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SOME RESULTS ON β-ALMOST SOLITONS ON ALMOST CO-KÄHLER MANIFOLDS

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Abstract. The object of the present paper is to study β-almost Yamabe solitons and β-almost Ricci solitons on almost co-Kähler manifolds. In this paper, we prove that if an almost co-Kähler manifold M with the Reeb vector field ξ admits a β-almost Yamabe solitons with the potential vector field ξ or $b\xi$, where b is a smooth function then manifold is K-almost co-Kähler manifold or the soliton is trivial, respectively. Also, we show if a closed (κ,μ)-almost co-Kähler manifold M^n with $n>1$ and $\kappa<0$ admits a β-almost Yamabe soliton then the soliton is trivial and expanding. Then we study an almost co-Kähler manifold admits a β-almost Yamabe soliton or β-almost Ricci soliton with V as the potential vector field, V is a special geometric vector field.
Keywords: Ricci soliton, Yamabe soliton, Contact manifold, Almost co-Kähler manifold.

1. Introduction

Over the last few years, the geometric flows have been an interesting topic of active research in both mathematics and physics. The Ricci flow was introduced by Hamilton [15], which is an evolution equation for metrics on a Riemannian manifold defined as follows

$$\frac{\partial g}{\partial t}=-2S, \qquad g(0)=g_0, \tag{1.1}$$

where S denotes the Ricci tensor. A Ricci soliton (M,g,V,λ) on a Riemannian manifold (M,g) is a special solution to the Ricci flow, a generalization of an Einstein

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metric and is defined by

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\mathcal{L}_V$ is the Lie derivative operator along the vector field V (called the potential vector field) on M and λ is a real number. The Ricci soliton is said to be shrinking, steady, or expanding depending on whether λ is negative, zero, and positive, respectively. If the vector field V is the gradient of a potential function f, then g is called a gradient Ricci soliton. If in Ricci soliton (1.2), λ be a function $\lambda : M \to \mathbb{R}$ it is then called almost Ricci soliton. In [12, 21] almost Ricci soliton has been investigated. Also, if there exists a function $\beta : M \to \mathbb{R}$ such that

$$\beta\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.3}$$

then this soliton is called β-almost Ricci soliton which studied in [12, 13]. The β-almost Ricci soliton is called expanding, steady and shrinking when $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. A β-almost Ricci soliton is said to be trivial if the potential vector field V is homothetic, i.e., $\mathcal{L}_V g = cg$, for some constant c. Otherwise, it is called non-trivial. Similarly, a Riemannian manifold (M, g) is said to be β-almost Yamabe soliton if there exists a vector field V on M (called the potential vector field), a soliton function $\lambda : M \to \mathbb{R}$ and a smooth function $\beta : M \to \mathbb{R}$ such that

$$\beta\mathcal{L}_V g = (\lambda - r)g, \tag{1.4}$$

where r is scalar curvature of M respect to metric g. When the function λ in (1.3) (and in (1.4)) is constant we simply say that it is a β-Ricci soliton (a β-Yamabe soliton). The β-almost Yamabe soliton is called expanding, steady or shrinking when $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. A β-almost Yamabe soliton is said to be trivial if the potential vector field V is Killing, i.e., $\mathcal{L}_V g = 0$. We say that β is defined signal whenever either $\beta > 0$ on M or $\beta < 0$ on M. During the last two decades, the geometry of Ricci soliton and other solitons have been the focus of attention of many mathematicians and physicists. For theoretical physicists the Ricci solitons are as quasi Einstein metrics and they have been looking into the equation of geometric sloitons in relation with topics of physics as string theory and general relativity [11, 25].

In contact geometry, gradient Ricci soliton have been studied by Sharma [22] as a K-contact and by Ghosh et al. [14] as a (κ, μ)-metric. In [8], Cho et al. and in [16] Hui et al. studied η-Ricci solitons on real hypersurfaces in a non-flat complex space form and on η-Einstein Kenmotsu manifolds, respectively. Also, in [23], Suh et al. investigated the Yamabe solitons and Ricci solitons on almost co-Kähler manifolds. In [17], Kar and Majhi studied β-almost Ricci soliton on almost co-Kähler manifolds with ξ belong to (κ, μ)-nullity distribution.

