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SOBOLEV SPACES OVER $\mathbb{R}_I^\infty$

Hemanta Kalita

Department of Mathematics, Assam Don Bosco University, Sonapur 782402, Assam, India

Abstract. Our goal in this article is to construct Sobolev spaces over $\mathbb{R}_I^\infty$. Completeness of the Sobolev space over $\mathbb{R}_I^\infty$ are discussed. In application we have constructed the Sobolev spaces on a separable Banach space B.
Keywords: Distribution on $\mathbb{R}_I^\infty$; Sobolev space; continuous embedding; Separable Banach spaces

1. Introduction and Preliminaries

One of the most important problems of mathematical physics in the 20th century was to find the solution to Dirichlet and Neuman problems for Laplace equation (see for instance [14]). This problem attracted famous scientists of that period, namely Hilbert, Courant, Weyl and many more. Russian Mathematician Sergei Sobolev in 1930 overcame the main difficulty of this problem and introduced a functional space called Sobolev space, given by functions in $L^p[\mathbb{R}^n]$ whose distributional derivatives of order upto to k exist and are in $L^p[\mathbb{R}^n]$. Today there are many information about Sobolev spaces $S^{k,p}[\mathbb{R}^n]$, where $p > 1$ and $k = 0, 1, 2, \ldots$, (see [11, 13, 14, 18]).

Definition 1.1. [10, 19] Let $\mathcal{B}[\mathbb{R}^n]$ be the Borel σ-algebra for $\mathbb{R}^n$, $I = [-\frac{1}{2}, \frac{1}{2}]$ and $I_n = \prod_{i=n+1}^{\infty} I$. For $\mathfrak{A} \in \mathcal{B}[\mathbb{R}^n]$ the set $\mathfrak{A}_n = \mathfrak{A} \times I_n$ is called n^{th} order box set in $\mathbb{R}^\infty$. We define

1. $\mathfrak{A}_n \cup \mathfrak{B}_n = (\mathfrak{A} \cup \mathfrak{B}) \times I_n$;

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Corresponding Author: Hemanta Kalita, Department of Mathematics, Assam Don Bosco University, Sonapur 782402, Assam, India | E-mail: hemanta30kalita@gmail.com
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2. $\mathfrak{A}_n \cap \mathfrak{B}_n = (\mathfrak{A} \cap \mathfrak{B}) \times I_n$;

3. $\mathfrak{B}_n^c = \mathfrak{B}^c \times I_n$.

Definition 1.2. [10, 19] Define $\mathbb{R}_I^n = \mathbb{R}^n \times I_n$. We denote $\mathcal{B}[\mathbb{R}_I^n]$ to be the Borel σ-algebra for $\mathbb{R}_I^n$, where the topology for $\mathbb{R}_I^n$ is defined via the class of open sets $\mathfrak{D}_n = \{\mathfrak{U} \times I_n : \mathfrak{U} \text{ is open in } \mathbb{R}^n\}$. For any $\mathfrak{A} \in \mathcal{B}[\mathbb{R}^n]$, we define $\lambda_\infty(\mathfrak{A}_n)$ on $\mathbb{R}_I^n$ by product measure $\lambda_\infty(\mathfrak{A}_n) = \lambda_n(\mathfrak{A}) \times \Pi_{i=n+1}^{\infty}\lambda_1(I) = \lambda_n(\mathfrak{A})$, where λ_n is Lebesgue measure on $\mathbb{R}^n$.

Theorem 1.1. *[8, 10] $\lambda_\infty(.)$ is a measure on $\mathcal{B}[\mathbb{R}_I^n]$, which is equivalent to n-dimensional Lebesgue measure on $\mathbb{R}^n$.*

[8, 10] The measure $\lambda_\infty(.)$ is both translationally and rotationally invariant on $(\mathbb{R}_I^n, \mathcal{B}[\mathbb{R}_I^n])$ for each $n \in \mathbb{N}$.

We can construct a theory on $\mathbb{R}_I^n$ that completely parallels that on $\mathbb{R}^n$. Since $\mathbb{R}_I^n \subset \mathbb{R}_I^{n+1}$, we have an increasing sequence, so we define $\widehat{\mathbb{R}}_I^\infty = \lim_{n\to\infty} \mathbb{R}_I^n = \bigcup_{n=1}^{\infty} \mathbb{R}_I^n$. In [10] it is shown that the measure $\lambda_\infty(.)$ can be extended to $\mathbb{R}^\infty$. Let $x = (x_1, x_2, \ldots) \in \mathbb{R}_I^\infty$. Also let $I_n = \Pi_{k=n+1}^{\infty}[\frac{-1}{2}, \frac{1}{2}]$ and let $h_n(\widehat{x}) = \chi_{I_n}(\widehat{x})$, where $\widehat{x} = (x_i)_{i=n+1}^{\infty}$. Recalling $\mathbb{R}_I^\infty$ is the closure of $\widehat{\mathbb{R}}_I^\infty$ in the induced topology from $\mathbb{R}^\infty$. From our construction, it is clear that a set of the form $\mathfrak{A} = \mathfrak{A}_n \times (\Pi_{k=n+1}^{\infty}\mathbb{R})$ is not in $\widehat{\mathbb{R}}_I^\infty$ for any n. So, $\widehat{\mathbb{R}}_I^\infty \neq \mathbb{R}^\infty$. The natural topology for $\mathbb{R}_I^\infty$ is that induced as a closed subspace of $\mathbb{R}^\infty$. Thus if $x = (x_n),\ y = (y_n)$ are sequences in $\mathbb{R}_I^\infty$, a metric d on $\mathbb{R}_I^\infty$, is defined as

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Remark 1.1. $\mathbb{R}_I^\infty = \mathbb{R}^\infty$ as sets but not as topological spaces.

We call $\mathbb{R}_I^\infty$ the essentially bounded version of $\mathbb{R}^\infty$. There are certain pathologies of $\mathbb{R}^\infty$ that are preserved to $\mathbb{R}_I^\infty$, for example, if $\mathcal{A}_i$ has measure $1 + \epsilon$ for all i then $\lambda_\infty(\mathcal{A}) = \Pi_{i=1}^{\infty}\lambda(\mathcal{A}_i) = \infty$. On the other hand, if each $\mathcal{A}_i$ has measure $1 - \epsilon$, then $\lambda_\infty(\mathcal{A}) = \Pi_{i=1}^{\infty}\lambda(\mathcal{A}_i) = 0$. Thus the class of sets $\mathcal{A} \in \mathcal{B}[\mathbb{R}_I^\infty]$ for which $0 < \lambda_\infty(\mathcal{A}) < \infty$ is relatively small. It follows that the sets of measure zero need not be small nor sets of infinite measure be large.

1.1. Measurable function

We discuss about measurable function on $\mathbb{R}_I^\infty$ as follows:
Let $x = (x_1, x_2, \ldots) \in \mathbb{R}_I^\infty$, $I_n = \Pi_{k=n+1}^{\infty}[\frac{-1}{2}, \frac{1}{2}]$ and let $h_n(\widehat{x}) = \chi_{I_n}(\widehat{x})$, where $\widehat{x} = (x_i)_{i=n+1}^{\infty}$.

Definition 1.3. [10] Let M^n be represented the class of measurable functions on $\mathbb{R}^n$. If $x \in \mathbb{R}_I^\infty$ and $f^n \in M^n$. Let $\overline{x} = (x_i)_{i=1}^n$ and define an essentially tame measurable function of order n (or e_n-tame) on $\mathbb{R}_I^\infty$ by

$$f(x) = f^n(\overline{x}) \otimes h_n(\widehat{x}).$$

We let $M_I^n = \{f(x) \ : f(x) = f^n(\overline{x}) \otimes h_n(\widehat{x}),\ x \in \mathbb{R}_I^\infty\}$ be the class of all e_n- tame functions.

Definition 1.4. A function $f : \mathbb{R}_I^\infty \to \mathbb{R}$ is said to be measurable and we write $f \in M_I$, if there is a sequence $\{f_n \in M_I^n\}$ of e_n- tame functions, such that

$$\lim_{n\to\infty} f_n(x) \to f(x)\ \lambda_\infty - (a.e.).$$

1.2. L^1-Theory in $\mathbb{R}_I^\infty$

Let $L^1[\mathbb{R}_I^n]$ be the class of integrable functions on $\mathbb{R}_I^n$. Since $\mathbb{R}_I^n \subset \mathbb{R}_I^{n+1}$ we define $L^1[\widehat{\mathbb{R}}_I^\infty] = \bigcup_{n=1}^{\infty} L^1[\mathbb{R}_I^n]$. We say that a measurable function $f \in L^1[\mathbb{R}_I^\infty]$ if there exists a Cauchy sequence $\{f_n\} \subset L^1[\widehat{\mathbb{R}}_I^\infty]$ with $f_n \in L^1[\mathbb{R}_I^n]$ and $\lim_{n\to\infty} f_n(x) = f(x)$, λ_∞-(a.e.).

