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MATRIX TRANSFORMS OF SUBSPACES OF SUMMABILITY DOMAINS OF NORMAL MATRICES DETERMINED BY SPEED

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Abstract. Let X, Y be two subspaces of summability domains of matrices with real or complex entries defined by speeds of the convergence, i.e.by monotonically increasing positive sequences λ and μ. In this paper, we give necessary and sufficient conditions for a matrix M (with real or complex entries) to map X into Y, where X is the subspace of summability domain of a normal matrix A defined by the speed λ and Y is the subspace of a lower triangular matrix B defined by the speed μ. As an application we consider the case if A is the Riesz method (R, p_n).
Keywords: Matrix transforms, Boundedness and summability with speed, Riesz method.

1. Introduction

Let X, Y be two sequence spaces and $M = (m_{nk})$ be an arbitrary matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x = (x_k) \in X$ the series

$$M_n x = \sum_k m_{nk} x_k$$

converge and the sequence $Mx = (M_n x)$ belongs to Y, we say that the matrix M transforms X into Y. By (X, Y) we denote the set of all matrices which transform

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X into Y. Let c and m be correspondingly the spaces of all convergent and bounded sequences, and c_0 the space of all sequences converging to zero.

Let throughout this paper $\lambda = (\lambda_k)$ be a positive monotonically increasing sequence, i.e.; the speed of convergence. Following Kangro [5], [6] a convergent sequence $x = (x_k)$ with

$$\lim_k x_k := \xi(x) \text{ and } l_k(x) = \lambda_k (x_k - \xi(x)) \tag{1.1}$$

is called bounded with the speed λ (shortly, λ-bounded) if $l_k(x) = O_x(1)$ (or $(l_k(x)) \in m$), and convergent with the speed λ (shortly, λ-convergent) if the finite limit

$$\lim_k l_k(x) := b(x)$$

exists (or $(l_k(x)) \in c$). We denote the set of all λ-bounded sequences by m^λ, and the set of all λ-convergent sequences by c^λ. It is not difficult to see that $c^\lambda \subset m^\lambda \subset c$. In addition, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$ we get $c^\lambda = m^\lambda = c$.

Let $A = (a_{nk})$ be a normal matrix (it means A is lower triangular, and $a_{nn} \neq 0$ for every n) and $B = (b_{nk})$ a lower triangular matrix. A sequence $x = (x_k)$ is said to be A^λ-bounded (A^λ-summable), if $Ax \in m^\lambda$ ($Ax \in c^\lambda$, respectively). The set of all A^λ-bounded sequences will be denoted by m_A^λ, and the set of all A^λ-summable sequences by c_A^λ. Let c_A be the summability domain of A, i.e.; the set of sequences x (with real or complex entries), for which the finite limit $\lim_n A_n x$ exists. It is easy to see that $c_A^\lambda \subset m_A^\lambda \subset c_A$, and, if λ is a bounded sequence, then $m_A^\lambda = c_A^\lambda = c_A$.

Let $\mu = (\mu_n)$ be another speed of convergence,

$$c_0^\mu := \{x = (x_n) \,:\, x \in c^\mu \text{ and } \lim_n \mu_n (x_n - \xi(x)) = 0\},$$

and

$$(c_0)_B^\mu := \{x \in c_B^\mu \,:\, Bx \in c_0^\mu\}.$$

Necessary and sufficient conditions for $M \in \left(m_A^\lambda, m_B^\mu\right)$ have been proved in [2], and for $M \in \left(c_A^\lambda, c_B^\mu\right)$ in [3]. Necessary and sufficient conditions for $M \in \left(c_A^\lambda, m_B^\mu\right)$ have been presented in [1], Exercises 9.3 and 9.4.

In this paper we describe necessary and sufficient conditions for $M \in \left(m_A^\lambda, c_B^\mu\right)$ and for $M \in \left(m_A^\lambda, (c_0)_B^\mu\right)$. As an application we consider the case if A is the Riesz method (R, p_n).

2. Auxiliary results

For the proof of the main results we need some auxiliary results.

