling [1].

$$A = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log \left[F(x_{(i)}) \left(1 - F(x_{(j)}) \right) \right] \right\}, \tag{2.13}$$

where $j = n - i + 1$. The AD estimators of β and λ are also obtained by solving the following nonlinear equations

$$\begin{aligned} \frac{\partial A}{\partial \beta} &= \sum_{i=1}^{n} (2i-1) \left[\frac{\Lambda_1(x_{(i)}, \beta, \lambda)}{F(x_{(i)}, \beta, \lambda)} - \frac{\Lambda_1(x_{(j)}, \beta, \lambda)}{F(x_{(j)}, \beta, \lambda)} \right] = 0, \\ \frac{\partial A}{\partial \lambda} &= \sum_{i=1}^{n} (2i-1) \left[\frac{\Lambda_2(x_{(i)}, \beta, \lambda)}{F(x_{(i)}, \beta, \lambda)} - \frac{\Lambda_2(x_{(j)}, \beta, \lambda)}{F(x_{(j)}, \beta, \lambda)} \right] = 0, \end{aligned} \tag{2.14}$$

respectively. Here, Λ_1 and Λ_2 are given in (2.7) and (2.8). Nonlinear equations given in (2.14) can be solved by using numerical methods. Substituting the AD estimators $\hat{\beta}^{(4)}$ and $\hat{\lambda}^{(4)}$ in equation (1.4), we can get the estimator of C_{pyk} as

$$\hat{C}_{pyk}^{(4)} = \min \left\{ \frac{\frac{1}{2} - \left(1 + \frac{U}{\hat{\lambda}^{(4)}}\right)^{-\hat{\beta}^{(4)}}}{\frac{1}{2} - \alpha_2}, \frac{\left(1 + \frac{L}{\hat{\lambda}^{(4)}}\right)^{\hat{\beta}^{(4)}} - \frac{1}{2}}{\frac{1}{2} - \alpha_1} \right\}. \tag{2.15}$$

2.5. Right-tail Anderson-Darling estimator

The RAD estimators $\hat{\beta}^{(5)}$ and $\hat{\lambda}^{(5)}$ of the parameters β and λ are obtained by minimizing the following equation with respect to the parameters of interest.

$$R = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{(i)}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log(1 - F(x_{(j)})). \tag{2.16}$$

The RAD estimators of β and λ are also obtained by solving the following nonlinear equations:

$$\begin{aligned} -2\sum_{i=1}^{n} \frac{\Lambda_1(x_{(i)},\beta,\lambda)}{F(x_{(i)},\beta,\lambda)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Lambda_1(x_{(i)},\beta,\lambda)}{1 - F(x_{(i)},\beta,\lambda)} &= 0, \\ -2\sum_{i=1}^{n} \frac{\Lambda_2(x_{(i)},\beta,\lambda)}{F(x_{(i)},\beta,\lambda)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Lambda_2(x_{(i)},\beta,\lambda)}{1 - F(x_{(i)},\beta,\lambda)} &= 0. \end{aligned} \tag{2.17}$$

where Λ_1 and Λ_2 are the same as given in (2.7) and (2.8), respectively. Substituting the RAD estimators $\hat{\beta}^{(5)}$ and $\hat{\lambda}^{(5)}$ in equation (1.4), we can get the estimator of C_{pyk} as

$$\hat{C}_{pyk}^{(5)} = \min \left\{ \frac{\frac{1}{2} - \left(1 + \frac{U}{\hat{\lambda}^{(5)}}\right)^{-\hat{\beta}^{(5)}}}{\frac{1}{2} - \alpha_2}, \frac{\left(1 + \frac{L}{\hat{\lambda}^{(5)}}\right)^{\hat{\beta}^{(5)}} - \frac{1}{2}}{\frac{1}{2} - \alpha_1} \right\}. \tag{2.18}$$

2.6. Cramér-von Mises estimators

CVM estimators of the parameters of Lomax distribution, $\hat{\beta}^{(6)}$ and $\hat{\lambda}^{(6)}$, are obtained by minimizing the following equation with respect to the parameters β and λ, respectively, see MacDonald [30].

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_{(i)},\beta,\lambda) - \frac{2i-1}{2n}\right)^2 \tag{2.19}$$

To obtain the CVM estimators of the parameters, we can also solve the following equations by using numerical methods.

$$\begin{aligned}\frac{\partial CVM}{\partial \beta} &= \sum_{i=1}^{n}\left(F(x_{(i)};\beta,\lambda)-\frac{2i-1}{2n}\right)\Lambda_1(x_{(i)};\beta,\lambda)=0,\\ \frac{\partial CVM}{\partial \lambda} &= \sum_{i=1}^{n}\left(F(x_{(i)};\beta,\lambda)-\frac{2i-1}{2n}\right)\Lambda_2(x_{(i)};\beta,\lambda)=0.\end{aligned} \tag{2.20}$$

Here, Λ_1 and Λ_2 are given in (2.7) and (2.8). Substituting the CVM estimators $\hat{\beta}^{(6)}$ and $\hat{\lambda}^{(6)}$ in equation (1.4), we can get the estimator of C_{pyk} as

$$\hat{C}_{pyk}^{(6)} = \min\left\{\frac{\frac{1}{2}-\left(1+\frac{U}{\hat{\lambda}^{(6)}}\right)^{-\hat{\beta}^{(6)}}}{\frac{1}{2}-\alpha_2}, \frac{\left(1+\frac{L}{\hat{\lambda}^{(6)}}\right)^{\hat{\beta}^{(6)}}-\frac{1}{2}}{\frac{1}{2}-\alpha_1}\right\}. \tag{2.21}$$