With the fact [9, Theorem 1.18] : $L^1[\widehat{\mathbb{R}}_I^\infty] = L^1[\mathbb{R}_I^\infty]$. The integral of $f \in L^1[\mathbb{R}_I^\infty]$ can be defined by

$$\int_{\mathbb{R}_I^\infty} f(x) d\lambda_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_I^\infty} f_n(x) d\lambda_\infty,$$

where $\{f_n\} \subset L^1[\mathbb{R}_I^\infty]$ is any Cauchy-sequence converges to $f(x)$-a.e. (see [9, Definition 1.19])

Let $C_c[\mathbb{R}_I^n]$ be the class of continuous function on $\mathbb{R}_I^n$ which vanish outside compact sets. We say that a measurable function $f \in C_c[\mathbb{R}_I^\infty]$, if there exists a Cauchy-sequence$\{f_n\} \subset \bigcup_{n=1}^{\infty} C_c[\mathbb{R}_I^n] = C_c[\widehat{\mathbb{R}}_I^\infty]$ such that $\lim_{n\to\infty} ||f_n - f||_\infty = 0$. We define $C_0[\mathbb{R}_I^\infty]$, the continuous functions that vanish at ∞, and $C_0^\infty[\mathbb{R}_I^\infty]$ the compactly supported smooth functions, in similar way (see [8, page 71]).

Remark 1.2. *1.* $L^1[\widehat{\mathbb{R}}_I^\infty] = L^1[\mathbb{R}_I^\infty]$.

2. $C_0^\infty[\widehat{\mathbb{R}}_I^\infty] = C_0^\infty[\mathbb{R}_I^\infty]$

3. $C_c[\mathbb{R}_I^\infty]$ *is dense in* $L^1[\mathbb{R}_I^\infty]$.

Theorem 1.2. $C_0^\infty[\mathbb{R}_I^\infty]$ *is dense in* $L^1[\mathbb{R}_I^\infty]$.

Proof. Since $C_0^\infty[\mathbb{R}_I^n] \subset L^1[\mathbb{R}_I^n]$ as dense. So $\cup C_0^\infty[\mathbb{R}_I^n] \subset \cup L^1[\mathbb{R}_I^n]$ as dense. This gives $C_0^\infty[\cup\mathbb{R}_I^n] \subset L^1[\cup\mathbb{R}_I^n]$ as dense. Now $\lim_{n\to\infty} C_0^\infty[\cup\mathbb{R}_I^n] \subset \lim_{n\to\infty} L^1[\cup\mathbb{R}_I^n]$. This implies $C_0^\infty[\lim_{n\to\infty} \cup\mathbb{R}_I^n] \subset L^1[\lim_{n\to\infty} \cup\mathbb{R}_I^n]$. So, $C_0^\infty[\widehat{\mathbb{R}}_I^\infty] \subset L^1[\widehat{\mathbb{R}}_I^\infty] = L^1[\mathbb{R}_I^\infty]$ as dense. $\square$

Remark 1.3. *In a similar fashion we can define the* $L^1_{loc}[\mathbb{R}_I^\infty]$.

1.3. L^p-Theory in $\mathbb{R}_I^\infty$

The L^p spaces are function spaces defined using a natural generalization of the p-norm for finite-dimensional vector spaces. They are sometimes called Lebesgue spaces. L^p spaces form an important class of Banach spaces in functional analysis and topological vector spaces. They have key role in the mathematical analysis of measure and probability spaces. Lebesgue spaces are also used in the theoretical discussion of problems in physics, statistics, finance, engineering, and other disciplines. We now construct the spaces $L^p[\mathbb{R}_I^\infty]$, $1 < p < \infty$, using the same approach that led to $L^1[\mathbb{R}_I^\infty]$. Since $L^p[\mathbb{R}_I^n] \subset L^p[\mathbb{R}_I^{n+1}]$, we define $L^p[\widehat{\mathbb{R}}_I^\infty] = \cup_{n=1}^\infty L^p[\mathbb{R}_I^n]$. We say that a measurable function $f \in L^p[\mathbb{R}_I^\infty]$, if there is a Cauchy-sequence $\{f_n\} \subset L^p[\widehat{\mathbb{R}}_I^\infty]$ such that $\lim_{n\to\infty} ||f_n - f||_p = 0$.

Similar to Theorem 1.2, we have that functions in $L^p[\widehat{\mathbb{R}}_I^\infty]$ differ from functions in its closure $L^p[\mathbb{R}_I^\infty]$, by sets of measure zero.

Theorem 1.3. $L^p[\widehat{\mathbb{R}}_I^\infty] = L^p[\mathbb{R}_I^\infty]$.

Definition 1.5. If $f \in L^p[\mathbb{R}_I^\infty]$, we define the integral of f by

$$\int_{\mathbb{R}_I^\infty} f(x) d\lambda_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_I^\infty} f_n(x) d\lambda_\infty(x). \tag{1.1}$$

where $\{f_n\} \subset L^p[\mathbb{R}_I^\infty]$ is any Cauchy-sequence converging to $f(x)$-a.e..

Theorem 1.4. *If* $f \in L^p[\mathbb{R}_I^\infty]$, *then above integral exists.*

Proof. The proof follows from the fact that the sequence in the Definition 1.5 is of L^p-Cauchy. $\square$

If f is a measurable function on $\mathbb{R}_I^\infty$ and $1 < p < \infty$, we define

$$||f||_p = \left[\int_{\mathbb{R}_I^\infty} |f|^p d\lambda_\infty(x)\right]^{\frac{1}{p}}.$$

Remark 1.4. *1. [9, Theorem 2.1] If* $f \in L^p[\mathbb{R}_I^\infty]$, *then the integral of (1.1) exists and all theorems that are true for* $f \in L^p[\mathbb{R}_I^n]$, *also hold for* $f \in L^p[\mathbb{R}_I^\infty]$.

2. *[8, Theorem 2.54]* $C_c[\mathbb{R}_I^\infty]$ *is dense in* $L^p[\mathbb{R}_I^\infty]$.

3. *Let* $\phi \in C_0^\infty[\mathbb{R}_I^\infty]$, $\phi \geq 0$ *and* $\int \phi(x)dx = 1$, *and define for* $\epsilon > 0$ $\phi_\epsilon(x) = \epsilon^{-1}\phi(\frac{x}{\epsilon})$. *If* $f \in L^p[\mathbb{R}_I^\infty]$ *with compact support, then* $\phi_\epsilon(x) * f$ *has compact support, is of class* $C^\infty[\mathbb{R}_I^\infty]$ *and* $\phi_\epsilon * f$ *converges to* f *in* $L^p[\mathbb{R}_I^\infty]$. *Hence,* $C_0^\infty[\mathbb{R}_I^\infty]$ *is dense in* $L^p[\mathbb{R}_I^\infty]$.

2. Meaning of $\mathbb{D}^k f(x)$ when $x \in \mathbb{R}_I^\infty$

Recalling in the set theory, for two sets A and B, $A \subset\subset B$ means that the closure of A is a relatively compact subset of B. For example:

$$(0,\infty) \subset \mathbb{R} \text{ but } (0,\infty) \not\subset\not\subset \mathbb{R} \text{ where as } (0,1) \subset \mathbb{R} \text{ and } (0,1) \subset\subset \mathbb{R}.$$

The test functions $\mathcal{D}[\mathbb{R}_I^n]$ on $\mathbb{R}_I^n$ are similar as test functions on $\mathbb{R}^n$, so ignore the detailed of the test functions on $\mathbb{R}_I^n$.

We denote test functions on $\mathbb{R}_I^\infty$ as $\mathcal{D}[\mathbb{R}_I^\infty]$, to construct this spaces we use the same approach that led to $L^1[\mathbb{R}_I^\infty]$ in subsection 1.2. Since $\mathcal{D}[\mathbb{R}_I^n] \subset \mathcal{D}[\mathbb{R}_I^{n+1}]$, we define $\mathcal{D}[\widehat{\mathbb{R}}_I^\infty] = \cup_{n=1}^\infty \mathcal{D}[\mathbb{R}_I^n]$.

Definition 2.1. We say that a measurable function $f \in \mathcal{D}[\mathbb{R}_I^\infty]$ if and only if there exists a sequence of functions $\{f_m\} \subset \mathcal{D}[\widehat{\mathbb{R}_I^\infty}] = \bigcup_{n=1}^{\infty} \mathcal{D}[\mathbb{R}_I^n]$ and a compact set $K \subset \mathbb{R}_I^\infty$, which contains the support of $f - f_m$ for all m, and $\mathbb{D}^\alpha f_m \to \mathbb{D}^\alpha f$ uniformly on K, for every multi index $\alpha \in \mathbb{N}_0^\infty$. We call the topology of $\mathcal{D}[\mathbb{R}_I^\infty]$ as the compact sequential limit topology.

Theorem 2.1. *For each* p, $1 \leq p \leq \infty$, *then test function* $\mathcal{D}[\mathbb{R}_I^n] \subset L^p[\mathbb{R}_I^n]$ *as a continuous embedding. Also the test function* $\mathcal{D}[\mathbb{R}_I^\infty] \subset L^p[\mathbb{R}_I^\infty]$ *as a continuous embedding.*

Proof. Proof is similar as the proof of the Theorem 3.47 of [8]. □

The mollifiers are used in distribution theory to create sequences of smooth functions that approximate non smooth functions via convolution. In 1938, Sergie Sobolev [17] used mollifier functions in his work to create Sobolev embedding theorem. Modern approach of mollifier was introduced by Kurt Otto Friedrichs [7] in 1944.

