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ON THE GENERALIZED DUAL FIBONACCI AND LUCAS OCTONIONS

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Abstract. Generalized number systems, particularly octonions with their algebraic structure, have drawn much interest in mathematics, physics, and computer technology. Therefore, in this paper, we introduce two new concepts, modified generalized dual Fibonacci and modified generalized dual Lucas octonions, to expand the topic of octonions. Additionally, we explore the well-known Catalan and Cassini identities, shedding light on the characteristics of these new constructs. Also, we give generating functions and the Binet formulas of the modified generalized dual Fibonacci and modified generalized dual Lucas octonions.
Keywords: Fibonacci Octonions, Lucas Octonions, Catalan identities.

1. Introduction

Two integer sequences that have great significance in the fields of number theory and mathematics are the Fibonacci and Lucas numbers. The Fibonacci numbers, the Lucas numbers and their generalizations play an important role in many areas of science. The classical Fibonacci numbers are defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geq 2. \tag{1.1}$$

The Lucas numbers are defined by the recurrence relation

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2}, \quad n \geq 2. \tag{1.2}$$

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In recent years, many researchers have studied the generalizations and applications of the Fibonacci and Lucas numbers (see [23, 16, 15, 26, 9, 13, 7, 25]). For example, in 2007, Falcon and Plaza [16] defined the $k-$Fibonacci sequence, $\{F_{k,n}\}_{n=0}^{\infty}$. In 2012, Yazlik and Taskara [26] presented the generalized $k-$Horadam sequence. A new generalization of the Fibonacci numbers, first referred to in the literature as the bi-periodic Fibonacci sequence, was defined by Edson and Yayenie [9]. This paper has been one of the most important studies in this area. Moreover, Yayenie [13] presented another significant study which is the modified generalized Fibonacci sequence as

$$Q_0 = 0, \quad Q_1 = 1, \quad Q_n = \begin{cases} aQ_{n-1} + cQ_{n-2}, & \text{if } n \text{ is even} \\ bQ_{n-1} + dQ_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2, \tag{1.3}$$

where a, b, c and d are real numbers. Later, Bilgici [7] introduced both the bi-periodic Lucas numbers and the modified generalized Lucas numbers, wherein he established the modified generalized Lucas sequence as

$$U_0 = \frac{d+1}{d}, \quad U_1 = a, \quad U_n = \begin{cases} bU_{n-1} + dU_{n-2}, & \text{if } n \text{ is even} \\ aU_{n-1} + cU_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2, \tag{1.4}$$

where a, b, c and d are real numbers. The generating functions of Q_n and U_n are given by

$$H(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{x\left(1 + ax - cx^2\right)}{1 - (ab + c + d)\, x^2 + cdx^4} \tag{1.5}$$

and

$$U(x) = \sum_{n=0}^{\infty} U_n x^n = \frac{1}{d}\left(\frac{d + 1 + adx - (ab + cd + c)\, x^2 + adx^3}{1 - (ab + c + d)\, x^2 + cdx^4}\right), \tag{1.6}$$

respectively. Additionally, the following formulas provide the Binet formulas for the sequences Q_n and U_n, respectively:

$$Q_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha^{\lfloor \frac{n}{2} \rfloor}(\alpha + d - c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta^{\lfloor \frac{n}{2} \rfloor}(\beta + d - c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha - \beta}\right) \tag{1.7}$$

and

$$\begin{aligned} U_n &= \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}}\left(\frac{(\alpha + d + 1)\,\alpha^{\lfloor \frac{n-1}{2} \rfloor}(\alpha + d - c)^{\lfloor \frac{n}{2} \rfloor}}{\alpha - \beta}\right. \\ &\qquad \left. - \frac{(\beta + d + 1)\,\beta^{\lfloor \frac{n-1}{2} \rfloor}(\beta + d - c)^{\lfloor \frac{n}{2} \rfloor}}{\alpha - \beta}\right), \end{aligned} \tag{1.8}$$

where $\alpha = \frac{ab+c-d+\sqrt{(ab+c-d)^2+4abd}}{2}$ and $\beta = \frac{ab+c-d-\sqrt{(ab+c-d)^2+4abd}}{2}$ are the roots of the polynomial $x^2 - (ab + c - d)\, x - abd = 0$ and $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$ is the parity function.

The dual numbers are a hypercomplex number system. Firstly, Clifford introduced dual numbers in a work he published in 1871 [12]. Dual numbers can be described as a mathematical extension of the real numbers, incorporating a new element denoted as ε (epsilon), whose square is equal to zero. In other words, dual numbers are of the form

$$d = a + \varepsilon a^*, \tag{1.9}$$

where $a, a^* \in \mathbb{R}$ and ε satisfies $\varepsilon^2 = 0$. The set of dual numbers can be denoted by $\mathbb{D}$. For $a, a^*, b, b^* \in \mathbb{R}, \varepsilon^2 = 0$ and $b \neq 0$, the following arithmetic operations are valid for dual numbers.

$$\begin{aligned}
\text{Addition:}&\quad (a+\varepsilon a^*) + (b+\varepsilon b^*) = (a+b) + \varepsilon(a^* + b^*)\\
\text{Subtraction:}&\quad (a+\varepsilon a^*) - (b+\varepsilon b^*) = (a-b) + \varepsilon(a^* - b^*)\\
\text{Multiplication:}&\quad (a+\varepsilon a^*)(b+\varepsilon b^*) = ab + \varepsilon(ab^* + a^*b)\\
\text{Division:}&\quad \tfrac{(a+\varepsilon a^*)}{(b+\varepsilon b^*)} = \tfrac{(a+\varepsilon a^*)(b-\varepsilon b^*)}{(b+\varepsilon b^*)(b-\varepsilon b^*)} = \tfrac{ab+\varepsilon(a^*b-ab^*)}{b^2}
\end{aligned}$$

Dual numbers are highly beneficial in mathematics as they offer a useful algebraic structure for computing derivatives. Furthermore, they find applications in various fields such as computer graphics, robotics, and optimization.