Lemma 2.1 (*[4], p. 44, see also [8], Proposition 12*). A matrix $A = (a_{nk}) \in (c_0, c)$ if and only if conditions

(I) $\lim_n a_{nk} := a_k$ for all k,

(II) $\sum_k |a_{nk}| = O(1)$

are satisfied. Moreover,

$$\lim_n A_n x = \sum_k a_k x_k. \tag{2.1}$$

Lemma 2.2 (*[4], p. 51, see also [7], p. 8, Theorem 1.2 or [8], Proposition 10*). The following statements are equivalent:

(a) $A = (a_{nk}) \in (m, c)$.

(b) The conditions (I), (II) are satisfied and

$$\lim_n \sum_k |a_{nk} - a_k| = 0. \tag{2.2}$$

(c) The condition (I) holds and

$$\textit{the series } \sum_k |a_{nk}| \textit{ converges uniformly in } n.$$

Moreover, if one of the statements (a)-(c) is satisfied, then the equation (2.1) holds.

Lemma 2.3 (*[8], Proposition 21*). A matrix $A = (a_{nk}) \in (m, c_0)$ if and only if condition

(III) $\lim_n \sum_k |a_{nk}| = 0$

is satisfied.

3. The sets $\left(m_A^\lambda, c_B^\mu\right)$ and $\left(m_A^\lambda, (c_0)_B^\mu\right)$

First we present necessary and sufficient conditions for existence of the transformation $y = Mx$ for every $x \in m_A^\lambda$. Let $A^{-1} := (\eta_{nk})$ be the inverse matrix of a normal matrix A. Then

$$\sum_{k=0}^{j} m_{nk} x_k = \sum_{k=0}^{j} m_{nk} \sum_{l=0}^{k} \eta_{kl} y_l = \sum_{l=0}^{j} h_{jl}^n y_l$$

for each $x := (x_k) \in m_A^\lambda$, where $y_l := A_l x$ and $H^n := \left(h_{jl}^n\right)$ is the lower triangular matrix for every fixed n, with

$$h_{jl}^n := \sum_{k=l}^{j} m_{nk} \eta_{kl}, \ l \le j.$$

This implies that the transformation $y = Mx$ exists for every $x \in m_A^\lambda$ if and only if the matrix $H^n := \left(h_{jl}^n\right) \in \left(m^\lambda, c\right)$ for every fixed n. Hence we can formulate the following result (see [1], Proposition 8.1 or [2], Lemma 1).

Proposition 3.1. Let $A = (a_{nk})$ be a normal method and $M = (m_{nk})$ an arbitrary matrix. Then the transformation $y = Mx$ exists for every $x \in m_A^\lambda$ if and only if

(IV) there exist finite limits $\lim_j h_{jl}^n := h_{nl}$ for every fixed l and n,

(V) $\lim_j \sum_{l=0}^{j} h_{jl}^n$ exists and is finite for every fixed n,

(VI) $\sum_l \frac{\left|h_{jl}^n\right|}{\lambda_l} = O_n(1)$ for every fixed n,

(VII) $\lim_j \sum_{l=0}^{j} \frac{\left|h_{jl}^n - h_{nl}\right|}{\lambda_l} = 0$ for every fixed n.

Also, condition (VI) can be replaced by the condition

(VIII) $\sum_l \frac{|h_{nl}|}{\lambda_l} = O_n(1)$ for every fixed n.

Remark 3.2. Using Lemma 2.2 c) it is possible to show that conditions (VI) and (VII) can be replaced by the condition

(IX) the series $\sum_l \frac{\left|h_{jl}^n\right|}{\lambda_l}$ converges uniformly in j for every fixed n.

Now we are able to prove the main results. Let $e = (1, 1, ...)$, $e^k = (0, ..., 0, 1, 0, ...)$, where 1 is in the k-th position, and $G = (g_{nk}) = BM$; i.e.,

$$g_{nk} := \sum_{l=0}^{n} b_{nl} m_{lk}.$$

Theorem 3.3. Let $A = (a_{nk})$ be a normal method, $B = (b_{nk})$ a triangular method and $M = (m_{nk})$ an arbitrary matrix. Then $M \in \left(m_A^\lambda, c_B^\mu\right)$ if and only if conditions (IV)-(VII) are satisfied and