Motivated by the above studies the object of he tpresent paper is to study β-almost Yamabe solitons and β-almost Ricci solitons on almost co-Kähler manifolds and we generalize the results of [23] and also we obtain other results of these solitons

on almost co-Kähler manifolds when the potential vector field of solitons is satisfied in certain conditions. This paper is organized as follows. In section 2., after a brief introduction, we study almost co-Kähler manifolds and give some formula that will be used in the next sections. In section 3., we consider β-almost Yamabe solitons on almost co-Kähler manifolds and prove if an almost co-Kähler manifold M with the Reeb vector field ξ admits a β-almost Yamabe solitons with the potential vector field ξ or $b\xi$, where b is a smooth function, and β is defined signal, then manifold M is K-almost co-Kähler manifold or the soliton is trivial, respectively. Also we prove several important results about the geometric fields and β-almost Yamabe solitons on almost co-Kähler manifolds. In the following part of this section, we study β-almost Yamabe solitons on (κ, μ)-almost co-Kähler manifolds. In the last section, we consider β-almost Ricci solitons with geometric vector fields on almost co-Kähler manifolds.

2. Preliminaries

In this section, we give some well known definitions and formulas on almost co-Kähler manifolds which will be useful in the later sections. A smooth $(2n+1)$ dimensional Riemannian manifold (M, g) is said to admit an almost contact metric structures (ϕ, ξ, η, g), if it admits a $(1,1)$ tensor field ϕ, a unit vector field ξ (called the Reeb vector field), and a 1-form η satisfying [1, 2],

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

or equivalently

$$g(\phi X, Y) = -g(X, \phi Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

for all vector fields X, Y on M. For an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, we can always define a 2-form Φ as $\Phi(X, Y) = g(X, \phi Y)$. An almost contact metric structure becomes a contact metric structure if $\Phi = d\eta$. In this case, 1-form η is a contact form, ξ is its characteristic vector field, and Φ is the fundamental 2-form. If, in addition, ξ is a Killing vector field, then M^{2n+1} is said to be K-contact manifold. The almost contact metric structure is said to be normal if $[\phi, \phi] = -2d\eta \otimes \xi$ where

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \tag{2.4}$$

for any vector fields X, Y on M. A normal almost contact metric manifold is said to be Sasakian, that is an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.5}$$

or equivalently

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.6}$$

for any vector fields X, Y on M (see [2]). An almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be almost co-Kähler manifold [2, 4] if both the 1-form η and the 2-form Φ are closed. If, in addition the associated almost contact structure is normal, which is also equivalent to $\nabla\Phi = 0$, or equivalently $\nabla\phi = 0$, then M is said to be co-Kähler manifold. There exists some examples of (almost) co-Kähler manifolds, for instance, the Riemannian product of a real line and a (almost) Kähler manifold admits a (almost) co-Kähler structure ([7, 18, 19, 20]). On an almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ we set $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $h' = h \circ \phi$. Then the following formulas also hold for a almost co-Kähler manifold [9, 10, 19, 20]

$$h\xi = 0, \quad h\phi + \phi h = 0, \quad trh = trh', \tag{2.7}$$

$$\nabla_\xi\phi = 0, \quad \nabla\xi = h', \quad \text{div}\xi = 0, \tag{2.8}$$

$$S(\xi, \xi) + ||h||^2 = 0. \tag{2.9}$$

If, in addition, we put $l = R(., \xi)\xi$, then we also get $\phi l\phi - l = 2h^2$, where R is the Riemannian curvature tensor. From the second term of (2.7) it is easy to see that

$$(\mathcal{L}_\xi g)(X, Y) = 2g(h'X, Y), \tag{2.10}$$

for any vector fields X, Y on M. Therefore, the Reeb vector field ξ on almost co-Kähler manifold is Killing if and only if the $(1, 1)$ tensor field h vanishes. An almost co-Kähler manifold is said to be a K-almost co-Kähler manifold if the Reeb vector field ξ is Killing.
A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose curvature tensor satisfies

$$R(X, Y)\xi = \kappa\left[\eta(Y)X - \eta(X)Y\right] + \mu\left[\eta(Y)hX - \eta(X)hY\right], \tag{2.11}$$

for any vector fields X, Y on M and $\kappa, \mu \in \mathbb{R}$ is called (κ, μ)-contact manifold and ξ is said to belong to the (κ, μ)-nullity distribution. Similarly, we have

Definition 2.1. An almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be a (κ, μ)-almost co-Kähler manifold if the Reeb vector field ξ satisfies the equation (2.11).