Now let's talk about the concept of quaternions, which is necessary to better explain our work. The quaternions are used in mathematics, physics, computer science and related areas. In 1843, the quaternions were first defined by William Rowan Hamilton. Generally, a quaternion q is defined by the following formula

$$q = q_0 + iq_1 + jq_2 + kq_3, \tag{1.10}$$

where q_0, q_1, q_2 and q_3 are real numbers and i, j, k are standard orthonormal basis in $\mathbb{R}^3$. Additionally, the standard orthonormal basis i, j, k satisfy the following multiplication rules as

$$i^2 = j^2 = k^2 = ijk = -1, \tag{1.11}$$

$$ij = -ji = k, jk = -kj = i, ki = -ik = j. \tag{1.12}$$

Furthermore, the conjugate of the quaternion $\bar{q}$ is defined by

$$\overline{q} = q_0 - iq_1 - jq_2 - kq_3. \tag{1.13}$$

In recent years, many studies about quaternions have been conducted. [17, 14, 4, 18, 1, 8, 3, 6, 5, 20]. In 1963, Horadam [1] presented the *nth* Fibonacci and Lucas quaternions as Q_n and P_n, respectively. In 2015, Ramirez [8] examined the $k-$Fibonacci and $k-$Lucas quaternions utilizing the characteristics of the $k-$Fibonacci and $k-$Lucas numbers. Recently, Tan [5] introduced the biperiodic Fibonacci quaternion. Then, Tan et al.[6] also presented the bi-periodic Lucas quaternion.

The idea of quaternions, which are an extension of complex numbers, is extended by dual quaternions. Dual quaternions, on the other hand, are pairs of quaternions written as (q, q^*), where q is the real part (similar to a standard quaternion) and q^* is a dual part. The general form of a dual quaternion is:

$$\hat{q} = q + \varepsilon q^* \tag{1.14}$$

where ε is a dual unit and q and q^* are quaternions. If $A_i = q_i + \varepsilon q_i^*$ and $A_i \in \mathbb{D}, i = 0, 1, 2, 3$, then the dual quaternion $\hat{q}$, can be denoted as;

$$\hat{q} = A_0 e_0 + A_1 e_1 + A_2 e_2 + A_3 e_3. \tag{1.15}$$

As a result, eight real parameters are used to generate every dual quaternion $\hat{q}$. Here $e_i, i = 0, 1, 2, 3$ are quaternion basis elements that obey the (1.11) and (1.12) multiplication rules. More information on dual numbers and dual quaternions can be found in [12] and [28]. In 2015, Nurkan and Güven [10] examined the dual Fibonacci quaternions. They studied relations between the dual Fibonacci and the dual Lucas quaternions and presented give the Binet and Cassini formulas for these quaternions.

The octonions in Clifford algebra C are a normed division algebra with eight dimensions. In 2015, Keçilioğlu ve Akkuş [11] presented Fibonaaci and Lucas octonions. They are defined by the recurrence relations:

$$Q_n = \sum_{l=0}^{7} F_{n+l} e_l, \quad n \geq 0, \tag{1.16}$$

and

$$T_n = \sum_{l=0}^{7} L_{n+l} e_l, \quad n \geq 0, \tag{1.17}$$

where F_n ve L_n are the nth classic Fibonacci and Lucas numbers, respectively. Multiplication rules for the basis are listed in the following table [24]:

x	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

It has been many important studies in this area. In 2015, in her work on dual Fibonacci octonions, Halici studied some basic algebraic properties of these octonions and presented Binet formulas and their generating functions [19]. İpek and Çimen defined the $(p, q)-$Fibonacci octonions [2]. In 2017, Ünal et al. studied some properties of dual Fibonacci and dual Lucas octonions [27].

Köme et. al. investigated modified generalized Fibonacci and Lucas quaternions [21]. Also, they are presented the matrix representations of the modified generalized Fibonacci and modified generalized Lucas quaternions. In 2020, Köme and Kirik defined the modified generalized Fibonacci and modified generalized Lucas 2^k-ions [22]. In this paper Köme and Kirik defined of the modified generalized Fibonacci 2^k-ions Θ_n and modified generalized Lucas 2^k-ions σ_n respectively as follows:

$$\Theta_n = \sum_{l=0}^{N-1} Q_{n+l} e_l, \tag{1.18}$$

and

$$\sigma_n = \sum_{l=0}^{N-1} U_{n+l} e_l, \tag{1.19}$$

where Q_n is the nth modified generalized Fibonacci numbers that is defined in (1.3) and U_n is the modified generalized Lucas numbers that is defined in (1.4).

It is clear that the modified generalized Fibonacci and Lucas 2^k-ions are the generalization of the modified generalized Fibonacci and Lucas octonions (for $N = 2^k = 8$). As a result of, the modified generalized Fibonacci and Lucas octonions are defined as follows respectively;

$$\Phi_n = \sum_{l=0}^{7} Q_{n+l} e_l \tag{1.20}$$

and

$$\vartheta_n = \sum_{l=0}^{7} U_{n+l} e_l \tag{1.21}$$

where Q_n is the nth modified generalized Fibonacci numbers that is defined in (1.3) and U_n is the modified generalized Lucas numbers that is defined in (1.4).

Generalized number systems have been the subject of the work of many mathematicians recently due to their applications in fields such as mathematics, physics and computer science. Particularly, octonions have emerged as an intriguing algebraic structure, displaying intricate mathematical features. Built upon this foundation, the aim of this article is to advance the field by developing the modified generalized dual Fibonacci and modified generalized dual Lucas octonions, respectively. We also aim to give the Catalan identity and Cassini identity, which sheds light on the key properties of modified generalized dual Fibonacci and modified generalized dual Lucas octonions. In conclusion, we hope that our article summarizes both the results of previous research and the introduction of the original results, thereby enriching the discourse on octonion generalizations.

2. Modified Generalized Dual Fibonacci Octonions

In this section, we present the definitions and theorems of the modified generalized dual Fibonacci octonions.

Definition 2.1. The modified generalized dual Fibonacci numbers $\tilde{Q}_n$ are

$$\tilde{Q}_n = Q_n + \varepsilon Q_{n+1}, \tag{2.1}$$

where Q_n, Q_{n+1} are the nth and $(n+1)th$ modified generalized Fibonacci numbers in Eq.(1.3), respectively.

Definition 2.2. The modified generalized dual Fibonacci octonions $\tilde{\Phi}_n$ are defined by

$$\tilde{\Phi}_n = \Phi_n + \varepsilon \Phi_{n+1}. \tag{2.2}$$

Here Φ_n is the nth modified generalized Fibonacci octonions in Eq.(1.20). From Definition (2.1), Eq.(2.2) can be denoted as;

$$\tilde{\Phi}_n = \sum_{l=0}^{7} \tilde{Q}_{n+l} e_l.$$

In the following theorem we give the Binet formula of the modified generalized dual Fibonacci octonion by the help of the Binet formula of Φ_n.