2.7. Maximum product of spacing estimator

The MPS estimation method is introduced by Cheng and Amin [4] and Ranneby [44]. The MPS estimator is obtained by maximizing the geometric mean of spacing defined as

$$D_i(\beta,\lambda)=F(x_{(i)};\beta,\lambda)-F(x_{(i-1)};\beta,\lambda), \quad i=1,2,...,n+1, \tag{2.22}$$

where $F(x_{(0)};\beta,\lambda)=0$ and $F(x_{(n+1)};\beta,\lambda)$=1. Then, the MPS estimators of parameters β and λ, say $\hat{\beta}^{(7)}$ and $\hat{\lambda}^{(7)}$ are obtained by maximizing the following function:

$$MPS=\frac{1}{n+1}\sum_{i=1}^{n}\log D_i(\beta,\lambda). \tag{2.23}$$

After taking partial derivatives of MPS function with respect to β and λ, following nonlinear equations are obtained as

$$\begin{aligned}\frac{\partial MPS}{\partial \beta} &= \frac{1}{n+1}\sum_{i=1}^{n}\frac{\Lambda_1(x_{(i)};\beta,\lambda)-\Lambda_1(x_{(i)};\beta,\lambda)}{D_i(\beta,\lambda)}=0,\\ \frac{\partial MPS}{\partial \lambda} &= \frac{1}{n+1}\sum_{i=1}^{n}\frac{\Lambda_2(x_{(i)};\beta,\lambda)-\Lambda_2(x_{(i)};\beta,\lambda)}{D_i(\beta,\lambda)}=0,\end{aligned} \tag{2.24}$$

where Λ_1 and Λ_2 are defined in equations (2.7) and (2.8), respectively. Substituting the MPS estimators in (1.4), we can get the estimator of C_{pyk} as

$$\hat{C}^{(7)}_{pyk} = \min\left\{ \frac{\frac{1}{2} - \left(1 + \frac{U}{\hat{\lambda}^{(7)}}\right)^{-\hat{\beta}^{(7)}}}{\frac{1}{2} - \alpha_2}, \frac{\left(1 + \frac{L}{\hat{\lambda}^{(7)}}\right)^{\hat{\beta}^{(7)}} - \frac{1}{2}}{\frac{1}{2} - \alpha_1} \right\}. \tag{2.25}$$

3. Bootstrap confidence intervals

In this section, we discuss SB, PB and BCPB for constructing BCIs of C_{pyk} based on proposed estimators. The bootstrap method is introduced by Efron [13]. BCIs have been widely used to obtain CIs for PCIs, for example Choi and Bai [6], Franklin and Wasserman [49], Dey et al. [12], Park et al. [36].

Let $x_1, x_2, ..., x_m$ be a sample of size m taken from a process. The bootstrap procedure can be defined as follows:

- A bootstrap sample, $x^*_1, x^*_2, ..., x^*_n$ of size n is chosen from an original sample with replacement.
- Calculate $\hat{C}_{pyk}$ from a new bootstrap sample, say $\hat{C}^{*(1)}_{pyk}$.
- Repeat this process B times and obtain $\hat{C}^{*(1)}_{pyk}, \hat{C}^{*(2)}_{pyk}, ..., \hat{C}^{*(B)}_{pyk}$.

It should be noted that a minimum $B = 1000$ bootstrap samples are needed for obtaining reliable CI estimates, see Efron and Tibshirani [14].

3.1. Standard bootstrap (SB) confidence interval

Let $\hat{C}^{*(j)}_{pyk}$ is the j-th bootstrap estimate of C_{pyk}. The sample mean and sample standard deviation of $\{\hat{C}^{*(j)}_{pyk}; j = 1, 2, ..., B$ are computed as

$$\bar{\hat{C}}^*_{pyk} = \frac{1}{B}\sum_{j=1}^{B} \hat{C}^{*(B)}_{pyk} \tag{3.1}$$

and

$$S^* = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B} (\hat{C}^{*(j)}_{pyk} \bar{\hat{C}}^*_{pyk})^2}, \tag{3.2}$$

respectively. The $100(1-\alpha)\%$ CI for C_{pyk} is given by

$$\left[\bar{\hat{C}}^*_{pyk} - z_{(\alpha/2)}S^*, \bar{\hat{C}}^*_{pyk} - z_{(\alpha/2)}S^*\right]. \tag{3.3}$$

Here $z_{(\alpha/2)}$ is the $(1-\alpha/2)$th quantile of the standard normal distribution.

3.2. Percentile bootstrap (PB) confidence intervals

Let $B = 1000$ $\hat{C}_{pyk}$ are ordered as $\hat{C}^{*(1)}_{pyk} \leq \hat{C}^{*(2)}_{pyk} \leq ... \leq \hat{C}^{*(B)}_{pyk} \leq$. The α level percentile bootstrap CI is given as

$$\left[\hat{C}^{*}_{pyk}([1000\alpha/2]), \hat{C}^{*}_{pyk}([1000(1-\alpha/2)])\right]. \tag{3.4}$$

3.3. Bias-corrected percentile bootstrap (BCPB) interval

Efron [13] introduced a BCPB method to overcome the potential bias of bootstrap distribution stated by Franklin and Gary [15].
Let order $\{\hat{C}^{*(j)}_{pyk}, j = 1, 2, ..., B$ in ascending order and compute the following probabilities:

$$p_0 = \frac{1}{B}\sum_{j=1}^{B} I(\hat{C}^{*(j)}_{pyk} \leq \hat{C}_{pyk}), \quad p_l = \Phi(2z_0 - z_{\alpha/2}) \quad \text{and} \quad p_u = \Phi(2z_0 + z_{\alpha/2}).$$

Here, $I(.)$ is the indicator function and $z_0 = \Phi^{-1}(p_0)$. The BCPB confidence interval is

$$\left[\hat{C}^{*}_{pyk}(1000p_l), \hat{C}^{*}_{pyk}(1000p_u)\right]. \tag{3.5}$$

4. Simulation study

In this section, we carry out a Monte-Carlo simulation study to compare the performance of estimators of C_{pyk} for Lomax distribution based on seven different estimators (ML, LS, WLS, AD, RAD, CVM, MPS) of parameters of Lomax distribution in terms of bias and mean squared error (MSE) criteria. We also perform a simulation study to assess the performance of the proposed BCIs (SB, PB, BCPB) of the C_{pyk} for Lomax distribution based on different methods of estimation with respect to the estimated ALs and CPs.

We use various sample sizes as $n = 20, 50, 200, 500$ and parameter settings as $\beta = (0.5, 2)$ and $\lambda = (0.1, 1, 3)$ during the simulation process. We set the lower and upper specification limits as 0.2 and 10.2, respectively. We also consider $\alpha_1 = 0.03$ and $\alpha_2 = 0.01$ for illustration, similar to a number of studies in the literature.

The simulations are performed using programs written in the open source statistical package R. All computations are done based on $[\![100.000/n]\!]$ Monte Carlo runs. Here $[\![.]\!]$ represents the greatest integer value function. We use $B = 1000$ replicates for bootstrap procedures.