Definition 2.2. (Friedrichs's Definition) Mollifier identified the convolution operator as

$$\phi_\epsilon(f)(x) = \int_{\mathbb{R}^n} \varphi_\epsilon(x-y) f(y) dy$$

where $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(\frac{x}{\epsilon})$ and φ is a smooth function satisfying

1. $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^n$.
2. $\varphi(x) = \mu(|x|)$ for some infinitely differentiable function $\mu : \mathbb{R}^+ \to \mathbb{R}$.

To construct mollifier in $\mathbb{R}_I^\infty$, for each $\epsilon > 0$, let $\varphi_\epsilon \in C_0^\infty[\mathbb{R}_I^\infty]$ be given with the property

$$\varphi_\epsilon \geq 0, \ \text{supp}(\varphi_\epsilon) \subset \{x \in \mathbb{R}_I^\infty \ : |x| \leq \epsilon\}, \ \int \varphi_\epsilon = 1$$

such functions can be constructed (see page 32 [15]), for example, by taking an appropiate multiple of

$$\varphi_\epsilon(x) = \begin{cases} \exp(|x|^2 - \epsilon^2)^{-1} \ , |x| < \epsilon; \\ 0 \ , |x| \geq \epsilon \end{cases}$$

Let $f \in L^1[\mathbb{G}]$, where $\mathbb{G}$ is open in $\mathbb{R}_I^\infty$. Suppose that the support of f satisfies $\text{supp}(f) \subset\subset \mathbb{G}$ (compact support), then the distance from $\text{supp}(f)$ to $\partial\mathbb{G}$ is a positive number Δ. We extend f as zero on complement of $\mathbb{G}$ and also we denote the extension in $L^1[\mathbb{R}_I^\infty]$ by f. Define for each ϵ the mollifier:

$$f_\epsilon(x) = \int_{\mathbb{R}_I^\infty} f(x-y)\varphi_\epsilon(y)d\lambda_\infty, \ x \in \mathbb{R}_I^\infty. \tag{2.1}$$

From now on we consider functions $f \in L^p[\mathbb{R}_I^\infty]$ so, $f = 0$ almost everywhere. We obtain the following lemma

Lemma 2.1. *1. For each $\epsilon > 0$, $supp(f_\epsilon) \subset supp(f) + \{y \ : |y| \leq \epsilon\}$ and $f_\epsilon \in C^\infty[\mathbb{R}_I^\infty]$.*

2. If $f \in C_0[\mathbb{G}]$, then $f_\epsilon \to f$ uniformly on $\mathbb{G}$. If $f \in L^p[\mathbb{G}]$, $1 \leq p < \infty$ then $||f_\epsilon||_{L^p[\mathbb{G}]} \leq ||f||_{L^p[\mathbb{G}]}$ and $f_\epsilon \to f$ in $L^p[\mathbb{G}]$.

Proof. For (1), the proof is similar to that of [15, Lemma 1.1].
For (2), use the fact that $L^p[\mathbb{G}]$ is dense as continuous embedding on $L^p[\mathbb{G}]$ and follow the proof of [15, Lemma 1.2]. □

Theorem 2.2. *$C_0^\infty[\mathbb{G}]$ is a dense subset of $L^2[\mathbb{G}]$ and $L^p[\mathbb{G}]$.*

Proof. Since $C_0^\infty[\mathbb{G}]$ is dense in $L^2[\mathbb{G}]$ and $L^p[\mathbb{G}]$, it follows that $C_0^\infty[\mathbb{G}$ is dense in $L^2[\mathbb{G}]$ and $L^p[\mathbb{G}]$. It follows the result. □

Definition 2.3. A distribution on $\mathbb{G}$ is a conjugate linear functional on $C_0^\infty[\mathbb{G}]$, that is $C_0^\infty[\mathbb{G}]^*$ is the linear space of distributions on $\mathbb{G}$.

Example 2.1. *The space $L^1_{loc}[\mathbb{G}] = \bigcap\{L^1[K] \ : K \subset\subset \mathbb{G}\}$ of locally integrable functions on $\mathbb{G}$ can be identified with a subspace of distributions on $\mathbb{G}$. That is , $f \in L^1_{loc}[\mathbb{G}]$ is assigned the distribution $T_f \in C_0^\infty[\mathbb{G}]^*$ defined by*

$$T_f(\varphi) = \int_{\mathbb{G}} f\varphi^c, \quad \varphi \in C_0^\infty[\mathbb{G}]$$

where the Lebesgue integral over the support of φ is used.

Remark 2.1. *We can find from the theorem (2.2) that* $T : L^1_{loc}[\mathbb{G}] \to C_0^\infty[\mathbb{G}]^*$ *is an injection. In particluar the equivalence functions in* $L^2[\mathbb{G}]$ *will be identified with a subspace of* $\mathcal{D}^*[\mathbb{G}]$.

Let $\alpha = (\alpha_1, \alpha_2, ..,)$ be multi-index of non negative integers with $|\alpha| = \sum_{k=1}^{\infty} \alpha_k$. We define the operators $\mathbb{D}_n^\alpha$ and $\mathbb{D}_{\alpha,n}$ by

$$\mathbb{D}_n^\alpha = \Pi_{k=1}^n \frac{\partial^{\alpha_k}}{\partial x^{\alpha_k}} \ \mathbb{D}_{\alpha,n} = \Pi_{k=1}^n \left(\frac{1}{2\pi i} \frac{1}{\partial x_k} \right)^{\alpha_k},$$

respectively.

Definition 2.4. 1. We say that a sequence of functions $\{f_m\} \subset C^\infty[\mathbb{R}_I^\infty]$ converges to a function $f \in C^\infty[\mathbb{R}_I^\infty]$ if and only if for all multi-indices α, $\mathbb{D}^\alpha f \in C[\mathbb{R}_I^\infty]$ and for $x \in \mathbb{R}_I^\infty$ for all $n \in \mathbb{N}$, such that

$$\lim_{m\to\infty} \sup[\sup_{\alpha} \sup_{||x|| \le \mathbb{N}} |\mathbb{D}^\alpha f(x) - \mathbb{D}^\alpha f_m(x)|] = 0.$$

2.1. Observation 1

: We say that a function $f \in C^\infty[\mathbb{R}_I^\infty]$ if and only if there exists a sequence of functions $\{f_m\} \subset C^\infty[\widehat{\mathbb{R}_I^\infty}] = \bigcup_{n=1}^{\infty} C^\infty[\mathbb{R}_I^n]$ such that for all $x \in \mathbb{R}_I^\infty$ and $n \in \mathbb{N}$,

$$\lim_{m\to\infty} \sup[\sup_{\alpha} \sup_{||x|| \le \mathbb{N}} |\mathbb{D}^\alpha f(x) - \mathbb{D}^\alpha f_m(x)|] = 0.$$

From the above we can say the set of all continuous linear functionals $T \in \mathcal{D}^*[\mathbb{R}_I^\infty]$ is called the space of distributions on $\mathbb{R}_I^\infty$. A family of distributions $\{T_i\} \subset \mathcal{D}^*[\mathbb{R}_I^\infty]$ is said to converge to $T \in \mathcal{D}^*[\mathbb{R}_I^\infty]$ if for every $\varphi \in \mathcal{D}[\mathbb{R}_I^\infty]$, the numbers $T_i(\varphi)$ converge to $T(\varphi)$.
We define derivatives of distributions in such a way that it agrees with the usual notion of derivative in those distributions which arise from continuously differentiable functions. We define $\partial^\alpha : \mathcal{D}^*[\mathbb{R}_I^\infty] \to \mathcal{D}^*[\mathbb{R}_I^\infty]$ as $\partial^\alpha(T_f) = T_{\mathbb{D}^\alpha f}$, $|\alpha| \le m$, $f \in C^m[\mathbb{R}_I^\infty]$. By integration by parts we obtain

$$T_{\mathbb{D}^\alpha f}(y) = (-1)^{|\alpha|} T_f(\mathbb{D}^\alpha \varphi), \quad \varphi \in C_0^\infty[\mathbb{R}_I^\infty]$$

and this identity suggest the following definition:
The α^{th} partial derivative of the distribution T is the distribution $\partial^\alpha T$ defined by

$$\partial^\alpha T(\varphi) = (-1)^{|\alpha|} T(\mathbb{D}^\alpha \varphi), \quad \varphi \in C_0^\infty[\mathbb{R}_I^\infty].$$

Since $\mathbb{D}^\alpha \in L(C_0^\infty[\mathbb{R}_I^\infty], C_0^\infty[\mathbb{R}_I^\infty])$, it follows that $\partial^\alpha T$ is linear. Every distribution has derivatives of all orders and so every function. For distribution theory one can see [1, 5, 6] and references therein.

