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ON NEW GENERALIZED FRACTIONAL MIDPOINT-TYPE INEQUALITIES FOR CO-ORDINATED CONVEX AND CO-ORDINATED CONCAVE FUNCTIONS

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Abstract. In this paper, we firstly obtain a new generalized identity for twice partially differentiable functions Riemann–Liouville fractional integrals. Then, using this equality, we obtain some midpoint-type inequalities for co-ordinated convex and co-ordinated concave functions. We also show that our result generalizes the give several inequalities obtained in earlier works.
Keywords: Riemann–Liouville fractional integrals, inequalities, convex function, concave function.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature. These inequalities state that if $\digamma : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$\digamma\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \digamma(\varkappa)d\varkappa \leq \frac{\digamma(a)+\digamma(b)}{2}. \tag{1.1}$$

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Both inequalities hold in the reversed direction if $\mathcal{F}$ is concave.

Over the years, numerous studies have focused on obtaining trapezoid-type and midpoint-type inequalities which give bounds for the right-hand side and left-hand side of the inequality (1.1), respectively. For example, Dragomir and Agarwal first obtained trapezoid-type inequalities for convex functions in [6], whereas Kirmacı first, established midpoint inequalities for convex functions in [10]. Moreover in [15], Qaisar and Hussain presented several generalized midpoint-type inequalities. Sarikaya et al. and Iqbal et al. proved some fractional trapezoid-type and midpoint-type inequalities for convex functions in [17] and [8], respectively. In [1] and [3], the authors established some generalized midpoint-type inequalities for Riemann–Liouville fractional integrals. On the other hand, Dragomir proved Hermite–Hadamard inequalities for co-ordinated convex mappings in [5]. In [11] and [18], the authors proved midpoint and trapezoid-type inequalities for co-ordinated convex functions, respectively. Moreover, Sarikaya obtained fractional Hermite–Hadamard inequalities and fractional trapezoid-type for functions with two variables in [19]. Tunç et al. presented some fractional midpoint-type inequalities for co-ordinated convex functions in [21]. For other similar inequalities, please refer to [2, 4, 12, 14, 16, 20].

This paper aims to prove some generalized midpoint-type inequalities for Riemann–Liouville fractional integrals by using co-ordinated convex and co-ordinated concave functions. The overall structure of the study takes the form of four sections including an introduction. The remaining part of the paper proceeds as follows: We first recall the definitions of Riemann–Liouville fractional integrals for single variable functions and two-variables functions. In Section 2, an identity for twice partially differentiable functions is presented. Then by using this equality we prove some midpoint-type inequalities for co-ordinated convex functions. Moreover, by utilizing Jensen integral inequality for functions with two variables, we obtain several midpoint-type inequalities for co-ordinated concave functions in Section 3. Finally, some conclusions and further directions of research are discussed in Section 4.

Definition 1.1. Let $\mathcal{F} \in L_1[a,b]$. The Riemann–Liouville integrals $\mathcal{J}^{\alpha}_{a+}\mathcal{F}$ and $\mathcal{J}^{\alpha}_{b-}\mathcal{F}$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$\mathcal{J}^{\alpha}_{a+}\mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_a^{\varkappa} (\varkappa - t)^{\alpha-1} \mathcal{F}(t)dt, \quad \varkappa > a$$

and

$$\mathcal{J}^{\alpha}_{b-}\mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{b} (t - \varkappa)^{\alpha-1} \mathcal{F}(t)dt, \quad \varkappa < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $\mathcal{J}^{0}_{a+}\mathcal{F}(\varkappa) = \mathcal{J}^{0}_{b-}\mathcal{F}(\varkappa) = \mathcal{F}(\varkappa)$.