Bias and MSE values are calculated by

$$Bias(\hat{C}_{pyk}) = \hat{E}(\hat{C}_{pyk} - C_{pyk}), \tag{4.1}$$

and

$$MSE(\hat{C}_{pyk}) = \hat{E}\Big\{(\hat{C}_{pyk} - C_{pyk})^2\Big\}. \tag{4.2}$$

$\hat{E}(.)$ denotes the means of the observed values.

In Table 6.1, we report bias and MSE values of $\hat{C}_{pyk}$ for Lomax distribution obtained by using mentioned estimation methods. It is observed from Table 6.1 that when $n = 20$ and 50, the MPS estimators of C_{pyk} have the smallest bias. However, the ML estimator of C_{pyk} for Lomax distribution has lower bias values for $n = 100$, 200 and 500. As we expect, when the sample size increases, bias values decrease and all the estimates have negligible bias.

In terms of the MSE criterion, the ML estimates of C_{pyk} are the most efficient estimator with the lowest MSE values for $n = 100$, 200 and 500. It is followed by the MPS method in most cases. Furthermore, the MPS estimators show the best performance with the smallest MSE values, when $n = 20$ and 50 for all scenarios. It should be noted that the CVM estimators of C_{pyk} for Lomax distribution show the worst performance with the highest bias and MSE values in almost all cases. Also it is clear from Table 6.1 that when the sample size increases, the MSE values decreases.

Table 6.2 demonstrates the estimated CPs and ALs of bootstrap CIs of the C_{pyk} for Lomax distribution using ML, LS, WLS, AD, RAD, CVM and MPS methods. The simulation results show that the BCPB confidence intervals for the MPS estimator provides the highest CP and the smallest AL for $n = 20$ and 50 for all selected parameter values. However, BCPB confidence intervals for th ML estimators of C_{pyk} demonstrates the strongest performance in most of the cases for $n = 100$, 200 and 500 in terms of both CP and AL criteria.

For $n = 100$ and 200, when $\beta = 2$ and $\alpha = 3$, SB confidence intervals of the ML estimates have the best performance with the highest CP and the smallest AL among the others.

Overall, we suggest using the MPS method for estimating C_{pyk} for Lomax distribution when the sample size is less and equal to 50. However, the ML method is more preferable for large values of the sample size.

We also may conclude that BCPB confidence intervals show overall better performance in terms of both CP and AL in most cases. It is also observed from Table 6.2 that as the sample size increases, CPs increase and ALs decrease for all BCIs.

5. Real data analysis

In this section, two real data sets are considered for illustrative purposes. First, we check whether the considered data set fits the Lomax distribution by using well-known and widely used goodness of fit test Kolmogorov-Smirnov (KS) test for each estimation method. According to KS statistics given in Table 6.3, we can say that the Lomax distribution with all proposed estimators of β and λ provides adequate fits to the data set I and data set II at the 5% level. These observations are supported by Q-Q plots shown in Figure 5.1 and Figure 5.2 for data set I and data set II, respectively.

We also consider four model selection criteria called log-likelihood (logL), Akaike information criterion (AIC), Bayesian information criterion (BCI), the root mean

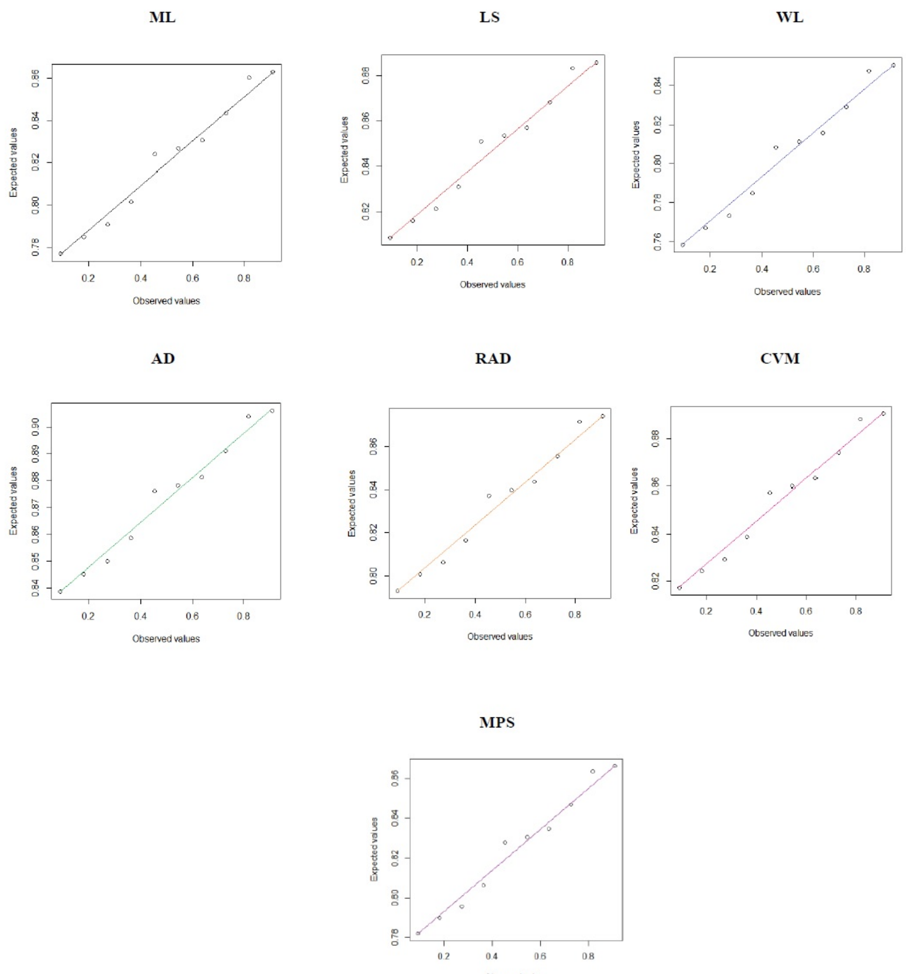

FIG. 5.1: Lomax QQ plot for the data set I.

square error (RMSE) to determine the most efficient estimation method. The values of estimators of the parameters β and α, estimators of C_{pyk} based on them, -logL, AIC, BIC, RMSE and KS test statistic D values are reported in Table 6.3.

It is clear from Table 6.3, values of all criteria are similar for each estimation method. However, the ML and MPS have similar fitting performance with the smaller with smaller model selection criteria values for each data set.