In a consequence of (2.11), we obtain

$$S(X, \xi) = 2n\kappa\eta(X), \tag{2.12}$$

and $Q\xi = 2n\kappa\xi$, where Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$.

3. β-almost Yamabe solitons on almost co-Kähler manifolds

In this section we study β-almost Yamabe solitons on almost co-Kähler manifolds $(M^{2n+1}, \phi, \xi, \eta, g)$. If we assume that the potential vector field of soliton be ξ, then we have:

Theorem 3.1. *If an almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits a β-almost Yamabe soliton with ξ as the potential vector field and β is defined signal, then manifold M is K-almost co-Kähler manifold.*

Proof. Let in almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ the metric g be a β-almost Yamabe soliton with the potential vector field ξ. Then, we have $\mathcal{L}_\xi g = \frac{\lambda - r}{\beta} g$. By taking trace of both-sides of last identity we get $2\text{div}\xi = \frac{\lambda - r}{\beta}(2n+1)$. Since $\text{div}\xi = 0$, we conclude $\frac{\lambda - r}{\beta} = 0$, which means ξ is a Killing vector field. This completes the proof of Theorem. □

Any three dimensional almost co-Kähler manifold is co-Kähler manifold if and only if it is K-almost co-Kähler manifold [19]. Hence, we get the following result.

Corollary 3.1. *If an almost co-Kähler manifold $(M^3, \phi, \xi, \eta, g)$ admits a β-almost Yamabe soliton with ξ as the potential vector field and β is defined signal, then manifold M is a co-Kähler manifold.*

Now, if we assume that the potential vector field of β-almost Yamabe soliton is pointwise collinear with the Reeb vector field, then we have the following theorem.

Theorem 3.2. *If an almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits a β-almost Yamabe soliton with $b\xi$ as the potential vector field where b is non-zero smooth function and β is defined signal, then the soliton is trivial.*

Proof. Using (2.8) we can write

$$\nabla_X(b\xi) = X(b)\xi + b\nabla X\xi = X(b)\xi + bh'X, \tag{3.1}$$

for any vector field X on M. On the other hand, the metric g is a β-almost Yamabe soliton with the potential vector field $b\xi$, then we have $\beta\mathcal{L}_{b\xi} g = (\lambda - r)g$ and this implies

$$\begin{aligned} (\lambda - r)g(X, Y) &= \beta g(\nabla_X(b\xi), Y) + \beta g(X, \nabla_Y(b\xi)) \\ &= \beta X(b)\eta(Y) + \beta Y(b)\eta(X) 2\beta b g(h'X, Y). \end{aligned} \tag{3.2}$$

For each point p in M, we consider a local ϕ-basis $\{e_i : 1 \leq i \leq 2n+1\}$ on the tangent space T_pM. Since

$$\sum_{i=1}^{2n+1} g(h'e_i, e_i) = \sum_{i=1}^{2n+1} g(\nabla_{e_i}\xi, e_i) = \text{div}\xi = 0, \tag{3.3}$$

taking $X = Y = e_i$ in (3.2) and summing over i, we derive

$$(\lambda - r)(2n+1) = 2\beta\xi(b). \tag{3.4}$$

Again, substituting $X = Y = \xi$ in (3.2), we get

$$(\lambda - r) = 2\beta\xi(b). \tag{3.5}$$

Equations (3.4) and (3.5) yield $\lambda = r$. Putting $\lambda = r$ in $\beta\mathcal{L}_{b\xi} g = (\lambda - r)g$, we obtain $\mathcal{L}_{b\xi} g = 0$. Thus, $b\xi$ is a Killing vector field and the soliton is trivial. □

Now we state the following lemma, which will be used in the next results when in a (κ,μ)-almost co-Kähler manifold $(M^{2n+1},\phi,\xi,\eta,g)$ with $n>1$ constant κ is negative.