Example 2.2. *It is clear that the derivatives ∂^α and $\mathbb{D}^\alpha$ are compatible with identifications of $C^\infty[\mathbb{R}_I^\infty]$ in $\mathcal{D}^*[\mathbb{R}_I^\infty]$. For example:*

1. *If $f \in C^1[\mathbb{R}_I^\infty]$ then*

$$\partial f(\varphi) = -f(\mathbb{D}\varphi) = -\int f(\mathbb{D}\varphi^c) = \int (\mathbb{D}f)\varphi^c = \mathbb{D}f(\varphi)$$

where the equality follows by integration by parts. In particular, if $f(x) = H(x)$, where H is the Heaveside function on $\mathbb{R}_I^\infty$,

$$H(x) = \begin{cases} 1 \text{ for } x_i \geq 0; \\ 0 \text{ for } x_i < 0,\ i \in \mathbb{N} \end{cases}$$

for $x = (x_1, x_2, x_3, \ldots) \in \mathbb{R}_I^\infty$, then

$$\begin{aligned} \int_{\mathbb{R}_I^\infty} \mathbb{D}H(x)\varphi(x)d\lambda_\infty(x) &= \int_{\mathbb{R}_I^\infty} H(x)\mathbb{D}\varphi(x)d\lambda_\infty(x) \\ &= \varphi(0) \\ &= \int_{\mathbb{R}_I^\infty} \partial_{\mathbb{R}_I^\infty}(x)\varphi(x)d\lambda_\infty(x). \end{aligned}$$

That is, in the generalized sense of distributions, $\mathbb{D}H(x) = \partial(x)$ the Dirac delta function on $\mathbb{R}_I^\infty$

2. *Let $f : \mathbb{R}_I^\infty \to K$ be satisfy $f_{|\mathbb{R}_I^\infty -} \in C^\infty(-\infty, 0]$ and $f_{|\mathbb{R}_I^\infty +} \in C^\infty[0, \infty)$ and denote the jump in the various derivatives at 0 by*

$$\sigma_m(f) = \mathbb{D}^m f(0^+) - \mathbb{D}^m f(0^-), \quad m \geq 0.$$

Then we obtain

$$\partial f(\varphi) = \mathbb{D}f(\varphi) + \sigma_0(f)\partial(\varphi), \quad \varphi \in C_0^\infty[\mathbb{R}_I^\infty].$$

That is $\partial f = \mathbb{D}f + \sigma_0(f)\delta$, we can compute derivatives of higher order as :
$\partial^2 f = \mathbb{D}^2 f + \sigma_1(f)\delta + \sigma_0(f)\partial\delta$
$\partial^3 f = \mathbb{D}^3 f + \sigma_2(f)\delta + \sigma_1(f)\partial\delta + \sigma_0(f)\partial^2\delta$
eg $\partial(H \cdot \sin) = H \cdot \cos$
$\partial(H \cdot \cos) = -H \cdot \sin + \delta$. *So,* $H \cdot \sin$ *is a generalized solution of the ODE* $(\delta^2 + 1)y = \delta$.

Definition 2.5. If α is a multi-index and $u, v \in L^1_{loc}[\mathbb{R}_I^\infty]$, we say that v is the α^{th} weak (or distributional) partial derivative of u and write $\mathbb{D}^\alpha u = v$ provided that

$$\int_{\mathbb{R}_I^\infty} u(\mathbb{D}^\alpha\varphi)d\lambda_\infty = (-1)^{|\alpha|}\int_{\mathbb{R}_I^\infty} \varphi v d\lambda_\infty$$

for all functions $\varphi \in C_c^\infty[\mathbb{R}_I^\infty]$. Thus v is in the dual space $\mathcal{D}^*[\mathbb{R}_I^\infty]$ of $\mathcal{D}[\mathbb{R}_I^\infty]$.

If $u \in L^1_{loc}[\mathbb{R}_I^\infty]$ and $\varphi \in \mathcal{D}[\mathbb{R}_I^\infty]$ then we can define $T_u(\cdot)$ by

$$T_u(\varphi) = \int_{\mathbb{R}_I^\infty} u\varphi d\lambda_\infty.$$

This is a linear functional on $\mathcal{D}[\mathbb{R}_I^\infty]$. If $\{\varphi_n\} \subset \mathcal{D}[\mathbb{R}_I^\infty]$ and $\varphi_n \to \varphi$ in $\mathcal{D}[\mathbb{R}_I^\infty]$, with the support of $\varphi_n - \varphi$ contained in a compact set $K \subset \mathbb{R}_I^\infty$, then we have

$$\begin{aligned} |T_u(\varphi_n) - T_u(\varphi)| &= \left| \int_{\mathbb{R}_I^\infty} u(x)[\varphi_n(x) - \varphi(x)] d\lambda_\infty(x) \right| \\ &\leq \sup_{x \in K} |\varphi_n(x) - \varphi(x)| \int_{\mathbb{R}_I^\infty} |u(x)| d\lambda_\infty(x). \end{aligned}$$

By uniform convergence on K, we see that T is continuous, so $T \in \mathcal{D}^*[\mathbb{R}_I^\infty]$. We assume

$$||\varphi|| = \sup_{x \in \mathbb{R}_I^\infty} \{|\mathbb{D}^\alpha \varphi(x)| \ : \alpha \in \mathbb{N}_0^\infty, \ |\alpha| \leq N\}.$$

Theorem 2.3. *Let $\mathcal{D}^*[\mathbb{R}_I^\infty]$ be the dual space of $\mathcal{D}[\mathbb{R}_I^\infty]$.*

1. *Every differentiable operator D^α, $\alpha \in \mathbb{N}_0^\infty$ defines a bounded linear operator on $\mathcal{D}[\mathbb{R}_I^\infty]$.*
2. *If $T \in \mathcal{D}^*[\mathbb{R}_I^\infty]$ and $\alpha \in \mathbb{N}_0^\infty$, then $D^\alpha T \in \mathcal{D}^*[\mathbb{R}_I^\infty]$ and*
$$(\mathbb{D}^\alpha T)(\varphi) = (-1)^{|\alpha|} T(\mathbb{D}^\alpha \varphi), \quad \varphi \in \mathcal{D}[\mathbb{R}_I^\infty].$$
3. *If $|T(\varphi)| \leq c||\varphi||_N$ for all $\varphi \in \mathcal{D}[K]$, for some compact set $K \subset \mathbb{R}_I^\infty$, then $|(\mathbb{D}^\alpha T)(\varphi)| \leq c||\varphi||_{N+|\varphi|}$ and $\mathbb{D}^\alpha \mathbb{D}^\beta T = \mathbb{D}^\beta \mathbb{D}^\alpha T$.*
4. *If $g = \mathbb{D}^\alpha f$ exists as a classical derivative and $g \in L^1_{loc}[\mathbb{R}_I^\infty]$, then $T_g \in \mathcal{D}^*[\mathbb{R}_I^\infty]$ and*
$$(-1)^{|\alpha|} \int_{\mathbb{R}_I^\infty} f(x)(\mathbb{D}^\alpha \varphi) d\lambda_\infty(x) = \int_{\mathbb{R}_I^\infty} g(x)\varphi(x) d\lambda_\infty(x)$$
for all $\varphi \in \mathcal{D}[\mathbb{R}_I^\infty]$.
5. *If $f \in C^\infty[\mathbb{R}_I^\infty]$ and $T \in \mathcal{D}^*[\mathbb{R}_I^\infty]$ then $fT \in \mathcal{D}^*[\mathbb{R}_I^\infty]$, with $fT(\varphi) = T(f\varphi)$ for all $\varphi \in \mathcal{D}[\mathbb{R}_I^\infty]$ and $\mathbb{D}^\alpha(fT) = \sum_{\beta \leq \alpha} C_{\alpha\beta}(\mathbb{D}^{\alpha-\beta} f)(\mathbb{D}^\beta T)$.*

Proof. The proofs are similar to those of $\mathbb{R}^n$. □

The weak and strong derivative for $L^p[\mathbb{R}_I^n]$ can be defined like the weak and strong derivative for $L^p[\mathbb{R}^n]$. For theory of the weak derivative and strong derivative for $L^p[\mathbb{R}^n]$ we follow the definition 29.15 of [2].

Theorem 2.4. *Strong differentiable implies weak differentiable in $L^p[\mathbb{R}_I^n]$.*

Proof. The proof is similar as (4) ⇒ (2) of [2, Theorem 29.18] those of $L^p[\mathbb{R}^n]$. □

We state the weak and strong derivative for $L^p[\mathbb{R}_I^\infty]$ as:

Definition 2.6. Let $v \in \mathbb{R}_I^\infty$ and $f \in L^p[\mathbb{R}_I^\infty]$ $(f \in L^1_{loc}[\mathbb{R}_I^\infty])$, then $\partial_v^w f$ is said to exists weakly in $L^p[\mathbb{R}_I^\infty](L^1_{loc}[\mathbb{R}_I^\infty])$ if there exists a function $g \in L^p[\mathbb{R}_I^\infty](g \in L^1_{loc}[\mathbb{R}_I^\infty])$ such that

$$< f, \partial_v \varphi >= - < g, \varphi >, \ \forall \varphi \in C_c^\infty[\mathbb{R}_I^\infty].$$

In this case $\partial_v^w f = g$.

Definition 2.7.