Theorem 2.1. *The Binet formula for the modified generalized dual Fibonacci octonions $\tilde{\Phi}_n$ is*

$$\begin{aligned}\tilde{\Phi}_n &= \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_{\xi(n)} \alpha^{\lfloor \frac{n}{2} \rfloor} (\alpha + d - c)^{n - \lfloor \frac{n}{2} \rfloor} - \beta_{\xi(n)} \beta^{\lfloor \frac{n}{2} \rfloor} (\beta + d - c)^{n - \lfloor \frac{n}{2} \rfloor}}{\alpha - \beta} \right) \\ &+ \varepsilon \frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\frac{\alpha_{\xi(n+1)} \alpha^{\lfloor \frac{n+1}{2} \rfloor} (\alpha + d - c)^{n+1 - \lfloor \frac{n+1}{2} \rfloor}}{\alpha - \beta} \right) \\ &- \varepsilon \frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\frac{\beta_{\xi(n+1)} \beta^{\lfloor \frac{n+1}{2} \rfloor} (\beta + d - c)^{n+1 - \lfloor \frac{n+1}{2} \rfloor}}{\alpha - \beta} \right)\end{aligned}$$

where

$$\alpha_{\xi(n)} = \sum_{l=0}^{7} \frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\lfloor \frac{l+\xi(n)}{2} \rfloor}} (\alpha + d - c)^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor} \alpha^{\lfloor \frac{l+\xi(n)}{2} \rfloor} e_l$$

and

$$\beta_{\xi(n)} = \sum_{l=0}^{7} \frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\lfloor \frac{l+\xi(n)}{2} \rfloor}} (\beta + d - c)^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor} \beta^{\lfloor \frac{l+\xi(n)}{2} \rfloor} e_l.$$

Proof. From the Eq.(2.2), we have

$$\tilde{\Phi}_n = \Phi_n + \varepsilon\Phi_{n+1}.$$

If $N = 8$ in Binet formula given in Theorem 2.2. of the modified generalized Fibonacci 2^k-ions in [22], we can obtain Binet formula of the modified generalized Fibonacci octonions. Thus, it can be easily seen that the following relations are correct.

$$\Phi_n = \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_{\xi(n)}\alpha^{\lfloor \frac{n}{2} \rfloor} (\alpha + d - c)^{n-\lfloor \frac{n}{2} \rfloor} - \beta_{\xi(n)}\beta^{\lfloor \frac{n}{2} \rfloor} (\beta + d - c)^{n-\lfloor \frac{n}{2} \rfloor}}{\alpha - \beta} \right)$$

and

$$\begin{aligned} \Phi_{n+1} &= \frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\frac{\alpha_{\xi(n+1)}\alpha^{\lfloor \frac{n+1}{2} \rfloor} (\alpha + d - c)^{n+1-\lfloor \frac{n+1}{2} \rfloor}}{\alpha - \beta} \right. \\ &\qquad \left. - \frac{\beta_{\xi(n+1)}\beta^{\lfloor \frac{n+1}{2} \rfloor} (\beta + d - c)^{n+1-\lfloor \frac{n+1}{2} \rfloor}}{\alpha - \beta} \right) \end{aligned}$$

thereby showing that indeed $\tilde{\Phi}_n$ holds. So, the Binet formula is obtained. □

Now, we give the generating function of the modified generalized dual Fibonacci octonions.

Theorem 2.2. *The generating function for the modified generalized dual Fibonacci octonions is*

$$\begin{aligned} G(t) &= \frac{\Phi_0 + (\Phi_1 - b\Phi_0)\, t + (a-b)R_1(t) + (c-d)R_2(t)}{1 - bt - dt^2} \\ &\quad + \epsilon \frac{\Phi_1 + (\Phi_2 - b\Phi_1)\, t + (a-b)S_1(t) + (c-d)S_2(t)}{1 - bt - dt^2} \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} R_1(t) &= e_0 t f(t) + e_1 \left(f(t) - Q_1 t\right) + e_2 \left(\frac{f(t) - Q_1 t}{t}\right) + e_3 \left(\frac{f(t) - Q_1 t - Q_3 t^3}{t^2}\right) \\ &\quad + e_4 \left(\frac{f(t) - Q_1 t - Q_3 t^3}{t^3}\right) + e_5 \left(\frac{f(t) - Q_1 t - Q_3 t^3 - Q_5 t^5}{t^4}\right) \\ &\quad + e_6 \left(\frac{f(t) - Q_1 t - Q_3 t^3 - Q_5 t^5}{t^5}\right) \\ &\quad + e_7 \left(\frac{f(t) - Q_1 t - Q_3 t^3 - Q_5 t^5 - Q_7 t^7}{t^6}\right), \end{aligned}$$

$$\begin{aligned}R_2(t) &= e_0 t^2 h(t) + e_1 t h(t) + e_2 h(t) + e_3\left(\frac{h(t)-Q_2t^2}{t}\right) + e_4\left(\frac{h(t)-Q_2t^2}{t^2}\right)\\ &+e_5\left(\frac{h(t)-Q_2t^2-Q_4t^4}{t^3}\right) + e_6\left(\frac{h(t)-Q_2t^2-Q_4t^4}{t^4}\right)\\ &+e_7\left(\frac{h(t)-Q_2t^2-Q_4t^4-Q_6t^6}{t^5}\right),\end{aligned}$$

$$\begin{aligned}S_1(t) &= e_0\left(f(t)-Q_1t\right) + e_1\left(\frac{f(t)-Q_1t}{t}\right) + e_2\left(\frac{f(t)-Q_1t-Q_3t^3}{t^2}\right)\\ &+e_3\left(\frac{f(t)-Q_1t-Q_3t^3}{t^3}\right) + e_4\left(\frac{f(t)-Q_1t-Q_3t^3-Q_5t^5}{t^4}\right)\\ &+e_5\left(\frac{f(t)-Q_1t-Q_3t^3-Q_5t^5}{t^5}\right) + e_6\left(\frac{f(t)-Q_1t-Q_3t^3-Q_5t^5-Q_7t^7}{t^6}\right)\\ &+e_7\left(\frac{f(t)-Q_1t-Q_3t^3-Q_5t^5-Q_7t^7}{t^7}\right),\end{aligned}$$

$$\begin{aligned}S_2(t) &= e_0 t h(t) + e_1 h(t) + e_2\left(\frac{h(t)-Q_2t^2}{t}\right) + e_3\left(\frac{h(t)-Q_2t^2}{t^2}\right)\\ &+e_4\left(\frac{h(t)-Q_2t^2-Q_4t^4}{t^3}\right) + e_5\left(\frac{h(t)-Q_2t^2-Q_4t^4}{t^4}\right)\\ &+e_6\left(\frac{h(t)-Q_2t^2-Q_4t^4-Q_6t^6}{t^5}\right) + e_7\left(\frac{h(t)-Q_2t^2-Q_4t^4-Q_6t^6}{t^6}\right),\end{aligned}$$

$$f(t) = \frac{t-ct^3}{1-(ab+d+c)\,t^2+cdt^4}$$

and

$$h(t) = \frac{at^2}{1-(ab+d+c)\,t^2+cdt^4}.$$

Proof. The generating function of the modified generalized dual Fibonacci octonions is $G(t)=\sum_{n=0}^{\infty}\tilde{\Phi}_n t^n$ and using the equations $btG(t)$ and $t^2G(t)$.