Lemma 3.1. *([24]) Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a (κ,μ)-almost co-Kähler manifold with $n>1$ and $\kappa<0$. Then the Ricci operator is given by*

$$Q=\mu h+2n\kappa\eta\otimes\xi, \tag{3.6}$$

where κ is a constant and μ is a smooth function satisfying $d\mu\wedge\eta=0$.

Theorem 3.3. *If a closed (κ,μ)-almost co-Kähler manifold $(M^{2n+1},\phi,\xi,\eta,g)$ with $n>1$ and $\kappa<0$ admits a β-almost Yamabe soliton and β is defined signal, then the soliton is trivial and expanding.*

Proof. Let the metric g of (κ,μ)-almost co-Kähler manifold $(M^{2n+1},\phi,\xi,\eta,g)$ be a β-almost Yamabe soliton with the potential vector field V, then we have $\beta\mathcal{L}_Vg=(\lambda-r)g$. Set $\rho:=\frac{\lambda-r}{\beta}$, from [26] we have

$$\mathcal{L}_Vr=-\rho r-2n\Delta\rho, \tag{3.7}$$

where $\Delta=\operatorname{div}\operatorname{grad}$ denotes the Laplace operator of g. On the other hand, from (3.6) of lemma 3.1 we derive

$$S(X,Y)=\mu g(hX,Y)+2n\kappa\eta(X)\eta(Y), \tag{3.8}$$

for any vector fields X,Y on M. Consider a local ϕ-basis $\{e_i:\ 1\le i\le 2n+1\}$ on the tangent space T_pM. Putting $X=Y=e_i$ in (3.8) and summing over i, $1\le i\le 2n+1$, we conclude $r=2n\kappa$. Hence the scalar curvature r is constant and negative. Thus $\mathcal{L}_Vr=0$ and equation (3.7) implies that $-\Delta\rho=\frac{r}{2n}\rho$. Multiplying both sides of this equation in function ρ, integrating over M and using divergence theorem we obtain

$$\int_M|\nabla\rho|^2\Omega=\frac{r}{2n}\int_M\rho^2\Omega, \tag{3.9}$$

where Ω is the volume form of M. If $\rho\neq0$ the (3.9) implies r is positive and this is a contradiction. Therefore, $\rho=0$, i.e. $\lambda=r$. So, the soliton is trivial and expanding. □

Now we recall the following definition which will be used in the next theorem.

Definition 3.1. A vector field V on a contact manifold $(M^{2n+1},\phi,\xi,\eta,g)$ is called a contact vector field if it satisfies

$$\mathcal{L}_V\eta=\psi\eta, \tag{3.10}$$

for some smooth function ψ on M. If $\psi=0$ on M, then the vector field V is called a strict contact vector field.

Theorem 3.4. *If an almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits a β-almost Yamabe soliton with V as the potential vector field, V is a contact vector field, i.e. $\mathcal{L}_V\eta = \psi\eta$, then ψ is constant and this soliton is shrinking, steady or expanding depending on whether $r + \frac{n+1}{2n+1}\beta\psi$ is negative, zero or positive, respectively. Moreover, if M be is closed manifold then $\lambda = r$.*

Proof. Since the metric g is β-almost Yamabe soliton with the potential vector field V, then

$$\beta(\mathcal{L}_V g)(X,Y) = (\lambda - r)g(X,Y). \tag{3.11}$$

This implies

$$\beta\mathcal{L}_V(g(X,Y)) - \beta g(\mathcal{L}_V X, Y) - \beta g(X, \mathcal{L}_V Y) = (\lambda - r)g(X,Y). \tag{3.12}$$

Putting ξ for X and Y in (3.12), we conclude

$$2\beta g(\mathcal{L}_V\xi, \xi) = \lambda - r. \tag{3.13}$$

Replacing $Y = \xi$ in (3.12) we deduce

$$\beta\mathcal{L}_V(\eta(X)) - \beta\eta(\mathcal{L}_V X) - \beta g(X, \mathcal{L}_V\xi) = (\lambda - r)g(X,\xi), \tag{3.14}$$

or equivalently

$$\beta(\mathcal{L}_V\eta)X - \beta g(X, \mathcal{L}_V\xi) = (\lambda - r)g(X,\xi). \tag{3.15}$$