1. For $v \in \mathbb{R}_I^\infty$, $h \in \mathbb{R} - \{0\}$ and a function $f : \mathbb{R}_I^\infty \to C$, let

$$\partial_{v^h} f(x) = \frac{f(x+hv) - f(x)}{h}$$

for those $x \in \mathbb{R}_I^\infty$ such that $x + hv \in \mathbb{R}_I^\infty$. When v is one of the standard basis elements e_i, for $1 \leq i \leq d$, we will write $\partial_i^h f(x)$ rather than $\partial_{e_i^h} f(x)$.

2. Let $v \in \mathbb{R}_I^\infty$ and $f \in L^p[\mathbb{R}_I^\infty]$, then it is said that $\partial_v^s f$ exists strongly in $L^p[\mathbb{R}_I^\infty]$, if $\lim\limits_{h\to 0} \partial_v^h f$ exists in $L^p[\mathbb{R}_I^\infty]$. In this case $\partial_v^s f = \lim\limits_{h\to 0} \partial_{v^h} f$.

Observation 2

Fix $n \in \mathbb{N}$ and let $\widehat{\mathbb{Q}}_I^\infty = \lim\limits_{n\to\infty} \mathbb{Q}_I^n = \bigcup\limits_{k=1}^{\infty} \mathbb{Q}_I^k$, where $\mathbb{Q}_I^n$ is the set $\{x \in \mathbb{R}_I^n :$ the coordinates of x are rational$\}$. Since this is a countable dense set in $\mathbb{R}_I^n$, we can arrange it as $\mathbb{Q}_I^n = \{x_1, x_2, ...\}$. For each k and i, let $\mathcal{B}_k(x_i)$ be a closed cube in $\mathbb{R}^n$ centered at x_i with sides parallel to the coordinate axes and edge $e_k = \frac{1}{2^k\sqrt{n}}$. Now choose the natural order which maps $\mathbb{N} \times \mathbb{N}$ bijectively to $\mathbb{N}$, and let $\{\mathcal{B}_k : k \in \mathbb{N}\}$ be the resulting set of (all) closed cubes

$$\{\mathcal{B}_k(x_i) | \ (k,i) \in \mathbb{N} \times \mathbb{N}\}$$

centered at a point in $\mathbb{Q}_I^n$. Let $\zeta_k(x)$ be the characteristic function of $\mathcal{B}_k$, so that $\zeta_k(x) \in L^p[\mathbb{R}_I^\infty] \cap L^\infty[\mathbb{R}_I^\infty]$ for $1 \leq p < \infty$.

Remark 2.2. *Any function in $L^\infty[\mathbb{R}_I^n]$ is weakly derivable in $L^p[\mathbb{R}_I^n]$ so*

$$\zeta_r(f) \in L^p[\mathbb{R}_I^n] \cap L^\infty[\mathbb{R}_I^n]$$

is also in the sense of weak if we consider in weak derivative.

Lemma 2.2. *Suppose $f \in L^1_{loc}[\mathbb{R}_I^\infty]$ and $\partial_v f$ exists weakly in $L^1_{loc}[\mathbb{R}_I^\infty]$. Then $supp_m(\partial_v f) \subset supp_m(f)$, where $supp_m(f)$ is essential support of f relative to Lebesgue measure.*

3. Sobolev spaces over $\mathbb{R}_I^n$

The function $f(x) = |x|$ is weak derivable in $L^p(\mathbb{R}_I^n)$ which is not strongly derivable in $L^p(\mathbb{R}_I^n)$. This type of functions motivate us to think in a space like Sobolev for $L^p(\mathbb{R}_I^n)$ and $L^p(\mathbb{R}_I^\infty)$.
In the one dimensional case the Sobolev space $S^{k,p}[\mathbb{R}]$ for $1 \le p \le \infty$ is defined as the subset of functions f in $L^p[\mathbb{R}]$ such that f and its weak derivatives upto order k have a finite L^p norm.
In one dimensional problem it is enough to assume that $f^{(k-1)}$, the $(k-1)th$ derivative of the function f is differentiable almost every where. That is

$$S^{k,p}[\mathbb{R}] = \{f(x) : \ \mathbb{D}^k f(x) \in L^p[\mathbb{R}]\}.$$

For multi-dimensional case the transition to multiple dimensions entails more difficulties, starting with the definition itself. The requirement that $f^{(k-1)}$ be the integral of $f^{(k)}$ does not generalize, and the simplest solution is to consider derivatives in the sense of distribution theory.
A formal definition we now state as: Let $k \in \mathbb{N}$, $\ 1 \le p \le \infty$. The Sobolev space $S^{k,p}[\mathbb{R}_I^n]$ is defined as the set of all functions f on $\mathbb{R}_I^n$ such that for every multi-index α with $|\alpha| \le k$, the mixed partial derivative

$$f^{(\alpha)} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots . \partial x_n^{\alpha_n}}$$

exists in the weak sense in $L^p[\mathbb{R}_I^n]$ that is $||f^{(\alpha)}||_{L^p} < \infty$.
Therefore the Sobolev space $S^{k,p}[\mathbb{R}_I^n]$ is the space

$$S^{k,p}[\mathbb{R}_I^n] = \{f \in L^p[\mathbb{R}_I^n] : \quad \mathbb{D}^\alpha f \in L^p[\mathbb{R}_I^n], \ \forall |\alpha| \le k\}.$$

We called k as the order of the Sobolev space $S^{k,p}[\mathbb{R}_I^n]$. We define a norm for $S^{k,p}[\mathbb{R}_I^n]$ as:

$$||f||_{WS^{k,p}[\mathbb{R}_I^n]} = \begin{cases} \left(\sum\limits_{|\alpha| \le k} ||\mathbb{D}^\alpha f||_{L^p}^p \right)^{\frac{1}{p}}, & \text{for } 1 \le p < \infty; \\ \max\limits_{|\alpha| \le k} ||\mathbb{D}^k f||_{L^\infty} & , \text{ for } p = \infty \end{cases}$$

For $k = 1$

$$||f||_{S^{1,p}[\mathbb{R}_I^n]} = \left(||f||_{L^p[\mathbb{R}_I^n]}^p + ||\mathbb{D}f||_{L^p[\mathbb{R}_I^n]} \right)^{\frac{1}{p}}$$

and

$$||f||_{S^{1,\infty}[\mathbb{R}_I^n]} = \sup_{r \ge 1} \left| \int_{\mathbb{R}_I^n} f(x) d\lambda_\infty(x) \right| + \sup_{r \ge 1} \left| \int_{\mathbb{R}_I^n} \mathbb{D}f(x) d\lambda_\infty(x) \right|.$$

We can consider equivalent norms

$$||f||_{S^{1,p}[\mathbb{R}_I^n]} = \left(||f||_{L^p[\mathbb{R}_I^n]}^p + \sum_{j=1}^n ||\mathbb{D}_j f||_{L^p[\mathbb{R}_I^n]}^p \right)^{\frac{1}{p}},$$

$$||f||_{S^{1,p}[\mathbb{R}_I^n]} \quad = \quad ||f||_{L^p[\mathbb{R}_I^n]} + \sum_{j=1}^{n} ||\mathbb{D}_j f||_{L^p[\mathbb{R}_I^n]}$$

when $1 \le p < \infty$ and

$$||f||_{S^{1,\infty}} = \max\{||f||_{L^\infty[\mathbb{R}_I^n]}, ||\mathbb{D}f||_{L^\infty[\mathbb{R}_I^n]}, \ldots, ||\mathbb{D}_n f||_{L^\infty[\mathbb{R}_I^n]}\}.$$

3.1. Completeness of Sobolev Spaces

A sequence (f_i) of functions $f_i \in S^{k,p}[\mathbb{R}_I^n]$ $i = 1, 2, ...$ converges to a function $f \in S^{k,p}[\mathbb{R}_I^n]$ if for every $\epsilon > 0$ there exists i_ϵ such that

$$||f_i - f||_{S^{k,p}[\mathbb{R}_I^n]} < \epsilon \text{ when } i \ge i_\epsilon.$$

Equivalently

$$\lim_{i\to\infty} ||f_i - f||_{S^{k,p}[\mathbb{R}_I^n]} = 0$$

A sequence (f_i) is a Cauchy sequence in $S^{k,p}[\mathbb{R}_I^n]$ if for every $\epsilon > 0$ there exists i_ϵ such that

$$||f_i - f_j||_{S^{k,p}[\mathbb{R}_I^n]} < \epsilon \text{ when } i, j \ge i_\epsilon.$$

Theorem 3.1. *$S^{k,p}[\mathbb{R}_I^n]$ is Banach space.*

Proof. First we prove $||.||_{S^{k,p}[\mathbb{R}_I^n]}$ is a norm.