(X) there exist the finite limits $\lim_n \gamma_{nl} := \gamma_l$,

(XI) there exist the finite limits $\lim_n \mu_n(\gamma_{nl} - \gamma_l) := S_l$,

(XII) $\sum_l \frac{|\gamma_{nl}|}{\lambda_l} = O(1)$,

(XIII) $\mu_n \sum\limits_l \frac{|\gamma_{nl}-\gamma_l|}{\lambda_l} = O(1)$,

(XIV) $\lim\limits_n \sum\limits_l \frac{|\mu_n(\gamma_{nl}-\gamma_l)-S_l|}{\lambda_l} = 0$,

where

$$\gamma_{nl} := \lim_j \gamma_{nl}^j,$$

and

(XV) $(\rho_n) \in c^\mu,\ \rho_n := \lim\limits_j \sum\limits_{l=0}^{j} \gamma_{nl}^j$,

where $\Gamma^n := \left(\gamma_{nl}^j\right)$ is the lower triangular matrix for every fixed n with

$$\gamma_{nl}^j := \sum_{k=l}^{j} g_{nk}\eta_{kl},\ l \leq j.$$

Also, condition (XII) can be replaced by the condition

(XVI) $\sum\limits_l \frac{|\gamma_l|}{\lambda_l} < \infty$,

and, if $\mu_n = O(1)$ and $\lambda_n \neq O(1)$, then it is necessary to replace the $O(1)$ in (XII) by $o(1)$.

Proof. *Necessity.* Assume that $M \in \left(m_A^\lambda, c_B^\mu\right)$. Then the transformation $y = Mx$ exists for every $x \in m_A^\lambda$. Hence conditions (IV) - (VII) hold by Proposition 3.1, and

$$B_n y = G_n x \tag{3.1}$$

for every $x \in m_A^\lambda$ because the change of the order of summation is allowed by the lower triangularity of B. From (3.1) we can conclude that $G \in \left(m_A^\lambda, c^\mu\right)$. In addition,

$$\sum_{k=0}^{j} g_{nk}\xi_k = \sum_{l=0}^{j} \gamma_{nl}^j A_l x \tag{3.2}$$

for every $x \in m_A^\lambda$. By the normality of A, there exists an $x \in m_A^\lambda$, such that $(A_l x) = e$. Consequently condition (XV) is satisfied by (3.2).

Assume now that $\lambda_n \neq O(1)$. Then, by the normality of A, for each bounded sequence (β_n) there exists an $x \in m_A^\lambda$, such that

$$\lim_n A_n x := \delta \text{ and } \beta_n = \lambda_n \left(A_n x - \delta\right). \tag{3.3}$$

Moreover, using (3.2) and (3.3) we obtain

$$\sum_{k=0}^{j} g_{nk}x_k = \delta \sum_{l=0}^{j} \gamma_{nl}^j + \sum_{l=0}^{j} \frac{\gamma_{nl}^j}{\lambda_l}\beta_l \tag{3.4}$$

for every $x \in m_A^\lambda$. As the series $G_n x$ are convergent for every $x \in m_A^\lambda$, and the finite limits ρ_n exist by (XV), then the matrix $\Gamma_\lambda^n := \left(\gamma_{nl}^j/\lambda_l\right) \in (m,c)$ for every n. Therefore, from (XV), we obtain, using Lemma 2.2 that

$$G_n x = \delta \rho_n + \sum_l \frac{\gamma_{nl}}{\lambda_l} \beta_l \tag{3.5}$$

for every $x \in m_A^\lambda$. In addition, the finite limit $\lim_n \rho_n := \rho$ exists by (XV). Therefore, from (3.5), we can conclude that the matrix $\Gamma_\lambda := (\gamma_{nl}/\lambda_l) \in (m,c)$. Consequently conditions (X), (XII) hold,

$$\lim_n \sum_l \frac{|\gamma_{nl} - \gamma_l|}{\lambda_l} = 0, \tag{3.6}$$

and

$$\lim_n G_n x = \delta \rho_n + \sum_l \frac{\gamma_l}{\lambda_l} \beta_l \tag{3.7}$$

for every $x \in m_A^\lambda$ by Lemma 2.2. Now it is clear that, for $\mu_n = O(1)$, it is necessary to replace $O(1)$ in (XIII) by $o(1)$; i.e., condition (XIII) is equivalent to (3.6).