Data set I: The first data set is considered in Hong et al. (2009), Wong (1998) and Nigm and Hamdy (1987). The data set represents the length of time in years for which a business operates until failure. The 10 observations are as follows:

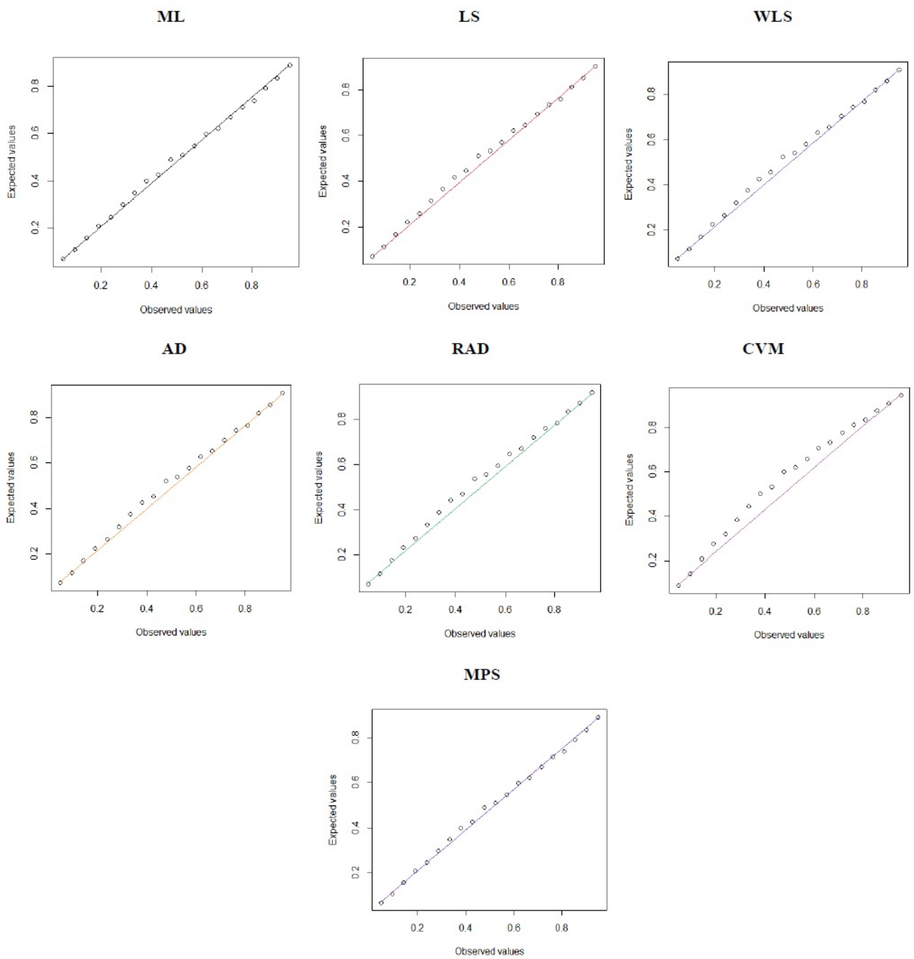

FIG. 5.2: Lomax QQ plot for the data set II.

$1.01, 1.05, 1.08, 1.14, 1.28, 1.30, 1.33, 1.43, 1.59, 1.62$. In the context of C_{pyk}, we take $L = 0.053$ and $U = 1.2$ for data set I.

Data set II: Our second data set involves the first failure times (in months) of 20 electric carts used for international transportation and delivery in large manufacturing facility. Here we use $L = 0.60$ and $U = 5$. The data set given as follows is reported by Zimmer et al. (1998) and Saha et al. (2019).

Table 6.3 shows that ML and MPS estimators of C_{pyk} are more or less the same for both data set I and data set II. Further, the widths of BCIs (SB, PB, BCPB) based on mentioned estimators are given in Table **??**. According to Table **??** that BCPB CIs based on MPS estimator of C_{pyk} is the narrowest among the others for both data sets, and it is followed by the BCPB confidence interval based on ML estimator of C_{pyk}.

6. Conclusion

In this paper, we have considered the estimation of generalized process capability C_{pyk} for Lomax distribution using different estimation methodologies, namely, ML, LS, WLS, AD, RAD, CVM and MPS. We have compared the performance estimated C_{pyk} based on these methods with different sample sizes and different parameter values of Lomax distribution using Monte Carlo simulation study in terms of bias and MSE criteria. Further, we have studied on three different BCIs called as SB, PB and BCPB based on estimated C_{pyk} for Lomax distribution obtained by using mentioned estimation methods. We also perform simulation study to compare the performances of BCIs. According to the simulation study, as MPS estimator of C_{pyk} shows the best performance for small sample sizes, the ML methodology is preferable for large value of sample size among the other estimation methods in terms of bias and MSE criteria.

The performance of BCIs for estimators of C_{pyk} based on all mentioned methods are compared in terms of CP and AL through a simulation study. Simulation results show that, BCPB confidence intervals achieve the best performance among the other BCIs in terms of CP and AL in most cases. BCPB confidence intervals based on MPS estimators of C_{pyk} show the strongest performance for small sample sizes for all chosen unknown parameter values of Lomax distribution. It is also concluded that BCPB confidence intervals based on the ML estimates of C_{pyk} have better performance than the other BCIs in terms of large sample sizes in most cases. Finally, two real life examples taken from literature are considered to support simulation results.

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Table 6.1: Bias and 100xMSE values of $\hat{C}_{pyk}$ obtained by using different estimation methods. C_{pyk} are given by (1.1).