$$G(t) = \tilde{\Phi}_0 + \tilde{\Phi}_1 t + ... + \tilde{\Phi}_n t^n + ... = \tilde{\Phi}_0 + \tilde{\Phi}_1 t + \sum_{n=2}^{\infty}\tilde{\Phi}_n t^n$$

$$btG(t) = b\tilde{\Phi}_0 t + b\tilde{\Phi}_1 t^2 + ... + b\tilde{\Phi}_n t^{n+1} + ... = b\tilde{\Phi}_0 t + \sum_{n=2}^{\infty} b\tilde{\Phi}_{n-1} t^n$$

$$dt^2G(t) = d\tilde{\Phi}_0 t^2 + d\tilde{\Phi}_1 t^3 + ... + d\tilde{\Phi}_n t^{n+2} + ... = \sum_{n=2}^{\infty} d\tilde{\Phi}_{n-2} t^n$$

and take into account the equation $\tilde{\Phi}_n = \Phi_n + \varepsilon\Phi_{n+1}$ in the above equations, we get

$$\begin{aligned} G(t) - btG(t) - dt^2G(t) &= \Phi_0 + \varepsilon\Phi_1 + (\Phi_1 + \varepsilon\Phi_2)t - b(\Phi_0 + \varepsilon\Phi_1)t \\ &\quad + \sum_{n=2}^{\infty}(\Phi_n - b\Phi_{n-1} - d\Phi_{n-2})t^n \\ &\quad + \varepsilon\sum_{n=2}^{\infty}(\Phi_{n+1} - b\Phi_n - d\Phi_{n-1})t^n \\ &= \Phi_0 + (\Phi_1 - b\Phi_0)t + \sum_{n=2}^{\infty}(\Phi_n - b\Phi_{n-1} - d\Phi_{n-2})t^n \\ &\quad + \varepsilon(\Phi_1 + (\Phi_2 - b\Phi_1)t + \sum_{n=2}^{\infty}(\Phi_{n+1} - b\Phi_n - d\Phi_{n-1})t^n) \end{aligned}$$

and if $N = 8$ in generating function given in Theorem 2.1. of the modified generalized Fibonacci 2^k-ions in [22], we can obtain generating function of the modified generalized Fibonacci octonions. Thus, it can be easily seen that the following relations are correct.

$$\sum_{n=2}^{\infty}(\Phi_n - b\Phi_{n-1} - d\Phi_{n-2})t^n = (a-b)R_1(t) + (c-d)R_2(t),$$
$$\sum_{n=2}^{\infty}(\Phi_{n+1} - b\Phi_n - d\Phi_{n-1})t^n = (a-b)S_1(t) + (c-d)S_2(t).$$

Finally, we can find the generating function of modified generalized dual Fibonacci octonions. □

Now, we give the Catalan's identity. Furthermore, we derive the Cassini's identity which is the special case of the Catalan's identity for $r = 1$.

Theorem 2.3. *(Catalan's identity) For $n, r \in \mathbb{N}_0$ and $r \leq n$, we have the identity*

$$\tilde{\Phi}_{2(n+r)+\xi(i)}\tilde{\Phi}_{2(n-r)+\xi(i)} - \tilde{\Phi}^2_{2n+\xi(i)}$$

$$\begin{aligned} &= \frac{(-c)^{\xi(i)}}{(ab)^{2r}(\alpha-\beta)^2} \times \Bigg[\alpha_{\xi(i)}\beta_{\xi(i)}\left((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r\right) \\ &\qquad + \beta_{\xi(i)}\alpha_{\xi(i)}\left((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r\right)\Bigg] \\ &+ \varepsilon\frac{(-c)^{\xi(i)}}{(ab)^{2r}(\alpha-\beta)^2} \times \Bigg[\alpha_{\xi(i)}\beta_{\xi(i+1)}\Bigg((ab)^{2r+\xi(i)}(cd)^n(\beta+d-c)^{\xi(i+1)}\beta^{\xi(i)} \end{aligned}$$

$$-(ab)^{2r+\xi(i)}(cd)^n(\beta+d-c)^{\xi(i+1)}\beta^{\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)^r\Bigg)$$

$$+\beta_{\xi(i+1)}\alpha_{\xi(i)}\Bigg((ab)^{2r+\xi(i)}(cd)^n(\beta+d-c)^{\xi(i+1)}\beta^{\xi(i)}$$

$$-(ab)^{2r+\xi(i)}(cd)^n(\beta+d-c)^{\xi(i+1)}\beta^{\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)^r\Bigg)$$

$$+\alpha_{\xi(i+1)}\beta_{\xi(i)}\Bigg((ab)^{2r+\xi(i)}(cd)^n(\alpha+d-c)^{\xi(i+1)}\alpha^{\xi(i)}$$

$$-(ab)^{2r+\xi(i)}(cd)^n(\alpha+d-c)^{\xi(i+1)}\alpha^{\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)^r\Bigg)$$

$$+\beta_{\xi(i)}\alpha_{\xi(i+1)}\Bigg((ab)^{2r+\xi(i)}(cd)^n(\alpha+d-c)^{\xi(i+1)}\alpha^{\xi(i)}$$

$$-(ab)^{2r+\xi(i)}(cd)^n(\alpha+d-c)^{\xi(i+1)}\alpha^{\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)^r\Bigg)\Bigg], \tag{2.4}$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.1 and $i \in \{0,1\}$.