We continue with the case $\mu_n \neq O(1)$, writing

$$\mu_n (G_n x - \lim_n G_n x) = \delta \mu_n (\rho_n - \rho) + \mu_n \sum_l \frac{\gamma_{nl} - \gamma_l}{\lambda_l} \beta_l \tag{3.8}$$

for every $x \in m_A^\lambda$. This implies that the matrix $\Gamma_{\lambda,\mu} := (\mu_n(\gamma_{nl} - \gamma_l)/\lambda_l) \in (m,c)$. Hence, using Lemma 2.2, we conclude that conditions (XI) and (XIV) hold.

If $\lambda_n = O(1)$, then the proof is similar to the case $\lambda_n \neq O(1)$, but now $\beta_l = o(1)$, and, instead of Lemma 2.2, it is necessary to use Lemma 2.1.

Finally, we note that the necessity of condition (XVI) follows from the validity of conditions (XII) and (XIII).

Sufficiency. Let all of the conditions of the present theorem be fulfilled. Then the transformation $y = Mx$ exists for every $x \in m_A^\lambda$ by Proposition 3.1, and equations (3.1) - (3.4) hold for every $x \in m_A^\lambda$. As in the proof of the necessity of the present theorem, we get, using (XV) and Lemma 2.2, that, from (3.4), follows the validity of (3.5) for every $x \in m_A^\lambda$. If $\lambda_n \neq O(1)$ and $\mu_n = O(1)$, then $\Gamma_\lambda^n \in (m,c)$ for every n by (X), (XII) and (3.6) (in this case, instead of (XIII), we have (3.6)); i.e., $M \in \left(m_A^\lambda, c_B\right)$.

If $\lambda_n \neq O(1)$ and $\mu_n \neq O(1)$, then the validity of (3.6) follows from the validity of (XIII). Thus, in that case, again $\Gamma_\lambda \in (m,c)$ by (X), (XII) and (3.6). Moreover, relation (3.7) holds for every $x \in m_A^\lambda$ by virtue of Lemma 2.2, and therefore relation (3.8) holds for every $x \in m^\lambda$. Hence $\Gamma_{\lambda,\mu} \in (m,c)$ by (XI), (XIII) and (XIV). Consequently, $M \in \left(m_A^\lambda, c_B^\mu\right)$ by (XV).

The proof for the case $\lambda_n = O(1)$ is analogous.

Condition (XII) can be replaced by (XVI) because the validity of (XII) follows from the validity of (XIII) and (XVI). □

Remark 3.4. Using (c) in Lemma 2.2 it is possible to show that conditions (XIII) and(XIV) in Theorem 3.2 can be replaced by the condition

(XVII) the series $\mu_n \sum_l \frac{|\gamma_{nl}-\gamma_l|}{\lambda_l}$ converges uniformly in n.

Using Lemma 2.3, from Theorem 3.3 we obtain the following result.

Corollary 3.5. Let $A = (a_{nk})$ be a normal method, $B = (b_{nk})$ a triangular method and $M = (m_{nk})$ an arbitrary matrix. Then $M \in \left(m_A^\lambda, (c_0)_B^\mu\right)$ if and only if $(\rho_n) \in c_0^\mu$, conditions (IV)-(VII), (X) and (XII) are satisfied, and

(XVIII) $\lim_n \sum_l \frac{|\mu_n(\gamma_{nl}-\gamma_l)|}{\lambda_l} = 0.$

Also, condition (XII) can be replaced by the condition (XVI) and, if $\mu_n = O(1)$ and $\lambda_n \neq O(1)$, then it is necessary to replace the $O(1)$ in (XII) by $o(1)$.