n	β	λ	C_{pyk}		ML	LS	WLS	AD	RAD	CVM	MPS
20	0.5	0.1	0.1646	Bias	-0.0183	-0.0072	-0.0042	-0.0082	-0.0181	-0.0165	0.0062
				MSE	4.4427	3.8587	3.9194	4.0506	4.3697	4.3613	3.6925
	0.5	1	0.4192	Bias	-0.0053	0.0188	0.0147	0.0081	0.0112	-0.0079	0.0044
				MSE	2.8895	2.9930	2.8810	2.7930	2.9349	3.3745	2.5882
	0.5	3	0.0485	Bias	0.0066	0.0052	0.0050	0.0054	0.0051	0.0063	0.0048
				MSE	3.8233	3.4345	3.4382	3.5365	3.8650	4.0088	3.0817
	2	0.1	-0.8274	Bias	0.0121	-0.0394	-0.0287	-0.0174	-0.0108	-0.0202	-0.0105
				MSE	1.4462	1.9400	1.6688	1.4884	1.4514	1.7037	1.3660
	2	1	0.4137	Bias	-0.0328	0.0323	0.0263	0.0254	0.0247	0.0371	0.0238
				MSE	2.8228	3.1046	3.0444	3.0284	2.8773	3.0327	2.7930
	2	3	0.8062	Bias	0.0217	0.0462	0.0383	0.0335	0.0254	0.0253	0.0205
				MSE	1.0743	1.2442	1.0928	1.0347	0.8452	0.9915	1.0133
50	0.5	0.1	0.1646	Bias	-0.0117	0.0068	0.0035	0.0030	-0.0029	0.0034	0.0025
				MSE	1.6487	1.6311	1.6076	1.6015	1.7789	1.7138	1.4982
	0.5	1	0.4192	Bias	-0.0040	0.0043	0.0021	-0.0018	0.0021	-0.0069	0.0039
				MSE	1.2015	1.3038	1.2354	1.2186	1.2388	1.3826	1.1441
	0.5	3	0.0485	Bias	0.0054	0.0050	0.0049	0.0052	0.0045	0.0062	0.0047
				MSE	1.5629	1.5196	1.5094	1.5280	1.6330	1.6184	1.4025
	2	0.1	-0.8274	Bias	0.0077	-0.0095	-0.0094	-0.0064	-0.0063	0.0085	-0.0060
				MSE	0.6740	0.8327	0.7108	0.6844	0.6724	0.8157	0.6507
	2	1	0.4137	Bias	-0.0121	0.0112	0.0108	0.0085	0.0083	0.0125	0.0077
				MSE	1.2569	1.3449	1.3140	1.3120	1.2638	1.3308	1.2392
	2	3	0.8062	Bias	0.0129	0.0141	0.0136	0.0134	0.0129	0.0217	0.0119
				MSE	0.3200	0.4066	0.3735	0.3697	0.3367	0.3742	0.3081
100	0.5	0.1	0.1646	Bias	-0.0096	0.0028	0.0026	0.0017	-0.0017	-0.0025	0.0024
				MSE	0.7876	0.7770	0.7715	0.7712	0.8405	0.7969	0.7462
	0.5	1	0.4192	Bias	-0.0014	0.0034	0.0020	0.0015	0.0016	-0.0022	0.0032
				MSE	0.5733	0.6606	0.6121	0.5962	0.6013	0.6792	0.5665
	0.5	3	0.0485	Bias	0.0033	0.0047	0.0043	0.0042	0.0040	0.0054	0.0045
				MSE	0.7704	0.7812	0.7670	0.7650	0.8190	0.8072	0.7238
	2	0.1	-0.8274	Bias	0.0046	-0.0057	-0.0058	-0.0052	-0.0049	0.0050	-0.0047
				MSE	0.2906	0.4398	0.3552	0.3202	0.3030	0.4429	0.3101
	2	1	0.4137	Bias	-0.0019	0.0046	0.0022	0.0036	0.0028	-0.0105	0.0021
				MSE	0.6334	0.7061	0.6820	0.6751	0.6561	0.7045	0.6432
	2	3	0.8062	Bias	0.0042	0.0054	0.0053	0.0049	0.0045	0.0089	0.0044
				MSE	0.1626	0.2032	0.1871	0.1853	0.1741	0.1964	0.1784
200	0.5	0.1	0.1646	Bias	-0.0087	-0.0025	-0.0017	-0.0016	-0.0015	-0.0022	-0.0020
				MSE	0.4232	0.4261	0.4200	0.4199	0.4553	0.4322	0.4092
	0.5	1	0.4192	Bias	0.0010	0.0031	0.0019	0.0012	0.0013	0.0021	0.0018
				MSE	0.2763	0.3196	0.2937	0.2919	0.2934	0.3231	0.2771
	0.5	3	0.0485	Bias	0.0026	0.0039	0.0034	0.0032	0.0029	0.0052	0.0036
				MSE	0.3777	0.3879	0.3787	0.3786	0.4041	0.3943	0.3646
	2	0.1	-0.8274	Bias	0.0028	-0.0037	-0.0039	-0.0032	-0.0032	0.0041	-0.0030
				MSE	0.1361	0.2166	0.1647	0.1585	0.1498	0.2170	0.1451
	2	1	0.4137	Bias	-0.0013	0.0022	0.0018	0.0014	0.0019	-0.0092	0.0017
				MSE	0.3093	0.3469	0.3319	0.3295	0.3196	0.3465	0.3124
	2	3	0.8062	Bias	0.0018	0.0025	0.0023	0.0027	0.0022	0.0028	0.0025
				MSE	0.0791	0.0972	0.0890	0.0885	0.0837	0.0954	0.0842
500	0.5	0.1	0.1646	Bias	0.0074	0.0018	0.0011	0.0011	0.0014	0.0011	0.0018
				MSE	0.1588	0.1629	0.1594	0.1590	0.1719	0.1635	0.1568
	0.5	1	0.4192	Bias	-0.0005	0.0022	-0.0012	-0.0009	-0.0008	-0.0011	0.0013
				MSE	0.1164	0.1374	0.1244	0.1239	0.1238	0.1383	0.1161
	0.5	3	0.0485	Bias	-0.0015	-0.0021	-0.0020	-0.0027	-0.0018	-0.0041	0.0017
				MSE	0.1459	0.1527	0.1478	0.1475	0.1568	0.1537	0.1437
	2	0.1	-0.8274	Bias	0.0012	-0.0017	-0.0018	-0.0019	-0.0019	0.0023	-0.0015
				MSE	0.0584	0.0950	0.0705	0.0694	0.0647	0.0950	0.0604
	2	1	0.4137	Bias	0.0009	0.0013	0.0016	0.0012	0.0012	0.0038	0.0011
				MSE	0.1160	0.1317	0.1251	0.1247	0.1225	0.1314	0.1183
	2	3	0.8062	Bias	0.0007	0.0020	0.0013	0.0015	0.0015	0.0013	0.0011
				MSE	0.0292	0.0360	0.0328	0.0327	0.0314	0.0356	0.0306

Table 6.2: Observed CP(AL) of 95% bootstrap confidence intervals of $\hat{C}_{pyk}$ obtained by using different estimation methods. C_{pyk} are given by (1.1).