Proof. From Equation (2.2) and Binet formula for the modified generalized dual Fiboancci octonions in Theorem 2.1 the proof is clear. □

Corollary 2.1. *(Cassini's identity) For $n \in \mathbb{N}_0$, we have the identity*

$$\tilde{\Phi}_{2(n+1)+\xi(i)}\tilde{\Phi}_{2(n-1)+\xi(i)} - \tilde{\Phi}^2_{2n+\xi(i)}$$

$$= \frac{(-c)^{\xi(i)}}{(ab)^2(\alpha-\beta)^2} \times \Bigg[\alpha_{\xi(i)}\beta_{\xi(i)}\Bigg((ab)^{2+\xi(i)}(cd)^n - (ab)^{2+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)\Bigg)$$

$$+\beta_{\xi(i)}\alpha_{\xi(i)}\Bigg((ab)^{2+\xi(i)}(cd)^n - (ab)^{2+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)\Bigg)\Bigg]$$

$$+ \quad \varepsilon\frac{(-c)^{\xi(i)}}{(ab)^2(\alpha-\beta)^2} \times \Bigg[\alpha_{\xi(i)}\beta_{\xi(i+1)}\Bigg((ab)^{2+\xi(i)}(cd)^n(\beta+d-c)^{\xi(i+1)}\beta^{\xi(i)}$$

$$-(ab)^{2+\xi(i)}(cd)^n(\beta+d-c)^{\xi(i+1)}\beta^{\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)\Bigg)$$

$$+\beta_{\xi(i+1)}\alpha_{\xi(i)}\Bigg((ab)^{2+\xi(i)}(cd)^n(\beta+d-c)^{\xi(i+1)}\beta^{\xi(i)}$$

$$-(ab)^{2+\xi(i)}(cd)^n(\beta+d-c)^{\xi(i+1)}\beta^{\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)\Bigg)$$

$$+\alpha_{\xi(i+1)}\beta_{\xi(i)}\Bigg((ab)^{2+\xi(i)}(cd)^n(\alpha+d-c)^{\xi(i+1)}\alpha^{\xi(i)}$$
$$-(ab)^{2+\xi(i)}(cd)^n(\alpha+d-c)^{\xi(i+1)}\alpha^{\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)\Bigg)$$
$$+\beta_{\xi(i)}\alpha_{\xi(i+1)}\Bigg((ab)^{2+\xi(i)}(cd)^n(\alpha+d-c)^{\xi(i+1)}\alpha^{\xi(i)}$$
$$-(ab)^{2+\xi(i)}(cd)^n(\alpha+d-c)^{\xi(i+1)}\alpha^{\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)\Bigg)\Bigg], \tag{2.5}$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.1 and $i \in \{0,1\}$.

Theorem 2.4. *$\tilde{\Phi}_n$ be a modified generalized dual Fibonacci octonion, $\overline{\tilde{\Phi}}_n$ be conjugate of $\tilde{\Phi}_n$, $\tilde{Q}_n$ a modified generalized dual Fibonacci number. Then the following equation can be given;*

$$\tilde{\Phi}_n + \overline{\tilde{\Phi}}_n = 2\tilde{Q}_n. \tag{2.6}$$

Proof. From Equation (2.2) *and Equation* (1.20)*, we get*

$$\begin{aligned}
&\tilde{\Phi}_n + \overline{\tilde{\Phi}}_n \\
=\ & (\Phi_n + \varepsilon\Phi_{n+1}) + \overline{(\Phi_n + \varepsilon\Phi_{n+1})} \\
=\ & \left(\tilde{Q}_n e_0 + \tilde{Q}_{n+1}e_1 + \tilde{Q}_{n+2}e_2 + \tilde{Q}_{n+3}e_3 + \tilde{Q}_{n+4}e_4 + \tilde{Q}_{n+5}e_5 + \tilde{Q}_{n+6}e_6 + \tilde{Q}_{n+7}e_7\right) \\
& + \left(\tilde{Q}_n e_0 - \tilde{Q}_{n+1}e_1 - \tilde{Q}_{n+2}e_2 - \tilde{Q}_{n+3}e_3 - \tilde{Q}_{n+4}e_4 - \tilde{Q}_{n+5}e_5 - \tilde{Q}_{n+6}e_6 - \tilde{Q}_{n+7}e_7\right) \\
=\ & 2\tilde{Q}_n.
\end{aligned}$$

□

3. Modified Generalized Dual Lucas Octonions

In this section, we present the definitions and theorems of the modified generalized dual Lucas octonions.

Definition 3.1. The modified generalized dual numbers Lucas $\tilde{U}_n$ are

$$\tilde{U}_n = U_n + \varepsilon U_{n+1}, \tag{3.1}$$

where U_n, U_{n+1} are the nth and $(n+1)th$ modified generalized Lucas numbers in Eq.(1.4), respectively.

Definition 3.2. The modified generalized dual Lucas octonions $\tilde{\vartheta}_n$ are defined by

$$\tilde{\vartheta}_n = \vartheta_n + \varepsilon\vartheta_{n+1}. \tag{3.2}$$

Here ϑ_n is the nth modified generalized Lucas octonions in Eq.(1.21). From Definition (3.1), Eq.(3.2) can be denoted as;

$$\tilde{\vartheta}_n = \sum_{l=0}^{7} \tilde{U}_{n+l} e_l.$$

In the following theorem we give the Binet formula of the modified generalized dual Lucas octonion by the help of the Binet formula of ϑ_n.

Theorem 3.1. *The Binet formula for the modified generalized dual Lucas octonion $\tilde{\vartheta}_n$ is*

$$\begin{aligned}\tilde{\vartheta}_n &= \frac{1}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \left(\frac{\alpha^*_{\xi(n)} \alpha^{\lfloor \frac{n-1}{2} \rfloor} (\alpha + d - c)^{\lfloor \frac{n}{2} \rfloor} (\alpha + d + 1)}{\alpha - \beta} \right. \\ &\qquad \left. - \frac{\beta^*_{\xi(n)} \beta^{\lfloor \frac{n-1}{2} \rfloor} (\beta + d - c)^{\lfloor \frac{n}{2} \rfloor} (\beta + d + 1)}{\alpha - \beta} \right) \\ &\quad + \varepsilon \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^*_{\xi(n+1)} \alpha^{\lfloor \frac{n}{2} \rfloor} (\alpha + d - c)^{\lfloor \frac{n+1}{2} \rfloor} (\alpha + d + 1)}{\alpha - \beta} \right. \\ &\qquad \left. - \frac{\beta^*_{\xi(n+1)} \beta^{\lfloor \frac{n}{2} \rfloor} (\beta + d - c)^{\lfloor \frac{n+1}{2} \rfloor} (\beta + d + 1)}{\alpha - \beta} \right)\end{aligned}$$

where

$$\alpha^*_{\xi(n)} = \sum_{l=0}^{7} \frac{a^{\xi(l+\xi(n))}}{(ab)^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor}} (\alpha + d - c)^{\lfloor \frac{l+\xi(n)}{2} \rfloor} \alpha^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor} e_l$$

and

$$\beta^*_{\xi(n)} = \sum_{l=0}^{7} \frac{a^{\xi(l+\xi(n))}}{(ab)^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor}} (\beta + d - c)^{\lfloor \frac{l+\xi(n)}{2} \rfloor} \beta^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor} e_l.$$

Proof. Assume that for n,

$$\tilde{\vartheta}_n = \vartheta_n + \varepsilon \vartheta_{n+1}.$$

If $N = 8$ in Binet formula given in Theorem 3.2. of the modified generalized Lucas 2^k-ions in [22], we can obtain Binet formula of the modified generalized Lucas octonions. Thus, it can be easily seen that the following relations are correct.