In view of (3.10), (3.15) yields

$$\beta\psi\eta(X) - \beta g(X, \mathcal{L}_V\xi) = (\lambda - r)g(X,\xi). \tag{3.16}$$

This implies

$$\beta\mathcal{L}_V\xi = (\beta\psi - \lambda + r)\xi, \tag{3.17}$$

and

$$\beta g(\mathcal{L}_V\xi, \xi) = \beta\psi - \lambda + r. \tag{3.18}$$

Applying (3.13) in (3.18) we infer

$$\beta\psi = \frac{3}{2}(\lambda - r). \tag{3.19}$$

Making use of (3.19) in (3.17) we obtain

$$\beta\mathcal{L}_V\xi = \frac{\lambda - r}{2}\xi. \tag{3.20}$$

From $\phi(\xi) = 0$ and (3.20) we conclude

$$\beta(\mathcal{L}_V\phi)\xi = \beta\mathcal{L}_V(\phi\xi) - \phi(\beta\mathcal{L}_V\xi) = 0. \tag{3.21}$$

On the other hand, on almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ we have

$$(\mathcal{L}_V d\eta)(X,Y) = (\mathcal{L}_V g)(X, \phi Y) + g(X, (\mathcal{L}_V\phi)Y), \tag{3.22}$$

for all vector fields X, Y on M. Multiplying both sides of (3.22) in β and using (3.11), we can write

$$\beta(\mathcal{L}_V d\eta)(X,Y) = (\lambda - r)g(X,\phi Y) + \beta g(X,(\mathcal{L}_V\phi)Y), \tag{3.23}$$

for any vector fields X, Y on M. Since V is a contact vector field, from (3.10) we find

$$\mathcal{L}_V d\eta = d\mathcal{L}_V\eta = d(\psi\eta) = (d\psi)\wedge\eta + \psi(d\eta). \tag{3.24}$$

This gives

$$(\mathcal{L}_V d\eta)(X,Y) = \frac{1}{2}\left[d\psi(X)\eta(Y) - d\psi(Y)\eta(X)\right] + \psi g(X,\phi Y). \tag{3.25}$$

In view of (3.23) and (3.25) we get

$$\begin{aligned} &\beta d\psi(X)\eta(Y) - \beta d\psi(Y)\eta(X) + 2\beta\psi g(X,\phi Y)\\ &\qquad = 2(\lambda - r)g(X,\phi Y) + 2\beta g(X,(\mathcal{L}_V\phi)Y), \end{aligned} \tag{3.26}$$

from which it follows that

$$2\beta(\mathcal{L}_V\phi)Y = 2\big(\beta\psi - \lambda + r\big)\phi Y + \beta\big(\eta(Y)D\psi - (Y\psi)\xi\big). \tag{3.27}$$

Substituting $Y = \xi$ in (3.27) we obtain

$$2\beta(\mathcal{L}_V\phi)\xi = \beta\big(D\psi - (\xi\psi)\xi\big). \tag{3.28}$$

Replacing (3.21) in (3.28) we derive

$$D\psi = (\xi\psi)\xi, \tag{3.29}$$

where we use β is defined signal. Taking inner product of (3.29) with respect to any vector field Y we have $d\psi(Y) = (\xi\psi)\eta(Y)$, then

$$d\psi = (\xi\psi)\eta. \tag{3.30}$$

Taking exterior derivative of (3.30) we conclude

$$0 = d^2\psi = d(\xi\psi)\wedge\eta + (\xi\psi)d\eta. \tag{3.31}$$

The wedge product of both sides of (3.31) with η implies

$$(\xi\psi)\eta\wedge d\eta = 0. \tag{3.32}$$

As $\Omega = \eta\wedge d\eta^n$ is the volume form, then $\eta\wedge d\eta \neq 0$ and the above equation yields $\xi\psi = 0$. Hence from (3.30) it follows that $d\psi = 0$, thus ψ becomes constant. Tracing (3.11) over X, Y, gives