Proof. As in Theorem 3.3, all relations (3.1) - (3.8) hold for $x \in m_A^\lambda$, $\Gamma_\lambda^n \in (m, c)$ for every n and $\Gamma_\lambda \in (m, c)$. Hence also conditions (IV) - (VII), (X), (XII) and (XIII) hold. Unlike Theorem 3.3, we now get the condition $(\rho_n) \in c_0^\mu$ instead of (XV), and $\Gamma_{\lambda,\mu} \in (m, c_0)$. Therefore instead of (XI) and (XIV) we obtain condition (XVIII) by Lemma 2.3. Moreover, the validity of condition (XIII) follows from (XVIII). □

4. The sets $\left(m_{(R,p_n)}^\lambda, c_B^\mu\right)$ and $\left(m_{(R,p_n)}^\lambda, (c_0)_B^\mu\right)$

In this section we apply results from Section 3 for the case if A is the Riesz matrix denoted by (R, p_n). Let (p_n) be a sequence of nonzero complex numbers and $P_n = p_0 + ... + p_n \neq 0$. Then (R, p_n), defined by a lower triangular matrix $A = (a_{nk})$, is given in sequence-to-sequence form by equalities ([1], p. 29 or p. 131)

$$a_{nk} = p_k/P_n, \ k \leq n.$$

The inverse matrix $A^{-1} = (\eta_{kl})$ of (R, p_n) is defined by ([1], p. 90)

$$\eta_{kl} = \begin{cases} P_k/p_k & (l = k), \\ -P_{k-1}/p_k & (l = k-1), \\ 0 & \text{otherwise}. \end{cases} \tag{4.1}$$

As in previous section, let M be an arbitrary matrix and B a lower triangular matrix with real or complex entries. Then, using (4.1) we obtain

$$h_{jl}^n = \begin{cases} h_{nl} & (l \leq j-1), \\ P_j m_{nj}/p_j & (l = j), \\ 0 & \text{otherwise}, \end{cases} \tag{4.2}$$

where

$$h_{nl} = P_l \Delta_l \frac{m_{nl}}{p_l}, \tag{4.3}$$

and

$$\gamma^j_{nl} = \begin{cases} \gamma_{nl} & (l \leq j-1), \\ P_j g_{nj}/p_j & (l = j), \\ 0 & \text{otherwise}, \end{cases} \tag{4.4}$$

where

$$\gamma_{nl} = P_l \Delta_l \frac{g_{nl}}{p_l}. \tag{4.5}$$

As

$$\sum_{l=0}^{j} h^n_{jl} = \sum_{l=0}^{j} \sum_{k=l}^{j} m_{nk} \eta_{kl} = \sum_{k=0}^{j} m_{nk} \eta_k,$$

where

$$\eta_k := \sum_{l=0}^{k} \eta_k = 1$$

by (4.1), then we get

$$\sum_{l=0}^{j} h^n_{jl} = \sum_{k=0}^{j} m_{nk}. \tag{4.6}$$

Similarly to (4.6) we obtain

$$\sum_{l=0}^{j} \gamma^n_{jl} = \sum_{k=0}^{j} g_{nk},$$

and then

$$\rho_n = \sum_k g_{nk}. \tag{4.7}$$

`Theorem 4.1.` Let $e^k \in m^{\lambda}_{(R,p_n)}$. Then $M \in \left(m^{\lambda}_{(R,p_n)}, c^{\mu}_B\right)$ if and only if

(XIX) the series $\sum_k m_{nk}$ is convergent for every n,

(XX) $\lim_l \frac{P_j}{p_j} \frac{m_{nj}}{\lambda_j} = 0$ for every n,

(XXI) $\sum_l \frac{1}{\lambda_l} \left| P_l \Delta_l \frac{m_{nl}}{p_l} \right| = O_n(1)$,

(XXII) there exist the finite limits $\lim_n g_{nl} := g_l$,

(XXIII) $e \in c^{\mu}_G$,

(XXIV) there exist the finite limits $\mu_n \Delta_l \frac{g_{nl} - g_l}{p_l} := G_l$,

(XXV) $\sum_l \frac{1}{\lambda_l} \left| P_l \Delta_l \frac{g_{nl}}{p_l} \right| = O(1)$,

(XXVI) $\mu_n \sum\limits_l \frac{1}{\lambda_l} \left| P_l \Delta_l \frac{g_{nl} - g_l}{p_l} \right| = O(1)$,

(XXVII) $\lim\limits_n \sum\limits_l \frac{1}{\lambda_l} \left| \mu_n P_l \Delta_l \frac{g_{nl} - g_l}{p_l} - G_l \right| = 0$.