$$\begin{aligned}\vartheta_n &= \frac{1}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \left(\frac{\alpha^*_{\xi(n)} \alpha^{\lfloor \frac{n-1}{2} \rfloor} (\alpha + d - c)^{\lfloor \frac{n}{2} \rfloor} (\alpha + d + 1)}{\alpha - \beta} \right. \\ &\qquad \left. - \frac{\beta^*_{\xi(n)} \beta^{\lfloor \frac{n-1}{2} \rfloor} (\beta + d - c)^{\lfloor \frac{n}{2} \rfloor} (\beta + d + 1)}{\alpha - \beta} \right)\end{aligned}$$

and

$$\vartheta_{n+1} = \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^*_{\xi(n+1)} \alpha^{\lfloor \frac{n}{2} \rfloor} (\alpha + d - c)^{\lfloor \frac{n+1}{2} \rfloor} (\alpha + d + 1)}{\alpha - \beta} - \frac{\beta^*_{\xi(n+1)} \beta^{\lfloor \frac{n}{2} \rfloor} (\beta + d - c)^{\lfloor \frac{n+1}{2} \rfloor} (\beta + d + 1)}{\alpha - \beta} \right)$$

thereby showing that indeed $\tilde{\vartheta}_n$ holds. So, the Binet formula is obtained. □

Now, we give the generating function of the modified generalized dual Lucas octonions.

Theorem 3.2. *The generating function for the modified generalized dual Lucas octonions is*

$$\begin{aligned} G(t) &= \frac{\vartheta_0 + (\vartheta_1 - a\vartheta_0)\, t + (b-a)R_1(t) + (d-c)R_2(t)}{1 - at - ct^2} \\ &\quad + \varepsilon \frac{\vartheta_1 + (\vartheta_2 - a\vartheta_1)\, t + (b-a)S_1(t) + (d-c)S_2(t)}{1 - at - ct^2} \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} R_1(t) &= e_0 t f(t) + e_1 \left(f(t) - U_1 t\right) + e_2 \left(\frac{f(t) - U_1 t}{t}\right) + e_3 \left(\frac{f(t) - U_1 t - U_3 t^3}{t^2}\right) \\ &\quad + e_4 \left(\frac{f(t) - U_1 t - U_3 t^3}{t^3}\right) + e_5 \left(\frac{f(t) - U_1 t - U_3 t^3 - U_5 t^5}{t^4}\right) \\ &\quad + e_6 \left(\frac{f(t) - U_1 t - U_3 t^3 - U_5 t^5}{t^5}\right) + e_7 \left(\frac{f(t) - U_1 t - U_3 t^3 - U_5 t^5 - U_7 t^7}{t^6}\right), \end{aligned}$$

$$\begin{aligned} R_2(t) &= e_0 t^2 h(t) + e_1 t (h(t) - U_0) + e_2 (h(t) - U_0) + e_3 \left(\frac{h(t) - U_0 - U_2 t^2}{t}\right) \\ &\quad + e_4 \left(\frac{h(t) - U_0 - U_2 t^2}{t^2}\right) + e_5 \left(\frac{h(t) - U_0 - U_2 t^2 - U_4 t^4}{t^3}\right) \\ &\quad + e_6 \left(\frac{h(t) - U_0 - U_2 t^2 - U_4 t^4}{t^4}\right) \\ &\quad + e_7 \left(\frac{h(t) - U_0 - U_2 t^2 - U_4 t^4 - U_6 t^6}{t^5}\right), \end{aligned}$$

$$\begin{aligned} S_1(t) &= e_0 \left(f(t) - U_1 t\right) + e_1 \left(\frac{f(t) - U_1 t}{t}\right) + e_2 \left(\frac{f(t) - U_1 t - U_3 t^3}{t^2}\right) \\ &\quad + e_3 \left(\frac{f(t) - U_1 t - U_3 t^3}{t^3}\right) + e_4 \left(\frac{f(t) - U_1 t - U_3 t^3 - U_5 t^5}{t^4}\right) \end{aligned}$$

$$+e_5\left(\frac{f(t)-U_1t-U_3t^3-U_5t^5}{t^5}\right)+e_6\left(\frac{f(t)-U_1t-U_3t^3-U_5t^5-U_7t^7}{t^6}\right)$$
$$+e_7\left(\frac{f(t)-U_1t-U_3t^3-U_5t^5-U_7t^7}{t^7}\right),$$

$$\begin{aligned}S_2(t) &= e_0t(h(t)-U_0)+e_1(h(t)-U_0)+e_2\left(\frac{h(t)-U_0-U_2t^2}{t}\right)\\ &+e_3\left(\frac{h(t)-U_0-U_2t^2}{t^2}\right)+e_4\left(\frac{h(t)-U_0-U_2t^2-U_4t^4}{t^3}\right)\\ &+e_5\left(\frac{h(t)-U_0-U_2t^2-U_4t^4}{t^4}\right)+e_6\left(\frac{h(t)-U_0-U_2t^2-U_4t^4-U_6t^6}{t^5}\right)\\ &+e_7\left(\frac{h(t)-U_0-U_2t^2-U_4t^4-U_6t^6}{t^6}\right),\end{aligned}$$

$$f(t)=\frac{at+at^3}{1-(ab+d+c)\,t^2+cdt^4}$$

and

$$h(t)=\frac{U_0+U_2t^2-(ab+d+c)U_0t^2}{1-(ab+d+c)\,t^2+cdt^4}.$$

Proof. The generating function of the modified generalized dual Lucas octonions is $G(t)=\sum_{n=0}^{\infty}\tilde{\vartheta}_nt^n$ and using the equations $btG(t)$ and $t^2G(t)$.