$$\beta\mathrm{div}V = (\lambda - r)(2n+1). \tag{3.33}$$

Taking Lie derivative of the volume form $\Omega = \eta \wedge d\eta^n$ along the vector field V and applying the formula $\mathcal{L}_V\Omega = (\mathrm{div}V)\Omega$ and (3.24) we obtain $(\mathrm{div}V)\Omega = (n+1)\psi\Omega$ and hence

$$\mathrm{div}V = (n+1)\psi. \tag{3.34}$$

From (3.33) and (3.34) we have

$$\lambda = r + \frac{n+1}{2n+1}\beta\psi. \tag{3.35}$$

Also, if manifold M is closed then integrating (3.34) and use divergence theorem we get $\psi = 0$ and $\lambda = r$.

□

In the following part, we recall definitions of some geometric vector fields.

Definition 3.2. ([3, 5, 6]) A vector field V on manifold M is called torse forming if it satisfies

$$\nabla_X V = fX + \theta(X)V \tag{3.36}$$

for all vector field X on M, where $f \in C^\infty(M)$ and θ is a 1-form. A torse forming vector field V is said to be recurrent if $f = 0$. If $\theta = 0$ then V is concircular. The vector field V is concurrent if it satisfies (3.36) with $f = 1$ and $\theta = 0$.

Definition 3.3. ([5, 6]) A vector field V is called parallel vector field if $\nabla_X V = 0$ for any vector field X on M.

Definition 3.4. ([5, 6]) A nowhere zero vector field V on Riemannian manifold is called a torqued vector field if it satisfies

$$\nabla_X V = fX + \theta(X)V, \qquad \theta(V) = 0. \tag{3.37}$$

Theorem 3.5. *If an almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits a β-almost Yamabe soliton with V as the potential vector field, V is a torse forming, then this soliton is shrinking, steady or expanding depending on whether $\frac{2}{2n+1}\beta\theta(V) + r + 2f\beta$ is negative, zero or positive, respectively. Moreover, if V is torqued vector field then $\lambda = r + 2f\beta$.*

Proof. Let the metric g of (κ, μ)-almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ be a β-almost Yamabe soliton with the potential vector field V. Then from (3.37), for any vector fields X, Y on M, we have

$$\begin{aligned} (\lambda - r)g(X,Y) &= \beta(\mathcal{L}_V g)(X,Y) = \beta g(\nabla_X V, Y) + \beta g(X, \nabla_Y V) \\ &= 2\beta f g(X,Y) + \beta\theta(X)g(V,Y) + \beta\theta(Y)g(V,X). \end{aligned} \tag{3.38}$$

Taking contraction of (3.38) over X and Y we get

$$(\lambda - r - 2f\beta)(2n+1) = 2\beta\theta(V). \tag{3.39}$$

If V is torqued vector field en $\theta(V) = 0$. This completes the proof of the theorem. □

4. β-almost Ricci solitons on almost co-Kähler manifolds

In this section we study the β-almost Ricci solitons on almost co-Kähler manifolds when the potential vector field of soliton is the Reeb vector field.

Theorem 4.1. *If a (κ,μ)-almost co-Kähler manifold $(M^{2n+1},\phi,\xi,\eta,g)$ admits a β-almost Ricci soliton with ξ as the potential vector field and β is defined signal, then ξ is a geodesic vector field and the soliton is expanding, steady or shrinking depending on whether κ is negative, zero or positive, respectively.*

Proof. Since (κ,μ)-almost co-Kähler manifold $(M^{2n+1},\phi,\xi,\eta,g)$ is a β-almost Ricci soliton with the potential vector field ξ, then for any vector fields X,Y on M we get

$$\beta(\mathcal{L}_\xi g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0. \tag{4.1}$$

The definition of Lie-derivative implies

$$\beta g(\nabla_X\xi,Y)+\beta g(X,\nabla_Y\xi)+2S(X,Y)+2\lambda g(X,Y)=0. \tag{4.2}$$

Putting $Y=\xi$ in the above equation, we obtain

$$\beta g(\nabla_X\xi,\xi)+\beta g(X,\nabla_\xi\xi)+2S(X,\xi)+2\lambda g(X,\xi)=0. \tag{4.3}$$