n	β	λ	C_{pyk}		ML	LS	WLS	AD	RAD	CVM	MPS
20				SB	0.908(0.2782)	0.918(0.2728)	0.903(0.2735)	0.916(0.2721)	0.917(0.2756)	0.910(0.2772)	0.922(0.2659)
	0.5	0.1	0.1646	PB	0.915(0.2739)	0.914(0.2693)	0.908(0.2700)	0.921(0.2685)	0.917(0.2716)	0.920(0.2734)	0.928(0.2647)
				BCPB	0.932(0.2946)	0.924(0.2698)	0.928(0.2675)	0.923(0.2687)	0.927(0.2650)	0.917(0.2740)	0.935(0.2635)
				SB	0.941(0.2678)	0.940(0.2797)	0.932(0.2777)	0.939(0.2765)	0.943(0.2735)	0.924(0.2920)	0.947(0.2673)
	0.5	1	0.4192	PB	0.940(0.6353)	0.939(0.6614)	0.941(0.6504)	0.938(0.6391)	0.942(0.6305)	0.928(0.7012)	0.946(0.6284)
				BCPB	0.948(0.2682)	0.940(0.2826)	0.946(0.2812)	0.943(0.2737)	0.947(0.2712)	0.931(0.3010)	0.952(0.2663)
				SB	0.938(0.2808)	0.930(0.3066)	0.936(0.2975)	0.941(0.2912)	0.942(0.2898)	0.925(0.3322)	0.945(0.2736)
	0.5	3	0.0485	PB	0.936(0.2788)	0.937(0.2953)	0.939(0.2860)	0.935(0.2825)	0.938(0.2777)	0.928(0.3106)	0.941(0.2734)
				BCPB	0.943(0.2795)	0.933(0.2935)	0.939(0.2917)	0.940(0.2899)	0.938(0.2785)	0.926(0.3010)	0.946(0.2709)
				SB	0.943(0.2808)	0.933(0.3366)	0.934(0.2975)	0.939(0.2912)	0.941(0.2898)	0.927(0.3322)	0.947(0.2836)
	2	0.1	0.8274	PB	0.939(0.2788)	0.928(0.2953)	0.931(0.2860)	0.933(0.2825)	0.936(0.2777)	0.925(0.3106)	0.943(0.2709)
				BCPB	0.946(0.2795)	0.936(0.2935)	0.937(0.2917)	0.938(0.2899)	0.941(0.2785)	0.928(0.3010)	0.948(0.2709)
				SB	0.926(0.2665)	0.913(0.2757)	0.923(0.2687)	0.925(0.2663)	0.922(0.2649)	0.920(0.2663)	0.935(0.2642)
	2	1	0.4137	PB	0.941(0.2639)	0.924(0.2728)	0.925(0.2715)	0.926(0.2733)	0.933(0.2686)	0.917(0.2934)	0.946(0.2608)
				BCPB	0.943(0.2664)	0.916(0.2742)	0.923(0.2689)	0.917(0.2749)	0.923(0.2641)	0.912(0.2951)	0.949(0.2612)
				SB	0.953(0.2087)	0.946(0.4210)	0.948(0.3897)	0.952(0.4208)	0.958(0.3783)	0.926(0.4028)	0.962(0.2060)
	2	3	0.8062	PB	0.954(0.3594)	0.923(0.3996)	0.943(0.3705)	0.927(0.3991)	0.926(0.3608)	0.919(0.3834)	0.957(0.3456)
				BCPB	0.931(0.4151)	0.901(0.4379)	0.918(0.4056)	0.911(0.4029)	0.907(0.4428)	0.904(0.4251)	0.941(0.4027)
50				SB	0.934(0.2562)	0.942(0.4942)	0.934(0.4971)	0.933(0.4924)	0.931(0.5110)	0.937(0.5092)	0.924(0.2490)
	0.5	0.1	0.1646	PB	0.923(0.2485)	0.931(0.2544)	0.923(0.2473)	0.927(0.2418)	0.923(0.2695)	0.922(0.2488)	0.930(0.2478)
				BCPB	0.935(0.2860)	0.925(0.2503)	0.926(0.2470)	0.927(0.2417)	0.932(0.2406)	0.921(0.2698)	0.941(0.2313)
				SB	0.943(0.2537)	0.942(0.2726)	0.933(0.2694)	0.941(0.2553)	0.945(0.2517)	0.928(0.2828)	0.947(0.2517)
	0.5	1	0.4192	PB	0.942(0.4550)	0.941(0.5015)	0.943(0.4303)	0.940(0.4153)	0.944(0.4260)	0.932(0.4520)	0.949(0.4145)
				BCPB	0.953(0.2671)	0.945(0.2785)	0.949(0.2743)	0.948(0.2729)	0.951(0.2706)	0.938(0.2945)	0.957(0.2652)
				SB	0.941(0.2741)	0.933(0.2991)	0.938(0.2846)	0.943(0.2888)	0.944(0.2854)	0.928(0.3304)	0.947(0.2706)
	0.5	3	0.0485	PB	0.943(0.2735)	0.940(0.2943)	0.942(0.2822)	0.940(0.2779)	0.942(0.2750)	0.933(0.3056)	0.947(0.2723)
				BCPB	0.946(0.2764)	0.935(0.2876)	0.942(0.2831)	0.945(0.2812)	0.941(0.2721)	0.933(0.2983)	0.950(0.2682)