1. $||f||_{S^{k,p}[\mathbb{R}_I^n]} = 0 \quad \Leftrightarrow f = 0$ a.e. in $\mathbb{R}_I^n$.
   $||f||_{S^{k,p}[\mathbb{R}_I^n]} = 0 \Rightarrow ||f||_{L^p[\mathbb{R}_I^n]} = 0$ which implies $f = 0$ a.e. in $\mathbb{R}_I^n$.
   Now $f = 0$ a.e. in $\mathbb{R}_I^n$, implies

   $$\int_{\mathbb{R}_I^n} \mathbb{D}^\alpha f \varphi d\lambda_\infty = (-1)^{|\alpha|} \int_{\mathbb{R}_I^n} f \mathbb{D}^\alpha \varphi d\lambda_\infty = 0 \text{ for all } \varphi \in C_0^\infty[\mathbb{R}_I^n].$$

   As $f \in L^1_{loc}[\mathbb{R}_I^n]$ satisfies $\int_{\mathbb{R}_I^n} f \varphi d\lambda_\infty = 0$ for every $\varphi \in C_0^\infty[\mathbb{R}_I^n]$ then $f = 0$ a.e. in $\mathbb{R}_I^n$. This implies $\mathbb{D}^\alpha f = 0$ a.e. in $\mathbb{R}_I^n$ for all α, $|\alpha| \le k$.

2. $||\alpha f||_{S^{k,p}[\mathbb{R}_I^n]} = |\alpha| ||f||_{S^{k,p}[\mathbb{R}_I^n]}$, $\alpha \in \mathbb{R}$.

3. The triangle inequality for $1 \le p < \infty$ follows from elementary inequality $(a + b)^\alpha \le a^\alpha + b^\alpha$, $0 < \alpha \le 1$ and Minkowski's inequality.

Now, let (f_i) be Cauchy sequence in $S^{k,p}[\mathbb{R}_I^n]$, since

$$||\mathbb{D}^\alpha f_i - \mathbb{D}^\alpha f_j||_{L^p[\mathbb{R}_I^n]} \le ||f_i - f_j||_{S^{k,p}[\mathbb{R}_I^n]} \ , |\alpha| \le k$$

it follows that $(\mathbb{D}^\alpha f_i)$ is Cauchy in $L^p[\mathbb{R}_I^n]$, $|\alpha| \le k$, next follow the completeness of $L^p[\mathbb{R}_I^n]$ implies that there exists $f_\alpha \in L^p[\mathbb{R}_I^n]$ such that $\mathbb{D}^\alpha f_i \to f_\alpha$ in $L^p[\mathbb{R}_I^n]$. □

Remark 3.1. *Sobolev space is a vector space of functions equipped with a norm that is a combination of L^p-norms of function together with its derivatives upto a given order.*

Theorem 3.2. $S^{k,p}[\mathbb{R}_I^n]$, $1 \leq p < \infty$ *is separable, however* $S^{1,\infty}[\mathbb{R}_I^n]$ *is not separable.*

Proof. In the case $k = 1$ consider the mapping $S^{1,p}[\mathbb{R}_I^n]$ to $L^p[\mathbb{R}_I^n] \times L^p[\mathbb{R}_I^n]$. The product space $L^p[\mathbb{R}_I^n] \times L^p[\mathbb{R}_I^n]$ is separable. From the [4, proposition 3.25] $T(S^{1,p})$ is also separable. Consequently $S^{1,p}$ is separable.
Let $\Omega = \Omega^{'} \times (0,1)$, $\Omega^{'} \subset \mathbb{R}_I^{N-1}$ is bounded. For $0 < z < 1$ choose $r_z > 0$ such that $I_z = (z - r_z, z + r_z) \subset (0,1)$ and

$$F_z(x^{'}, x_N) = \int_0^{x_N} \int_0^{t_1} .... \int_0^{t_{k-1}} \chi_{I_z} ds....dt_{k-1}$$

where $x = (x^{'}, x_N) \in \Omega = \Omega^{'} \times (0.1)$. Then $F_z \in S^{k,\infty}[\Omega]$ and the set $(U_z)_{z\in I}$ is uncountable, pairwise disjoint, open and non empty subset of $S^{k,\infty}[\Omega]$ where

$$U_z = \big\{ f \in S^{k,\infty}[\Omega] : \ ||f - F_z||_{S^{k,\infty}} < \frac{1}{2} \big\}.$$

This means, $f \in U_{z_1} \cap U_{z_2}$ implies $||F_{z_1} - F_{z_2}||_{S^{k,\infty}[\Omega]} < 1$. So, $||\partial_N^k (F_{z_1} - F_{z_2}||_{L^\infty[\Omega]} < 1$. Hence, $||\chi_{I_{z_1}} - \chi_{I_{z_2}}||_{L^\infty[0,1]} < 1$ implies $z_1 = z_2$. Therefore, $S^{k,\infty}[\Omega]$ is not separable. □

The space $S^{k,2}[\mathbb{R}_I^n]$ is a Hilbert space with the inner product

$$< f, g >_{S^{k,2}[\mathbb{R}_I^n]} = \sum_{|\alpha| \leq k} < \mathbb{D}^\alpha f, \mathbb{D}^\alpha g >_{L^2[\mathbb{R}_I^n]},$$

where

$$< \mathbb{D}^\alpha f, \mathbb{D}^\alpha g >_{L^2[\mathbb{R}_I^n]} = \int_{\mathbb{R}_I^n} \mathbb{D}^\alpha f \mathbb{D}^\alpha g d\lambda_\infty(x).$$

Observe that $||f||_{S^{k,2}[\mathbb{R}_I^n]} =< f, f >^{\frac{1}{2}}_{S^{k,2}[\mathbb{R}_I^n]}$.

Theorem 3.3. *For* $1 \leq p < \infty$, *we have*

1. *If* $1 < p < \infty$ *then* $S^{k,p}[\mathbb{R}_I^n]$ *is uniformly convex.*
2. *If* $1 < p < \infty$ *then* $S^{k,p}[\mathbb{R}_I^n]$ *is reflexive.*
3. $S^{k,\infty}[\mathbb{R}_I^n] \subset S^{k,p}[\mathbb{R}_I^n]$ *for* $1 \leq p < \infty$.

Proof. (1) Let $T : S^{k,p}[\mathbb{R}_I^n] \to L^p[\mathbb{R}_I^n]$, defined as $x \to (\mathbb{D}^\alpha x)_{|\alpha|\leq k}$, be a closed and isometric embedding. Since $L^p[\mathbb{R}_I^n]$ is uniformly convex for $1 < p < \infty$, so is any closed subspace, and hence as $S^{k,p}[\mathbb{R}_I^n]$ is isometric to its image under T, it follows that $S^{k,p}[\mathbb{R}_I^n]$ is uniformly convex for these p.
(2) Follows from part (1).
(3) Let $f \in S^{k,\infty}[\mathbb{R}_I^n]$. This implies that $|\int_{\mathbb{R}_I^n} \mathbb{D}^\alpha f(x) d\lambda_\infty(x)|$ is uniformly bounded for all n. Then $|\int_{\mathbb{R}_I^n} \mathbb{D}^\alpha f(x) d\lambda_\infty(x)|^p$ is uniformly bounded for each $p,\ 1 \leq p < \infty$. So, it is clear

$$\left[\left|\int_{\mathbb{R}_I^n} \mathbb{D}^\alpha f(x) d\lambda_\infty(x)\right|^p\right]^{\frac{1}{p}} < \infty.$$

Therefore, $f \in S^{k,p}[\mathbb{R}_I^n]$. □

Theorem 3.4. $S^{1,p}[\mathbb{R}_I^n] \to L^p[\mathbb{R}_I^n]$ *as continuous embedding for* $1 \leq p < \infty$.

Proof. As [18], we have $S^{1,p}[\mathbb{R}_I^n] \to L^q[\mathbb{R}_I^n]$ for $1 \leqslant p \leq q < \infty$. Also, $L^q[\mathbb{R}_I^n] \subset L^p[\mathbb{R}_I^n]$ as continuous dense for $1 \leq p < \infty$. So, $S^{1,p}[\mathbb{R}_I^n] \to L^p[\mathbb{R}_I^n]$ for $1 \leq p < \infty$. We need to prove $S^{1,p}[\mathbb{R}_I^n] \to L^p[\mathbb{R}_I^n]$. For this we find

$$||f||_{L^p[\mathbb{R}_I^n]} \leq ||f||_{S^{1,p}[\mathbb{R}_I^n]}$$

for $f \in S^{1,p}[\mathbb{R}_I^n]$, which gives our result. □

3.2. Sobolev Spaces on $\mathbb{R}^\infty$

In this section we will discuss $S^{k,p}[\mathbb{R}_I^\infty]$. As $S^{k,p}[\mathbb{R}_I^n] \subset S^{k,p}[\mathbb{R}_I^{n+1}]$, we can define

$$S^{k,p}[\widehat{\mathbb{R}_I^\infty}] = \bigcup_{n=1}^{\infty} S^{k,p}[\mathbb{R}_I^n].$$

Definition 3.1. We say that for $1 \leq p < \infty$, a measurable function $f \in S^{k,p}[\mathbb{R}_I^\infty]$, if there exists a Cauchy sequence $\{f_n\} \subset S^{k,p}[\widehat{\mathbb{R}_I^\infty}]$ with $f_n \in S^{k,p}[\mathbb{R}_I^n]$ and

$$\lim_{n\to\infty} \mathbb{D}^\alpha f_n(x) = \mathbb{D}^\alpha f(x), \quad \lambda_\infty - a.e.$$

Definition 3.2. Let $f \in S^{k,p}[\mathbb{R}_I^\infty]$, we define the integral by

$$\int_{\mathbb{R}_I^\infty} \mathbb{D}^\alpha f(x) d\lambda_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_I^\infty} \mathbb{D}^\alpha f_n(x) d\lambda_\infty(x)$$

where $f_n \in S^{k,p}[\mathbb{R}_I^\infty]$ for all n and the family $\{f_n\}$ is a Cauchy sequence.