Since, $g(\nabla_X\xi,\xi)=0$ and $S(X,\xi)=2n\kappa g(X,\xi)$, the above equation gives

$$\beta\nabla_\xi\xi=(4n\kappa+2\lambda)\xi. \tag{4.4}$$

Also, if we set $X=\xi$ in (4.3) then we infer $\lambda=-2n\kappa$ and $\nabla_\xi\xi=0$. Therefore, ξ is a geodesic vector field and this completes the proof of theorem. □

Definition 4.1. A contact manifold $(M^{2n+1},\phi,\xi,\eta,g)$ is said to be η-Einstein if its Ricci tensor S satisfies

$$S=ag+b\eta\otimes\eta$$

where a and b are smooth function on M.

Theorem 4.2. *If a (κ,μ)-almost co-Kähler manifold $(M^{2n+1},\phi,\xi,\eta,g)$ admits a β-almost Ricci soliton with ξ as the potential vector field, ξ is a torse forming, then ξ is geodesic vector field, η is closed, and M is η-Einstein. Moreover, if f is constant then $\kappa\leq 0$ and $f^2=-2\kappa$.*

Proof. Let the metric g of (κ,μ)-almost co-Kähler manifold $(M^{2n+1},\phi,\xi,\eta,g)$ be β-almost Ricci soliton with the potential vector field ξ and ξ is a torse forming. Then $\eta(\xi)=1$ and torse forming of ξ imply

$$0=g(\nabla_X\xi,\xi)=f\eta(X)+\theta(X), \tag{4.5}$$

for any vector field X on M and hence we have $\theta = -f\eta$. Consequently (3.36) reduces to

$$\nabla_X \xi = f\left(X - \eta(X)\xi\right), \tag{4.6}$$

for any vector field X on M. Equation (4.6) implies that ξ is a geodesic vector field that is $\nabla_\xi \xi = 0$ and $\nabla_X \xi$ is collinear to $\phi^2 X$ for all X and hence $d\eta = 0$, that is η is closed. Using (4.6) in (4.2), we obtain

$$S(X,Y) = -(f\beta + \lambda)g(X,Y) + f\beta\eta(X)\eta(Y), \tag{4.7}$$

hence manifold M is η-Einstein. By definition of Ricci curvature tensor, we have

$$R(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi. \tag{4.8}$$

Substituting (3.36) in (4.8) implies

$$R(X,Y)\xi = \big(X(f)Y - Y(f)X\big) - \big(X(f)\eta(Y) - Y(f)\eta(X)\big)\xi + f^2\big(\eta(X)Y - \eta(Y)X\big). \tag{4.9}$$

Now, if the function f be a constant, then

$$R(X,Y)\xi = f^2\big(\eta(X)Y - \eta(Y)X\big), \tag{4.10}$$

and

$$S(X,\xi) = -2nf^2\eta(X). \tag{4.11}$$

From (2.12), we have

$$S(X,\xi) = 4n\kappa\eta(X). \tag{4.12}$$

Comparing (4.11) and (4.12) we infer $f^2 = -2\kappa$ and κ is nonpositive. □

Corollary 4.1. *If a (κ,μ)-almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits a β-almost Ricci soliton with ξ as the potential vector field, ξ is a recurrent torse forming, then ξ is parallel and Killing vector field.*

Proof. Since ξ is recurrent vector field, therefore $f = 0$. So equation (4.6) yields $\nabla_X \xi = 0$, for all vector field X on M, which means that ξ is concurrent vector field. Also,

$$(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0 \tag{4.13}$$

for all vector fields X, Y on M, that means ξ is Killing vector field. □

Corollary 4.2. *If a (κ,μ)-almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits a β-almost Ricci soliton with ξ as the potential vector field, ξ is a concircular vector field, then ξ is parallel and Killing vector field.*

Proof. Let the metric g of (κ,μ)-almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ be β-almost Ricci soliton with the potential vector field ξ and ξ is a concircular vector field. Then $\eta(\xi) = 1$ and $\nabla_X \xi = fX$ imply

$$0 = g(\nabla_X \xi, \xi) = f\eta(X). \tag{4.14}$$

Substituting $X = \xi$ in above identity we have $f = 0$. Therefore ξ is parallel and Killing vector field. □

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