				SB	0.948(0.2783)	0.938(0.3310)	0.939(0.3041)	0.942(0.2905)	0.945(0.2887)	0.932(0.3311)	0.952(0.2705)
	2	0.1	0.8274	PB	0.958(0.2759)	0.933(0.2946)	0.935(0.2927)	0.943(0.2947)	0.941(0.2879)	0.929(0.3185)	0.948(0.2687)
				BCPB	0.965(0.2775)	0.955(0.2791)	0.956(0.2762)	0.942(0.2691)	0.946(0.2690)	0.932(0.2937)	0.972(0.2682)
				SB	0.929(0.2461)	0.922(0.2480)	0.926(0.2465)	0.928(0.2641)	0.927(0.2632)	0.925(0.2476)	0.932(0.2409)
	2	1	0.4137	PB	0.945(0.2607)	0.928(0.2653)	0.930(0.2642)	0.927(0.2713)	0.939(0.2618)	0.925(0.2840)	0.948(0.2595)
				BCPB	0.951(0.2627)	0.922(0.2711)	0.925(0.2611)	0.926(0.2713)	0.935(0.2641)	0.918(0.2903)	0.953(0.2615)
				SB	0.973(0.2069)	0.950(0.2497)	0.954(0.2305)	0.958(0.2403)	0.963(0.2246)	0.930(0.2429)	0.975(0.2055)
	2	3	0.8062	PB	0.956(0.2121)	0.944(0.2419)	0.948(0.2236)	0.939(0.2329)	0.943(0.2177)	0.927(0.2346)	0.960(0.2103)
				BCPB	0.948(0.2256)	0.926(0.2557)	0.933(0.2358)	0.933(0.2293)	0.928(0.2467)	0.915(0.2493)	0.950(0.2243)
100				SB	0.949(0.2359)	0.943(0.3567)	0.945(0.3559)	0.946(0.3523)	0.939(0.3652)	0.943(0.3617)	0.948(0.2412)
	0.5	0.1	0.1646	PB	0.953(0.2306)	0.932(0.2348)	0.926(0.2341)	0.931(0.2335)	0.927(0.2503)	0.932(0.2348)	0.933(0.2340)
				BCPB	0.947(0.2198)	0.927(0.2492)	0.924(0.2343)	0.928(0.2380)	0.934(0.2340)	0.923(0.2472)	0.943(0.2281)
				SB	0.952(0.2473)	0.948(0.2532)	0.942(0.2598)	0.943(0.2515)	0.947(0.2506)	0.937(0.2657)	0.953(0.2492)
	0.5	1	0.4192	PB	0.955(0.3126)	0.943(0.3553)	0.947(0.3332)	0.943(0.2998)	0.948(0.3029)	0.938(0.3190)	0.950(0.2902)
				BCPB	0.962(0.2593)	0.949(0.2717)	0.953(0.2741)	0.951(0.2699)	0.953(0.2633)	0.941(0.2895)	0.960(0.2602)
				SB	0.952(0.2551)	0.941(0.2865)	0.944(0.2814)	0.945(0.2765)	0.950(0.2753)	0.931(0.3244)	0.950(0.2602)
	0.5	3	0.0485	PB	0.951(0.2675)	0.942(0.2894)	0.945(0.2833)	0.947(0.2737)	0.948(0.2703)	0.940(0.3045)	0.949(0.2692)
				BCPB	0.955(0.2646)	0.939(0.2801)	0.949(0.2778)	0.948(0.2785)	0.943(0.2711)	0.936(0.2960)	0.953(0.2672)
				SB	0.955(0.2660)	0.943(0.2839)	0.945(0.2950)	0.946(0.2822)	0.948(0.2757)	0.934(0.3125)	0.954(0.2695)
	2	0.1	0.8274	PB	0.965(0.2017)	0.938(0.2469)	0.943(0.2193)	0.946(0.2169)	0.948(0.2121)	0.933(0.2447)	0.955(0.2019)
				BCPB	0.982(0.2017)	0.957(0.2480)	0.960(0.2405)	0.950(0.2175)	0.952(0.2127)	0.947(0.2562)	0.973(0.2092)
				SB	0.942(0.2260)	0.938(0.2366)	0.933(0.2311)	0.937(0.2328)	0.937(0.2306)	0.935(0.2368)	0.939(0.2305)
	2	1	0.4137	PB	0.950(0.2561)	0.932(0.2583)	0.934(0.2531)	0.938(0.2550)	0.941(0.2588)	0.928(0.2807)	0.949(0.2577)
				BCPB	0.955(0.2509)	0.933(0.2701)	0.932(0.2554)	0.940(0.2673)	0.936(0.2607)	0.923(0.2811)	0.955(0.2611)
				SB	0.979(0.1539)	0.953(0.1708)	0.957(0.1614)	0.960(0.1650)	0.967(0.1573)	0.935(0.1687)	0.978(0.1544)
	2	3	0.8062	PB	0.962(0.1499)	0.944(0.1664)	0.951(0.1573)	0.946(0.1608)	0.952(0.1534)	0.948(0.1639)	0.961(0.1513)
				BCPB	0.955(0.1540)	0.935(0.1711)	0.948(0.1616)	0.945(0.1574)	0.947(0.1661)	0.929(0.1695)	0.951(0.1585)