$$G(t)=\tilde{\vartheta}_0+\tilde{\vartheta}_1t+...+\tilde{\vartheta}_nt^n+...=\tilde{\vartheta}_0+\tilde{\vartheta}_1t+\sum_{n=2}^{\infty}\tilde{\vartheta}_nt^n$$

$$atG(t)=a\tilde{\vartheta}_0t+a\tilde{\vartheta}_1t^2+...+a\tilde{\vartheta}_nt^{n+1}+...=a\tilde{\vartheta}_0t+\sum_{n=2}^{\infty}a\tilde{\vartheta}_{n-1}t^n$$

$$ct^2G(t)=c\tilde{\vartheta}_0t^2+c\tilde{\vartheta}_1t^3+...+c\tilde{\vartheta}_nt^{n+2}+...=\sum_{n=2}^{\infty}c\tilde{\vartheta}_{n-2}t^n$$

and take into account the equation $\tilde{\vartheta}_n=\vartheta_n+\varepsilon\vartheta_{n+1}$ in the above equations, we get

$$\begin{aligned}G(t)-atG(t)-ct^2G(t) &= \vartheta_0+\varepsilon\vartheta_1+(\vartheta_1+\varepsilon\vartheta_2)t-a(\vartheta_0+\varepsilon\vartheta_1)t\\ &+\sum_{n=2}^{\infty}(\vartheta_n-a\vartheta_{n-1}-c\vartheta_{n-2})t^n\\ &+\varepsilon\sum_{n=2}^{\infty}(\vartheta_{n+1}-a\vartheta_n-c\vartheta_{n-1})t^n\end{aligned}$$

$$= \vartheta_0 + (\vartheta_1 - a\vartheta_0)t + \sum_{n=2}^{\infty}(\vartheta_n - a\vartheta_{n-1} - c\vartheta_{n-2})t^n$$
$$+\varepsilon(\vartheta_1 + (\vartheta_2 - a\vartheta_1)t + \sum_{n=2}^{\infty}(\vartheta_{n+1} - a\vartheta_n - c\vartheta_{n-1})t^n)$$

and if $N = 8$ in generating function given in Theorem 3.1. of the modified generalized Lucas 2^k-ions in [22], we can obtain generating function of the modified generalized Lucas octonions. Thus, it can be easily seen that the following relations are correct.

$$\sum_{n=2}^{\infty}(\vartheta_n - a\vartheta_{n-1} - c\vartheta_{n-2})t^n = (b-a)R_1(t) + (d-c)R_2(t),$$
$$\sum_{n=2}^{\infty}(\vartheta_{n+1} - a\vartheta_n - c\vartheta_{n-1})t^n = (b-a)S_1(t) + (d-c)S_2(t).$$

Finally, we can find the generating function of modified generalized dual Lucas octonions. □

Now, we give the Catalan's identity. Furthermore, we derive the Cassini's identity which is the special case of the Catalan's identity for $r = 1$.

Theorem 3.3. *(Catalan's identity) For $n, r \in \mathbb{N}_0$ and $r \leq n$, we have the identity*

$$\tilde{\vartheta}_{2(n+r)+\xi(i)}\tilde{\vartheta}_{2(n-r)+\xi(i)} - \tilde{\vartheta}^2_{2n+\xi(i)} = \frac{(-c)^{\xi(i+1)}(\alpha+d+1)(\beta+d+1)}{(ab)^{2r}(\alpha-\beta)^2}$$
$$\times\left[\alpha^*_{\xi(i)}\beta^*_{\xi(i)}\left((ab)^{2r+\xi(i+1)}(cd)^{n-\xi(i+1)} - (ab)^{2r+\xi(i+1)}(cd)^{n-\xi(i+1)}\left(\frac{\alpha+d}{\beta+d}\right)^r\right)\right.$$
$$\left.+\beta^*_{\xi(i)}\alpha^*_{\xi(i)}\left((ab)^{2r+\xi(i+1)}(cd)^{n-\xi(i+1)} - (ab)^{2r+\xi(i+1)}(cd)^{n-\xi(i+1)}\left(\frac{\beta+d}{\alpha+d}\right)^r\right)\right]$$
$$+\varepsilon\frac{(-c)^{\xi(i+1)}(\alpha+d+1)(\beta+d+1)}{(ab)^{2r}(\alpha-\beta)^2}$$
$$\times\left[\alpha^*_{\xi(i)}\beta^*_{\xi(i+1)}\left((ab)^{2r}(cd)^{n-\xi(i+1)}\beta^{\xi(i+1)}(\alpha+d-c)^{\xi(i)}\right.\right.$$
$$\left.-(ab)^{2r}(cd)^{n-\xi(i+1)}\beta^{\xi(i+1)}(\alpha+d-c)^{\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)^r\right)$$
$$+\beta^*_{\xi(i+1)}\alpha^*_{\xi(i)}\left((ab)^{2r}(cd)^{n-\xi(i+1)}\beta^{\xi(i+1)}(\alpha+d-c)^{\xi(i)}\right.$$
$$\left.-(ab)^{2r}(cd)^{n-\xi(i+1)}\beta^{\xi(i+1)}(\alpha+d-c)^{\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)^r\right)$$

$$+\alpha^*_{\xi(i+1)}\beta^*_{\xi(i)}\Bigg((ab)^{2r}(cd)^{n-\xi(i+1)}\alpha^{\xi(i+1)}(\beta+d-c)^{\xi(i)}$$

$$-(ab)^{2r}(cd)^{n-\xi(i+1)}\alpha^{\xi(i+1)}(\beta+d-c)^{\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)^r\Bigg)$$

$$+\beta^*_{\xi(i)}\alpha^*_{\xi(i+1)}\Bigg((ab)^{2r}(cd)^{n-\xi(i+1)}\alpha^{\xi(i+1)}(\beta+d-c)^{\xi(i)}$$

$$-(ab)^{2r}(cd)^{n-\xi(i+1)}\alpha^{\xi(i+1)}(\beta+d-c)^{\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)^r\Bigg)\Bigg],$$

where $\alpha^*_{\xi(i)}$ *and* $\beta^*_{\xi(i)}$ *are defined in Theorem 3.1 and* $i \in \{0,1\}$*.*

Proof. From Equation (3.2) and Binet formula for the modified generalized dual Lucas octonions in Theorem 3.1 the proof is clear. □

Corollary 3.1. *(Cassini's identity) For* $n, r \in \mathbb{N}_0$*, we have the identity*

$$\tilde{\vartheta}_{2(n+1)+\xi(i)}\tilde{\vartheta}_{2(n-1)+\xi(i)} - \tilde{\vartheta}^2_{2n+\xi(i)} = \frac{(-c)^{\xi(i+1)}(\alpha+d+1)(\beta+d+1)}{(ab)^2(\alpha-\beta)^2}$$

$$\times\Bigg[\alpha^*_{\xi(i)}\beta^*_{\xi(i)}\Bigg((ab)^{2+\xi(i+1)}(cd)^{n-\xi(i+1)} - (ab)^{2+\xi(i+1)}(cd)^{n-\xi(i+1)}\left(\frac{\alpha+d}{\beta+d}\right)\Bigg)$$

$$+\beta^*_{\xi(i)}\alpha^*_{\xi(i)}\Bigg((ab)^{2+\xi(i+1)}(cd)^{n-\xi(i+1)} - (ab)^{2+\xi(i+1)}(cd)^{n-\xi(i+1)}\left(\frac{\beta+d}{\alpha+d}\right)\Bigg)\Bigg]$$