Definition 1.2. [19] Let $\mathcal{F} \in L_1([a,b] \times [c,d])$. The Riemann–Liouville fractional integrals $\mathcal{J}^{\alpha,\beta}_{a+,c+}$, $\mathcal{J}^{\alpha,\beta}_{a+,d-}$, $\mathcal{J}^{\alpha,\beta}_{b-,c+}$, $\mathcal{J}^{\alpha,\beta}_{b-,d-}$ are defined by

$$\mathcal{J}^{\alpha,\beta}_{a+,c+}\mathcal{F}\left(\varkappa, \gamma\right)$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^{\varkappa}\int_c^{\gamma}(\varkappa-t)^{\alpha-1}(\gamma-s)^{\beta-1}\digamma(t,s)\,dsdt,\ \ \varkappa>a,\ \ \gamma>c,$$

$$\mathcal{J}^{\alpha,\beta}_{a+,d-}\digamma(\varkappa,\gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^{\varkappa}\int_{\gamma}^{d}(\varkappa-t)^{\alpha-1}(s-\gamma)^{\beta-1}\digamma(t,s)\,dsdt,\ \ \varkappa>a,\ \ \gamma<d,$$

$$\mathcal{J}^{\alpha,\beta}_{b-,c+}\digamma(\varkappa,\gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{b}\int_c^{\gamma}(t-\varkappa)^{\alpha-1}(\gamma-s)^{\beta-1}\digamma(t,s)\,dsdt,\ \ \varkappa<b,\ \ \gamma>c,$$

and

$$\mathcal{J}^{\alpha,\beta}_{b-,d-}\digamma(\varkappa,\gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{b}\int_{\gamma}^{d}(t-\varkappa)^{\alpha-1}(s-\gamma)^{\beta-1}\digamma(t,s)\,dsdt,\ \ \varkappa<b,\ \ \gamma<d,$$

where Γ is the Gamma function.

For more information and several properties of Riemann–Liouville fractional integrals, please refer to [7, 9, 13].

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.3. A function $\digamma : \Delta := [a,b]\times[c,d] \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(\varkappa,u),(\gamma,v)\in\Delta$ and $t,s\in[0,1]$, if it satisfies the following inequality:

$$\digamma(t\varkappa+(1-t)\,\gamma, su+(1-s)\,v) \leq ts\,\digamma(\varkappa,u)+t(1-s)\digamma(\varkappa,v)+s(1-t)\digamma(\gamma,u)+(1-t)(1-s)\digamma(\gamma,v). \tag{1.2}$$

The mapping $\digamma$ is a co-ordinated concave on Δ if the inequality (1.2) holds in reversed direction for all $t,s\in[0,1]$ and $(\varkappa,u),(\gamma,v)\in\Delta$.

2. Generalized Midpoint-type Inequalities for Co-ordinated Convex Functions

In this section, we establish several fractional midpoint-type inequalities for co-ordinated convex functions via Riemann–Liouville fractional integrals. We first prove the following lemma which will be used frequently.

Lemma 2.1. *Let $F : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be a twice partially differentiable mapping on Δ°. If $\frac{\partial^2 F}{\partial t \partial s} \in L(\Delta)$, then for all $(\varkappa, \gamma) \in \Delta$ we have the following equality for generalized fractional integrals,*

$$\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma) \tag{2.1}$$

$$\begin{aligned}
&= \frac{(b-\varkappa)^2 (d-\gamma)^2}{(b-a)(d-c)} \\
&\quad \times \int_0^1 \int_0^1 t^\alpha s^\beta \frac{\partial^2 F}{\partial s \partial t}\left(t(a+b-\varkappa)+(1-t)a, s(c+d-\gamma)+(1-s)c\right) dtds \\
&\quad + \frac{(\varkappa-a)^2 (\gamma-c)^2}{(b-a)(d-c)} \\
&\quad \times \int_0^1 \int_0^1 t^\alpha s^\beta \frac{\partial^2 F}{\partial s \partial t}\left(t(a+b-\varkappa)+(1-t)b, s(c+d-\gamma)+(1-s)d\right) dtds \\
&\quad - \frac{(b-\varkappa)^2 (\gamma-c)^2}{(b-a)(d-c)} \\
&\quad \times \int_0^1 \int_0^1 t^\alpha s^\beta \frac{\partial^2 F}{\partial s \partial t}\left(t(a+b-\varkappa)+(1-t)a, s(c+d-\gamma)+(1-s)d\right) dtds \\
&\quad - \frac{(\varkappa-a)^2 (d-\gamma)^2}{(b-a)(d-c)} \\
&\quad \times \int_0^1 \int_0^1 t^\alpha s^\beta \frac{\partial^2 F}{\partial s \partial t}\left(t(a+b-\varkappa)+(1-t)b, s(c+d-\gamma)+(1-s)c\right) dtds
\end{aligned}$$