Table 6.2 (continued)

n	β	λ	C_{pyk}		ML	LS	WLS	AD	RAD	CVM	MPS
200				SB	0.948(0.2465)	0.943(0.2481)	0.936(0.2469)	0.938(0.2468)	0.938(0.2553)	0.942(0.2501)	0.945(0.2247)
	0.5	0.1	0.1646	PB	0.955(0.2513)	0.945(0.2325)	0.944(0.2228)	0.947(0.2251)	0.945(0.2236)	0.946(0.2256)	0.947(0.2255)
				BCPB	0.948(0.2080)	0.930(0.2486)	0.923(0.2324)	0.934(0.2472)	0.935(0.2303)	0.931(0.2451)	0.945(0.2164)
				SB	0.956(0.2116)	0.951(0.2274)	0.948(0.2175)	0.945(0.2182)	0.951(0.2162)	0.943(0.2586)	0.955(0.2186)
	0.5	1	0.4192	PB	0.957(0.2069)	0.948(0.2225)	0.951(0.2127)	0.945(0.2113)	0.952(0.2123)	0.942(0.2242)	0.954(0.2015)
				BCPB	0.967(0.2471)	0.951(0.2623)	0.959(0.2533)	0.957(0.2616)	0.960(0.2577)	0.946(0.2842)	0.962(0.2521)
				SB	0.954(0.2424)	0.947(0.2824)	0.948(0.2731)	0.949(0.2717)	0.952(0.2685)	0.936(0.3169)	0.952(0.2403)
	0.5	3	0.0485	PB	0.958(0.2532)	0.948(0.2805)	0.947(0.2747)	0.949(0.2671)	0.953(0.2633)	0.944(0.2919)	0.952(0.2465)
				BCPB	0.959(0.2376)	0.942(0.2756)	0.951(0.2683)	0.951(0.2675)	0.952(0.2638)	0.938(0.2821)	0.955(0.2389)
				SB	0.963(0.1477)	0.946(0.1850)	0.951(0.1806)	0.952(0.1588)	0.959(0.1546)	0.945(0.1850)	0.959(0.1542)
	2	0.1	0.8274	PB	0.976(0.1447)	0.942(0.1806)	0.946(0.1568)	0.947(0.1551)	0.952(0.1513)	0.939(0.1803)	0.966(0.1498)
				BCPB	0.985(0.1438)	0.960(0.1811)	0.963(0.1771)	0.954(0.1654)	0.958(0.1513)	0.950(0.1907)	0.975(0.1442)
				SB	0.948(0.2166)	0.939(0.2276)	0.936(0.2227)	0.940(0.2230)	0.941(0.2190)	0.939(0.2277)	0.945(0.2187)
	2	1	0.4137	PB	0.952(0.2117)	0.935(0.2220)	0.936(0.2177)	0.941(0.2180)	0.942(0.2139)	0.930(0.2221)	0.951(0.2118)
				BCPB	0.959(0.2123)	0.937(0.2229)	0.938(0.2185)	0.946(0.2189)	0.936(0.2144)	0.932(0.2230)	0.956(0.2136)
				SB	0.981(0.1088)	0.957(0.1204)	0.963(0.1144)	0.961(0.1157)	0.971(0.1118)	0.939(0.1297)	0.980(0.1097)
	2	3	0.8062	PB	0.967(0.1062)	0.946(0.1176)	0.956(0.1116)	0.953(0.1128)	0.957(0.1092)	0.951(0.1167)	0.964(0.1085)
				BCPB	0.963(0.1074)	0.938(0.1193)	0.950(0.1133)	0.947(0.1105)	0.956(0.1146)	0.934(0.1188)	0.942(0.1095)
500				SB	0.975(0.1505)	0.953(0.1609)	0.947(0.1599)	0.955(0.1594)	0.957(0.1652)	0.962(0.1614)	0.965(0.1506)
	0.5	0.1	0.1646	PB	0.956(0.1561)	0.947(0.1574)	0.948(0.1567)	0.952(0.1556)	0.952(0.1615)	0.947(0.1579)	0.954(0.1568)
				BCPB	0.953(0.1573)	0.942(0.1579)	0.940(0.1576)	0.944(0.1559)	0.946(0.1617)	0.937(0.2082)	0.951(0.1586)
				SB	0.963(0.1333)	0.955(0.1435)	0.953(0.1371)	0.952(0.1366)	0.958(0.1349)	0.951(0.1452)	0.961(0.1345)
	0.5	1	0.4192	PB	0.968(0.1305)	0.950(0.1403)	0.957(0.1342)	0.953(0.1336)	0.960(0.1338)	0.948(0.1407)	0.962(0.1303)
				BCPB	0.971(0.1306)	0.955(0.1403)	0.962(0.1344)	0.964(0.1336)	0.964(0.1340)	0.952(0.1419)	0.966(0.1311)
				SB	0.968(0.1518)	0.950(0.1549)	0.952(0.1531)	0.953(0.1523)	0.958(0.1520)	0.947(0.1553)	0.966(0.1520)
	0.5	3	0.0485	PB	0.961(0.1465)	0.950(0.1515)	0.952(0.1511)	0.951(0.1496)	0.955(0.1490)	0.949(0.1520)	0.959(0.1485)
				BCPB	0.968(0.1416)	0.944(0.1516)	0.955(0.1512)	0.957(0.1536)	0.960(0.1534)	0.941(0.1553)	0.962(0.1483)
				SB	0.971(0.0933)	0.949(0.1175)	0.962(0.1111)	0.954(0.1005)	0.964(0.0980)	0.948(0.1177)	0.968(0.0945)
	2	0.1	0.8274	PB	0.983(0.0913)	0.944(0.1149)	0.956(0.0989)	0.953(0.0984)	0.959(0.0957)	0.944(0.1150)	0.972(0.0976)
				BCPB	0.988(0.0904)	0.964(0.1150)	0.967(0.0990)	0.959(0.1085)	0.961(0.0959)	0.956(0.1153)	0.977(0.0908)
				SB	0.955(0.1374)	0.952(0.1448)	0.952(0.1415)	0.948(0.1413)	0.953(0.1396)	0.947(0.1450)	0.951(0.1388)
	2	1	0.4137	PB	0.954(0.1344)	0.944(0.1415)	0.946(0.1386)	0.945(0.1379)	0.947(0.1364)	0.941(0.1416)	0.953(0.1355)
				BCPB	0.963(0.1342)	0.941(0.1417)	0.944(0.1389)	0.951(0.1383)	0.942(0.1367)	0.938(0.1419)	0.960(0.1358)
				SB	0.989(0.0662)	0.964(0.0760)	0.975(0.0725)	0.972(0.0728)	0.976(0.0711)	0.956(0.0779)	0.985(0.0672)
	2	3	0.8062	PB	0.971(0.0677)	0.952(0.0742)	0.958(0.0710)	0.955(0.0711)	0.961(0.0696)	0.953(0.0741)	0.969(0.0693)
				BCPB	0.966(0.0681)	0.940(0.0746)	0.957(0.0715)	0.954(0.0699)	0.960(0.0716)	0.945(0.0747)	0.961(0.0682)

Table 6.3: Estimates of the parameters and C_{pyk}, AIC, BIC,RMSE and KS statistic.

	Method	$(\hat{\beta}, \hat{\lambda})$	$\hat{C}_{pyk}$	$-$LogL	AIC	BIC	RMSE	KS
Data set I	ML	$(1.52, 0.60)$	0.7087	18.6973	41.3946	41.9998	0.3819	0.2667
	LS	(1.55,0.53)	0.7087	20.4682	44.9363	45.5415	0.4198	0.2718
	WLS	(1.56,0.68)	0.6585	19.5702	43.1404	46.5299	0.3994	0.2791
	AD	(1.55,0.45)	0.7266	21.8383	47.6767	48.2818	0.4427	0.2748
	RAD	(1.51,0.55)	0.6788	19.9597	43.9194	44.5246	0.4076	0.2702
	CVM	(1.50,0.48)	0.7235	20.9624	45.9248	46.5299	0.4261	0.2726
	MPS	(1.51,0.58)	0.7094	18.3773	41.3546	41.3598	0.3857	0.2686
Data set II	ML	(1.98,26.01)	0.4292	74.4338	154.5676	155.8591	0.0306	0.0751
	LS	(2.06,24.9)	0.3874	75.2334	154.9669	156.4583	0.0379	0.0825
	WLS	(2.1,24.68)	0.3725	75.1752	154.9504	156.3419	0.0376	0.0863
	AD	(2.13,25.27)	0.3766	75.1543	154.9985	156.3000	0.0368	0.0859
	RAD	(2.22,24.95)	0.3472	75.0649	154.9712	156.1212	0.0367	0.0794
	CVM	(2.45,22.88)	0.2420	75.1749	154.9734	156.3412	0.0781	0.1245
	MPS	(1.99,25.81)	0.4229	74.3996	154.4992	155.7906	0.0307	0.0714