Theorem 3.5. $S^{k,p}[\widehat{\mathbb{R}_I^\infty}] = S^{k,p}[\mathbb{R}_I^\infty]$.

We define Sobolev space $S^{k,p}[\mathbb{R}_I^\infty]$ as

$$S^{k,p}[\mathbb{R}_I^\infty] = \{f \in L^p[\mathbb{R}_I^\infty] : \ \mathbb{D}^\alpha f \in L^p[\mathbb{R}_I^\infty], \ \forall |\alpha| \le k\}.$$

We called k as the order of the Sobolev space $S^{k,p}[\mathbb{R}_I^\infty]$. We define a norm for $S^{k,p}[\mathbb{R}_I^\infty]$ as:

$$||f||_{S^{k,p}[\mathbb{R}_I^\infty]} = \begin{cases} \left(\sum_{|\alpha| \le k} ||\mathbb{D}^\alpha f||_{L^p}^p \right)^{\frac{1}{p}}, & \text{for } 1 \le p < \infty; \\ \max_{|\alpha| \le k} ||\mathbb{D}^k f||_{L^\infty}, & \text{for } p = \infty \end{cases}$$

For $k = 1$

$$||f||_{S^{1,p}[\mathbb{R}_I^\infty]} = \left(||f||_{L^p[\mathbb{R}_I^\infty]}^p + ||\mathbb{D} f||_{L^p[\mathbb{R}_I^\infty]} \right)^{\frac{1}{p}}$$

and

$$||f||_{S^{1,\infty}[\mathbb{R}_I^\infty]} = \sup_{r \ge 1} \left| \int_{\mathbb{R}_I^\infty} f(x) d\lambda_\infty(x) \right| + \sup_{r \ge 1} \left| \int_{\mathbb{R}_I^\infty} \mathbb{D} f(x) d\lambda_\infty(x) \right|.$$

We can consider equivalent norms

$$||f||_{S^{1,p}[\mathbb{R}_I^\infty]} = \left(||f||_{L^p[\mathbb{R}_I^\infty]}^p + \sum_{j=1}^n ||\mathbb{D}_j f||_{L^p[\mathbb{R}_I^\infty]}^p \right)^{\frac{1}{p}},$$

$$||f||_{S^{1,p}[\mathbb{R}_I^\infty]} = ||f||_{L^p[\mathbb{R}_I^\infty]} + \sum_{j=1}^n ||\mathbb{D}_j f||_{L^p[\mathbb{R}_I^\infty]}$$

when $1 \le p < \infty$ and

$$||f||_{S^{1,\infty}} = \max\{||f||_{L^\infty[\mathbb{R}_I^\infty]}, ||\mathbb{D} f||_{L^\infty[\mathbb{R}_I^\infty]}, .., ||\mathbb{D}_n f||_{L^\infty[\mathbb{R}_I^\infty]}\}.$$

Remark 3.2. *The remark 3.1 follows that the functions of $S^{k,p}[\mathbb{R}_I^\infty]$ are equal almost everywhere.*

Theorem 3.6. *Let $f \in S^{k,p}[\mathbb{R}_I^\infty]$, then*

$$\int_{\mathbb{R}_I^\infty} \mathbb{D}^\alpha f(x) d\lambda_\infty(x) = \lim_{n \to \infty} \int_{\mathbb{R}_I^\infty} \mathbb{D}^\alpha f_n(x) d\lambda_\infty(x)$$

where $f_n \in S^{k,p}[\mathbb{R}_I^\infty]$ for all n and the family $\{f_n\}$ is a Cauchy sequence.

Proof. Since the family of functions $\{f_n\}$ is Cauchy, it follows that if the integral exists, it is unique. To prove the existence, follow the standard argument and first assume that $f(x) \ge 0$. In this case, the sequence can always be chosen to be increasing, so that the integral exists. The general case now follows by the standard decomposition. $\square$

Theorem 3.7. *For $1 \leq p < \infty$, we have*

1. *If $1 < p < \infty$ then $S^{k,p}[\mathbb{R}_I^\infty]$ is uniformly convex.*
2. *If $1 < p < \infty$ then $S^{k,p}[\mathbb{R}_I^\infty]$ is reflexive.*
3. *$S^{k,\infty}[\mathbb{R}_I^\infty] \subset S^{k,p}[\mathbb{R}_I^\infty]$ for $1 \leq p < \infty$.*

Proof. (1) As $S^{k,p}[\mathbb{R}_I^n]$ is uniformly convex for each n and that is dense and compactly embedded in $S^{k,p}[\mathbb{R}_I^\infty]$ for all $p,\ 1 \leq p \leq \infty$. So, $\bigcup\limits_{n=1}^{\infty} S^{k,p}[\mathbb{R}_I^n]$ is uniformly convex for each n and that is dense and compactly embedded in $\bigcup\limits_{n=1}^{\infty} S^{k,p}[\mathbb{R}_I^\infty]$ for all $p,\ 1 \leq p \leq \infty$.
However $S^{k,p}[\widehat{\mathbb{R}_I^\infty}] = \bigcup\limits_{n=1}^{\infty} S^{k,p}[\mathbb{R}_I^n]$. That is $S^{k,p}[\widehat{\mathbb{R}_I^\infty}]$ is uniformly convex, dense and compactly embedded in $S^{k,p}[\widehat{\mathbb{R}_I^\infty}]$ for all $p,\ 1 \leq p \leq \infty$. As $S^{k,p}[\mathbb{R}_I^\infty]$ is closure of $S^{k,p}[\widehat{\mathbb{R}_I^\infty}]$. Therefore $S^{k,p}[\mathbb{R}_I^\infty]$ is uniformly convex.
(2) From (1) we have $S^{k,p}[\mathbb{R}_I^\infty]$ is reflexive for $1 < p < \infty$.
(3) Let $f \in S^{k,p}[\mathbb{R}_I^\infty]$. This implies

$$\left| \int_{\mathbb{R}_I^\infty} \mathbb{D}^\alpha f(x) d\lambda_\infty(x) \right|$$

is uniformly bounded for all r. It follows that $\left| \int_{\mathbb{R}_I^\infty} \mathbb{D}^\alpha f(x) d\lambda_\infty(x) \right|^p$ is uniformly bounded for $1 \leq p < \infty$. It is clear from the definition of $S^{k,p}[\mathbb{R}_I^\infty]$ that

$$\left[\left| \int_{\mathbb{R}_I^\infty} \mathbb{D}^\alpha f(x) d\lambda_\infty(x) \right|^p \right]^{\frac{1}{p}} \leq M ||f||_{S^{k,p}[\mathbb{R}_I^\infty]} < \infty.$$

So, $f \in S^{k,p}[\mathbb{R}_I^\infty]$. $\square$

Theorem 3.8. $S^{1,p}[\mathbb{R}_I^\infty] \to L^p[\mathbb{R}_I^\infty]$ *as continuous embedding for $1 \leq p < \infty$.*

Proof. As $S^{1,p}[\mathbb{R}_I^n] \longrightarrow L^p[\mathbb{R}_I^n]$ as continuous embedding for $1 \leq p < \infty$. So, $\bigcup\limits_{n=1}^{\infty} S^{1,p}[\mathbb{R}_I^n] \longrightarrow \bigcup\limits_{n=1}^{\infty} L^p[\mathbb{R}_I^n]$ as continuous embedding for $1 \leq p < \infty$. Therefore $S^{1,p}[\widehat{\mathbb{R}_I^\infty}] \longrightarrow L^p[\widehat{\mathbb{R}_I^\infty}]$ for $1 \leq p < \infty$. Hence, $S^{1,p}[\mathbb{R}_I^\infty] \longrightarrow L^p[\mathbb{R}_I^\infty]$ as continuous embedding for $1 \leq p < \infty$. $\square$

4. Application on $\mathbb{R}_I^\infty$

In this section, as an application of $\mathbb{R}_I^\infty$, we will construct a Sobolev spaces on an separable Banach spaces B. Let B be a Banach space with S-basis. We can find from the definition of a Schauder basis that, for any sequence (x_n) of scalars associated with a $x \in B$, $\lim_{n\to\infty} x_n = 0$.