where

(2.2)

$$\begin{aligned}
&\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma) \\
&= F(a+b-\varkappa, c+d-\gamma) - \frac{\Gamma(\beta+1)}{d-c}\Big[(d-\gamma)^{1-\beta} \mathcal{J}^{\beta}_{(c+d-\gamma)^-} F(a+b-\varkappa, c) \\
&\quad + (\gamma-c)^{1-\beta} \mathcal{J}^{\beta}_{(c+d-\gamma)^+} F(a+b-\varkappa, d)\Big] \\
&\quad - \frac{\Gamma(\alpha+1)}{b-a}\Big[(b-\varkappa)^{1-\alpha} \mathcal{J}^{\alpha}_{(a+b-\varkappa)^-} F(a, c+d-\gamma) \\
&\quad + (\varkappa-a)^{1-\alpha} \mathcal{J}^{\alpha}_{(a+b-\varkappa)^+} F(b, c+d-\gamma)\Big] \\
&\quad + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)(d-c)}\left[\frac{\mathcal{J}^{\alpha,\beta}_{(a+b-\varkappa)^-,(c+d-\gamma)^-} F(a,c)}{(b-\varkappa)^{1-\alpha}(d-\gamma)^{1-\beta}} + \frac{\mathcal{J}^{\alpha,\beta}_{(a+b-\varkappa)^+,(c+d-\gamma)^+} F(b,d)}{(\varkappa-a)^{1-\alpha}(\gamma-c)^{1-\beta}}\right. \\
&\quad \left. + \frac{\mathcal{J}^{\alpha,\beta}_{(a+b-\varkappa)^-,(c+d-\gamma)^+} F(a,d)}{(b-\varkappa)^{1-\alpha}(\gamma-c)^{1-\beta}} + \frac{\mathcal{J}^{\alpha,\beta}_{(a+b-\varkappa)^+,(c+d-\gamma)^-} F(b,c)}{(\varkappa-a)^{1-\alpha}(d-\gamma)^{1-\beta}}\right].
\end{aligned}$$

Proof. Using the integration by parts, we get

$$\begin{aligned}
(2.3)\qquad & \int_0^1 \int_0^1 t^{\alpha} s^{\beta} \frac{\partial^2 F}{\partial s \partial t} \left(t\left(a+b-\varkappa\right)+\left(1-t\right)a, s\left(c+d-\gamma\right)+\left(1-s\right)c\right) dtds \\
= \quad & \frac{F\left(a+b-\varkappa, c+d-\gamma\right)}{\left(b-\varkappa\right)\left(d-\gamma\right)} \\
& - \frac{\Gamma\left(\beta+1\right)}{\left(b-\varkappa\right)\left(d-\gamma\right)^{\beta+1}} \mathcal{J}^{\beta}_{\left(c+d-\gamma\right)^-} F\left(a+b-\varkappa, c\right) \\
& - \frac{\Gamma\left(\alpha+1\right)}{\left(b-\varkappa\right)^{\alpha+1}\left(d-\gamma\right)} \mathcal{J}^{\alpha}_{\left(a+b-\varkappa\right)^-} F\left(a, c+d-\gamma\right) \\
& + \frac{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}{\left(b-\varkappa\right)^{\alpha+1}\left(d-\gamma\right)^{\beta+1}} \mathcal{J}^{\alpha,\beta}_{\left(a+b-\varkappa\right)^-,\left(c+d-\gamma\right)^-} F\left(a, c\right),
\end{aligned}$$