$$+ \quad \varepsilon\frac{(-c)^{\xi(i+1)}(\alpha+d+1)(\beta+d+1)}{(ab)^2(\alpha-\beta)^2}$$

$$\times\Bigg[\alpha^*_{\xi(i)}\beta^*_{\xi(i+1)}\Bigg((ab)^2(cd)^{n-\xi(i+1)}\beta^{\xi(i+1)}(\alpha+d-c)^{\xi(i)}$$

$$-(ab)^2(cd)^{n-\xi(i+1)}\beta^{\xi(i+1)}(\alpha+d-c)^{\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)\Bigg)$$

$$+\beta^*_{\xi(i+1)}\alpha^*_{\xi(i)}\Bigg((ab)^2(cd)^{n-\xi(i+1)}\beta^{\xi(i+1)}(\alpha+d-c)^{\xi(i)}$$

$$-(ab)^2(cd)^{n-\xi(i+1)}\beta^{\xi(i+1)}(\alpha+d-c)^{\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)\Bigg)$$

$$+\alpha^*_{\xi(i+1)}\beta^*_{\xi(i)}\Bigg((ab)^2(cd)^{n-\xi(i+1)}\alpha^{\xi(i+1)}(\beta+d-c)^{\xi(i)}$$

$$-(ab)^2(cd)^{n-\xi(i+1)}\alpha^{\xi(i+1)}(\beta+d-c)^{\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)\Bigg)$$

$$+\beta^*_{\xi(i)}\alpha^*_{\xi(i+1)}\left((ab)^2(cd)^{n-\xi(i+1)}\alpha^{\xi(i+1)}(\beta+d-c)^{\xi(i)}\right.$$
$$\left.\left.-(ab)^2(cd)^{n-\xi(i+1)}\alpha^{\xi(i+1)}(\beta+d-c)^{\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)\right)\right],$$

where $\alpha^*_{\xi(i)}$ *and* $\beta^*_{\xi(i)}$ *are defined in Theorem 3.1 and* $i\in\{0,1\}$.

Theorem 3.4. $\tilde{\vartheta}_n$ *be a modified generalized dual Lucas octonion,* $\overline{\tilde{\vartheta}}_n$ *be conjugate of* $\tilde{\vartheta}_n$*,* $\tilde{U}_n$ *a modified generalized dual Lucas number. Then the following equation can be given;*

$$\tilde{\vartheta}_n+\overline{\tilde{\vartheta}}_n=2\tilde{U}_n.$$

Proof. From Equation (3.1) and Equation (1.21), we get

$$\begin{aligned}\tilde{\vartheta}_n+\overline{\tilde{\vartheta}}_n &= (\vartheta_n+\varepsilon\vartheta_{n+1})+\overline{(\vartheta_n+\varepsilon\vartheta_{n+1})}\\ &= \left(\tilde{U}_ne_0+\tilde{U}_{n+1}e_1+\tilde{U}_{n+2}e_2+\tilde{U}_{n+3}e_3+\tilde{U}_{n+4}e_4+\tilde{U}_{n+5}e_5+\tilde{U}_{n+6}e_6+\tilde{U}_{n+7}e_7\right)\\ &\quad+\left(\tilde{U}_ne_0-\tilde{U}_{n+1}e_1-\tilde{U}_{n+2}e_2-\tilde{U}_{n+3}e_3-\tilde{U}_{n+4}e_4-\tilde{U}_{n+5}e_5-\tilde{U}_{n+6}e_6-\tilde{U}_{n+7}e_7\right)\\ &= 2\tilde{U}_n.\end{aligned}$$

□

Theorem 3.5. $\tilde{\Phi}_n$ *be a modified generalized dual Fibonacci octonion and* $\tilde{\vartheta}_n$ *be a modified generalized dual Lucas octonion.Then the following equation can be given;*

$$\tilde{\vartheta}_n=\tilde{\Phi}_{n-1}+\tilde{\Phi}_{n+1}.$$

Proof. From Equation (2.2), Equation (3.1) and Theorem 20 in [7] , we get

$$\begin{aligned}&\tilde{\Phi}_{n-1}+\tilde{\Phi}_{n+1}\\ =\;& (\Phi_{n-1}+\varepsilon\Phi_n)+(\Phi_{n+1}+\varepsilon\Phi_{n+2})\\ =\;& (\Phi_{n-1}+\Phi_{n+1})+\varepsilon\left(\Phi_n+\Phi_{n+2}\right)\\ =\;& \left(\sum_{l=0}^{7}Q_{n-1+l}e_l+\sum_{l=0}^{7}Q_{n+1+l}e_l\right)+\varepsilon\left(\sum_{l=0}^{7}Q_{n+l}e_l+\sum_{l=0}^{7}Q_{n+2+l}e_l\right)\\ =\;& \left(\sum_{l=0}^{7}\left(Q_{n-1+l}+Q_{n+1+l}\right)e_l\right)+\varepsilon\left(\sum_{l=0}^{7}\left(Q_{n+l}+Q_{n+2+l}\right)e_l\right)\\ =\;& \left((Q_{n-1}+Q_{n+1})e_0+(Q_n+Q_{n+2})e_1+...+(Q_{n+6}+Q_{n+8})e_7\right)\\ &+\varepsilon\left((Q_n+Q_{n+2})e_0+(Q_{n+1}+Q_{n+3})e_1+...+(Q_{n+7}+Q_{n+9})e_7\right)\end{aligned}$$

$$
\begin{aligned}
&= \Big(U_n e_0 + U_{n+1} e_1 + U_{n+2} e_2 + ... + U_{n+7} e_7\Big) \\
&\quad +\varepsilon\Big(U_{n+1} e_0 + U_{n+2} e_1 + U_{n+3} e_2 + ... + U_{n+8} e_7\Big) \\
&= \vartheta_n + \varepsilon \vartheta_{n+1} \\
&= \tilde{\vartheta}_n .
\end{aligned}
$$

□

4. Conclusion

The importance of generalized number systems covering various fields such as mathematics, physics and computer science has increased. For this reason, there have been many studies on octonions recently [17, 8, 3, 5, 6, 4, 27, 2, 11]. In this article, we first introduced the concepts of modified generalized dual Fibonacci and modified generalized dual Lucas octonions. In this way, we offered a broader and different perspective to the field of octonions. We then obtained the Catalan and Cassini identities of these new octonions. In addition, we present the generator functions and Binet formulas of modified generalized dual Fibonacci and modified generalized dual Lucas octonions. The data obtained as a result of all these studies we have done has the potential to help discover new connections that can lead to important conclusions in the field of hypercomplex numbers and algebraic structures. In addition, this study contributes to the literature as it is a generalization of many studies in the literature.

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