Recalling from [8], $J_k = \left[-\frac{1}{2In(k+1)}, \frac{1}{2In(k+1)}\right]$ and $J^n = \Pi_{k=n+1}^\infty J_k,\ J = \Pi_{k=1}^\infty J_k$. If $\{e_k\}$ be an S-basis for B and let $x = \sum_{n=1}^\infty x_n e_n$. Recalling that $P_n(x) = \sum_{k=1}^n x_n e_k$ and define $Q_n x = (x_1, x_2, .., x_n)$, we define B_J^n by

$$B_J^n = \{Q_n(x) :\ x \in B\} \times J^n$$

with norm

$$||(x_k)||_{B_J^n} = \max_{1\le k\le n} ||\sum_{i=1}^k x_i e_i|| = \max_{1\le k\le n} ||P_n(x)||_B.$$

Since $B_J^n \subset B_J^{n+1}$ we set $B_J^\infty = \bigcup_{n=1}^\infty B_J^n$. We define B_J by

$$B_J = \{(x_1, x_2, ..) :\ \sum_{k=1}^\infty x_k e_k \in B\} \subset B_J^\infty$$

and define a norm on B_J by

$$||x||_{B_J} = \sup_n ||P_n(x)||_B = |||x|||_B.$$

Let $\mathfrak{B}(B_J^\infty)$ be the smallest $\sigma-$algebra containing B_J^∞ and define $\mathfrak{B}(B_J) = \mathfrak{B}(B_J^\infty) \cap B_J$. Using the [8, Theorem 1.61] gives that,

$$|||x|||_B = \sup_n ||\sum_{k=1}^n x_k e_k||_B \tag{4.1}$$

is an equivalent norm on B. When B carries the equivalent norm (4.1), the operator $T : (B, |||.|||_B) \to (B_J, ||.||_{B_J})$ defined by $T(x) = (x_k)$ is an isometric isomorphism from B onto B_J. B_J is called canonical representation of B (see [8, page 67].

Definition 4.1. [8, Definition 2.42] Define $\overline{v}_k,\ \overline{\gamma}_k$ on $A \in \mathfrak{B}(\mathbb{R})$ by $\overline{v}_k(A) = \frac{\mu(A)}{\mu(J_k)},\ \overline{\gamma}_k(A) = \frac{\mu(A\cap J_k)}{\mu(J_k)}$ for elementary sets $A = \Pi_{k=1}^\infty B_k,\ A \in \mathfrak{B}(B_J^n)$, define $\overline{v}_J^n$ by:

$$\overline{v}_J^n(A) = \Pi_{k=1}^n \overline{v}_k(A_k) \times \Pi_{k=n+1}^\infty \overline{\gamma}_k(B_k).$$

If B is a Banach space with an $S-$basis and $A \in \mathfrak{B}_J(B)$. We define $\mu_B(A) = v_J(T(A))$.

Let ν be any probability measure on $\mathfrak{B}(\mathbb{R})$ with density f. For each $x \in B_J^n$, and each $A \in \mathfrak{B}_J(B)$, define $f_B^n(x)$ by

$$f_B^n(x) = \left(\otimes_{k=1}^n f(x_k)\right) \otimes \left(\otimes_{k=n+1}^\infty \chi_I(x_k)\right)$$

and $\overline{v}_J^n$ on $\mathfrak{B}_J(B)$ by

$$\overline{v}_J^n(A) = \int_{T(A)\cap B_J^n} f_B^n(x) d\mu_B(x)$$

where T is the isometric isomorphism between B and B_J also $\mathbb{R}_I^\infty \subseteq B_J$.

4.1. Test Functions and weak derivatives

Definition 4.2. We define the set of test functions (or C^∞-functions with compact support on B as

$$D_t(B) = \left\{\phi \in C^\infty(B) : \ supp(\phi) = \overline{\left\{x : \ \phi(x) \neq 0\right\}} \subseteq B \ \ is\ compact.\right.$$

We call $supp(\phi)$ the support of ϕ.

Lemma 4.1. *The space of test functions $D_t(B)$ is dense in $L^p(B)$ for $1 \leq p < \infty$.*

Let $\mathcal{N}_0^\alpha$ be the set of all multi-index infinite tuples $\alpha = (\alpha_1, \alpha_2, ..)$ with $\alpha_i \in \mathcal{N}$ and all but a finite number of entries are zero.
We define the operator D^α and D_α by

$$D^\alpha = \Pi_{k=1}^\infty \frac{\partial^{\alpha_k}}{\partial {x_i}^{\alpha_k}}$$

and

$$D_\alpha = \Pi_{k=1}^\infty \left(\frac{1}{2\pi i}\frac{\partial}{\partial x_k}\right)^{\alpha_k}$$

Definition 4.3. [8, Definition 2.84] If α is a multi-index and $u, v \in L^1_{loc}(B), \ v$ is the α^{th} weak partial derivative of u provided that

$$\int_B u(D^\alpha \phi) d\mu_B = (-1)^{|\alpha|} \int_B \phi v d\mu_B$$

for all functions $\phi \in C_c^\infty(B)$.

Lemma 4.2. *$C_0^\infty(B')$ is dense in $L^p(B')$.*

Proof. Taking $\phi \in C_0^\infty(B'), \ \phi \geq 0$ and $\int_{B'} \phi d\mu_B = 1$. Define $\phi_\epsilon(x) = \epsilon^{-1}\phi(\frac{x}{\epsilon})$. If $f \in L^p(B')$ with compact support then $\phi_\epsilon * f$ has compact support is of the class $C^\infty(B')$ and $\phi_\epsilon * f$ converges to f in $L^p(B')$. □

Theorem 4.1. *(Fundamental lemma of the Calculus of variations) If $f \in L^1_{loc}(B)$ satisfies $\int_B f\phi d\mu_B = 0$ for every $\phi \in C_0^\infty(B)$, then $f = 0$ a.e. in B.*

Proof. Let $v_1, \ v_2 \in L^1_{loc}(B)$ are weak αth partial derivatives of u, then

$$\begin{aligned} \int_B u D^\alpha \phi d\mu_B &= (-1)^{|\alpha|} \int_B v_1 \phi d\mu_B \\ &= (-1)^{|\alpha|} \int_B v_2 \phi d\mu_B \end{aligned}$$

for every $\phi \in C_0^\infty(B)$. We have now,

$$\int_B (v_1 - v_2)\phi d\mu_B = 0 \quad \textit{for every } \phi \in C_0^\infty(B).$$

Let B' is open and $\overline{B}'$ is a compact subset of B. Since $C_0^\infty(B')$ is dense in $L^p(B')$ then there exists a sequence of function $\phi_i \in C_0^\infty(B')$ such that $|\phi_i| \le \alpha$ in B' and $\phi_i \to sgn(v_1 - v_2)$ a.e. in B' as $i \to \infty$. Now from dominated convergence theorem, with the majorant $|v_1 - v_2)\phi_i| \le 2(|v_1| + |v_2|) \in L^1(B')$, gives

$$\begin{aligned} 0 &= \lim_{i\to\infty} \int_{B'} (v_1 - v_2)\phi_i d\mu_B \\ &= \int_{B'} \lim_{i\to\infty} (v_1 - v_2)\phi_i d\mu_B \\ &= \int_{B'} (v_1 - v_2) sgn(v_1 - v_2) d\mu_B \\ &= \int_{B'} |v_1 - v_2| d\mu_B \end{aligned}$$

This implies that $v_1 = v_2$ a.e. in B' for every $B' \Subset B$. Thus $v_1 = v_2$ a.e. in B. Consequently, if $f \in L^1_{loc}(B)$ satisfies $\int_B f\phi d\mu_B = 0$ for every $\phi \in C_0^\infty(B)$ then $f = 0$ a.e. in B. □

Definition 4.4. [8, Definition 2.87] A function $f \in C^\infty(B)$ is called a Schwartz function, or $f \in \mathbb{S}(B)$, iff, for all multi-indices α and β in $\mathcal{N}_0^\alpha$, the seminorm $\rho_{\alpha,\beta}(f)$ is finite, where

$$\rho_{\alpha,\beta}(f) = \sup_{x\in B} |x^\alpha D^\beta f(x)|$$

$\mathbb{S}(B)$ (respectively $\mathbb{S}(B')$) is a Fréchet space, which is dense in $C_0(B)$. The test function space $D_t(B)$ is subspace of $\mathbb{S}(B)$ so from the Lemma 4.1, $\mathbb{S}(B)$ is dense in $L^p(B)$.

4.2. Sobolev space on separable Banach spaces

In this sub section, we discuss Sobolev space $S^{k,2}(B)$ on separable Banach space B.

Definition 4.5. 1. The Sobolev space $S^{k,2}(B)$ consists of functions $u \in L^2(B)$ such that for every multi-index α with $|\alpha| \le k$, the weak derivative $D^\alpha u$ exists and $D^\alpha u \in L^2(B)$. Thus

$$S^{k,2}(B) = \{u \in L^2(B) : \ D^\alpha u \in L^2(B), \ |\alpha| \le k\}.$$

2. We assume the inner product on $S^{k,2}(B)$ as:

$$\langle f \mid g \rangle_{S^{k,2}} = \sum_{|\alpha|\le m} \langle D^{(\alpha)} f \mid D^{(\alpha)} g \rangle_{L^2} \tag{4.2}$$

$$H^{k,2}(\overline{B}) = \overline{C^k(\overline{B}) \cap S^{k,2}(B)},$$

where the closure is with respect to the norm induced by $< . \mid . >_{S^{k,2}}$.

3. $H_0^{k,2}(B) = \overline{D_t(B)}$ is with respect to the induced norm on $S^{k,2}$.

Theorem 4.2. *$S^{k,2}(B)$ is a Hilbert space with the inner product (4.2).*

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