Proof. *Necessity.* Assume that $M \in \left(m^{\lambda}_{(R,p_n)}, c^{\mu}_B \right)$. Then, using equations (4.3) and (4.5) - (4.7), we obtain by Theorem 3.3 that correspondingly conditions (XXI), (XXV), (XIX) and (XXIII) are satisfied. Now it is not difficult to see that

$$\sum_l \frac{\left| h^n_{jl} - h_{nl} \right|}{\lambda_l} = \left| \frac{P_j m_{nj}}{p_j \lambda_j} - \frac{P_j}{\lambda_j} \Delta_j \frac{m_{nj}}{p_j} \right| + \sum_{l=j+1}^{\infty} \frac{1}{\lambda_l} \left| P_l \Delta_l \frac{m_{nl}}{p_l} \right|. \tag{4.8}$$

From (4.8) we get by condition (XXI) that

$$\lim_j \frac{1}{\lambda_j} \left| P_j \Delta_l \frac{m_{nj}}{p_j} \right| = 0 \tag{4.9}$$

and

$$\lim_j \sum_{l=j+1}^{\infty} \frac{1}{\lambda_l} \left| P_l \Delta_l \frac{m_{nl}}{p_l} \right| = 0 \tag{4.10}$$

Hence condition (XX) holds by Theorem 3.3.

As $e^k \in m^{\lambda}_{(R,p_n)}$ and equation (3.1) holds for every $x \in m^{\lambda}_{(R,p_n)}$, then condition (XXII) is satisfied. Finally, using the validity of (4.5) and (XXII), we have that conditions (XXIV), (XXVI) and (XXVII) hold correspondingly by conditions (XI), (XIII) and (XIV) of Theorem 3.3.

Sufficiency. Let all of the conditions of the present theorem be satisfied. We show that all conditions of Theorem 3.3 are satisfied for $A = (R, p_n)$. First, conditions (IV) and (V) hold by (4.2), (4.3) (4.6) and (XIX). Conditions (XX) and (XXI) imply the validity of condition (VI) by (4.2) and (4.3). Then the validity of condition (VII) follows from (XX) and (XXI) by (4.8) - (4.10).

Using equations (4.4), (4.5) and (4.7) we conclude that conditions (XXII) - (XXVII) imply the validity of conditions (X) - (XV). □

`Remark 4.2.` Condition (XXV) in Theorem 4.1 can be replaced by the condition

(XXVIII) $\sum\limits_l \frac{1}{\lambda_l} \left| P_l \Delta_l \frac{g_l}{p_l} \right| < \infty$,

since conditions (XXVI) and (XXVIII) imply the validity of (XXV).

`Remark 4.3.` Using Remark 3.4 we obtain that conditions (XXVI) and (XXVII) can be replaced by condition

(XXIX) the series $\mu_n \sum\limits_l \frac{1}{\lambda_l} \left| P_l \Delta_l \frac{g_{nl} - g_l}{p_l} \right|$ converges uniformly in n, Using Corollary 3.5, from Theorem 4.1 we immediately get the following result.

Corollary 4.4. Let $e^k \in m^{\lambda}_{(R,p_n)}$. Then $M \in \left(m^{\lambda}_{(R,p_n)}, (c_0)^{\mu}_B\right)$ if and only if $e \in (c_0)^{\mu}_G$, conditions (XIX) - (XX), (XXV) are satisfied and

$$\text{(XXX)} \ \lim_n \sum_l \frac{1}{\lambda_l} \left| \mu_n P_l \Delta_l \frac{g_{nl} - g_l}{p_l} \right| = 0.$$

5. Conclusions

In this paper we continued the investigations started in [2] and [3] (see also [1]), where we studied the matrix transformations of subspaces of summability domains of matrices with real or complex entries defined by speeds of convergence, i.e.; by monotonically increasing positive sequences λ and μ. Now we found necessary and sufficient conditions for a matrix M (with real or complex entries) to map the λ-boundedness domain of a normal matrix A into the μ-convergence domain (or into the specific subdomain of μ-convergence domain) of lower triangular matrix B. As an application we considered the case if A is the Riesz method (R, p_n). Further we intend to study matrix transforms between the specific subdomains of λ-boundedness (or λ-convergence) domain of matrix A and μ-boundedness (or μ-convergence) domain of matrix B.

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