$$\begin{aligned}
(2.4)\qquad & \int_0^1 \int_0^1 t^{\alpha} s^{\beta} \frac{\partial^2 F}{\partial s \partial t} \left(t\left(a+b-\varkappa\right)+\left(1-t\right)b, s\left(c+d-\gamma\right)+\left(1-s\right)d\right) dtds \\
= \quad & \frac{F\left(a+b-\varkappa, c+d-\gamma\right)}{\left(\varkappa-a\right)\left(\gamma-c\right)} \\
& - \frac{\Gamma\left(\beta+1\right)}{\left(\varkappa-a\right)\left(\gamma-c\right)^{\beta+1}} \mathcal{J}^{\beta}_{\left(c+d-\gamma\right)^+} F\left(a+b-\varkappa, d\right) \\
& - \frac{\Gamma\left(\alpha+1\right)}{\left(\varkappa-a\right)^{\alpha+1}\left(\gamma-c\right)} \mathcal{J}^{\alpha}_{\left(a+b-\varkappa\right)^+} F\left(b, c+d-\gamma\right) \\
& + \frac{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}{\left(\varkappa-a\right)^{\alpha+1}\left(\gamma-c\right)^{\beta+1}} \mathcal{J}^{\alpha,\beta}_{\left(a+b-\varkappa\right)^+,\left(c+d-\gamma\right)^+} F\left(b, d\right),
\end{aligned}$$

$$\begin{aligned}
(2.5)\qquad & \int_0^1 \int_0^1 t^{\alpha} s^{\beta} \frac{\partial^2 F}{\partial s \partial t} \left(t\left(a+b-\varkappa\right)+\left(1-t\right)a, s\left(c+d-\gamma\right)+\left(1-s\right)d\right) dtds \\
= \quad & -\frac{F\left(a+b-\varkappa, c+d-\gamma\right)}{\left(b-\varkappa\right)\left(\gamma-c\right)} \\
& + \frac{\Gamma\left(\beta+1\right)}{\left(b-\varkappa\right)\left(\gamma-c\right)^{\beta+1}} \mathcal{J}^{\beta}_{\left(c+d-\gamma\right)^+} F\left(a+b-\varkappa, d\right) \\
& + \frac{\Gamma\left(\alpha+1\right)}{\left(b-\varkappa\right)^{\alpha+1}\left(\gamma-c\right)} \mathcal{J}^{\alpha}_{\left(a+b-\varkappa\right)^-} F\left(a, c+d-\gamma\right) \\
& - \frac{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}{\left(b-\varkappa\right)^{\alpha+1}\left(\gamma-c\right)^{\beta+1}} \mathcal{J}^{\alpha,\beta}_{\left(a+b-\varkappa\right)^-,\left(c+d-\gamma\right)^+} F\left(a, d\right)
\end{aligned}$$

and

$$(2.6)\qquad \int_0^1 \int_0^1 t^{\alpha} s^{\beta} \frac{\partial^2 F}{\partial s \partial t} \left(t\left(a+b-\varkappa\right)+\left(1-t\right)b, s\left(c+d-\gamma\right)+\left(1-s\right)c\right) dtds$$

$$
\begin{aligned}
= & -\frac{F(a+b-\varkappa, c+d-\gamma)}{(\varkappa-a)(d-\gamma)} \\
& +\frac{\Gamma(\beta+1)}{(\varkappa-a)(d-\gamma)^{\beta+1}} \mathcal{J}_{(c+d-\gamma)^{-}}^{\beta} F(a+b-\varkappa, c) \\
& +\frac{\Gamma(\alpha+1)}{(\varkappa-a)^{\alpha+1}(d-\gamma)} \mathcal{J}_{(a+b-\varkappa)^{+}}^{\alpha} F(b, c+d-\gamma) \\
& -\frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{(\varkappa-a)^{\alpha+1}(d-\gamma)^{\beta+1}} \mathcal{J}_{(a+b-\varkappa)^{+},(c+d-\gamma)^{-}}^{\alpha, \beta} F(b, c) .
\end{aligned}
$$

By the identities (2.3)-(2.6) we obtain the required result (2.1). □

Theorem 2.1. *Let $F:\Delta\to\mathbb{R}$ be twice partially differentiable mapping on Δ°. If $\frac{\partial^2 F}{\partial s\partial t}\in L(\Delta)$ and $\left|\frac{\partial^2 F}{\partial s\partial t}\right|$, is a co-ordinated convex, then for all $(\varkappa,\gamma)\in\Delta$ we have the following inequality for fractional integrals,*

$$
\begin{aligned}
(2.7) \qquad & \left|\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)\right| \\
\leq & \frac{1}{(\alpha+2)(\beta+2)(b-a)(d-c)} \\
& \times\left\{\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c+d-\gamma)\right|\right. \\
& \times\left[(b-\varkappa)^2+(\varkappa-a)^2\right]\left[(d-\gamma)^2+(\gamma-c)^2\right] \\
& +\frac{1}{(\beta+1)}\left[\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c)\right|(d-\gamma)^2\left((b-\varkappa)^2+(\varkappa-a)^2\right)\right. \\
& \left.+\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,d)\right|(\gamma-c)^2\left((b-\varkappa)^2+(\varkappa-a)^2\right)\right] \\
& +\frac{1}{(\alpha+1)}\left[\left|\frac{\partial^2 F}{\partial s\partial t}(a,c+d-\gamma)\right|(b-\varkappa)^2\left((d-\gamma)^2+(\gamma-c)^2\right)\right. \\
& \left.+\left|\frac{\partial^2 F}{\partial s\partial t}(b,c+d-\gamma)\right|(\varkappa-a)^2\left((d-\gamma)^2+(\gamma-c)^2\right)\right] \\
& +\frac{1}{(\alpha+1)(\beta+1)} \\
& \times\left((b-\varkappa)^2(d-\gamma)^2\left|\frac{\partial^2 F}{\partial s\partial t}(a,c)\right|+(\varkappa-a)^2(\gamma-c)^2\left|\frac{\partial^2 F}{\partial s\partial t}(b,d)\right|\right. \\
& \left.\left.+\,(b-\varkappa)^2(\gamma-c)^2\left|\frac{\partial^2 F}{\partial s\partial t}(a,d)\right|+(\varkappa-a)^2(d-\gamma)^2\left|\frac{\partial^2 F}{\partial s\partial t}(b,c)\right|\right)\right\}
\end{aligned}
$$

where $\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)$ is defined as in (2.2).

Proof. By taking modulus in Lemma 2.1, we have

$$(2.8)\ \left|\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)\right|$$

$$
\begin{aligned}
\leq \; & \frac{(b-\varkappa)^2(d-\gamma)^2}{(b-a)(d-c)} \\
& \times \int_0^1\int_0^1 \left|t^\alpha s^\beta\right| \left|\frac{\partial^2 F}{\partial s\partial t}\left(t(a+b-\varkappa)+(1-t)a, s(c+d-\gamma)+(1-s)c\right)\right| dtds \\
& + \frac{(\varkappa-a)^2(\gamma-c)^2}{(b-a)(d-c)} \\
& \times \int_0^1\int_0^1 \left|t^\alpha s^\beta\right| \left|\frac{\partial^2 F}{\partial s\partial t}\left(t(a+b-\varkappa)+(1-t)b, s(c+d-\gamma)+(1-s)d\right)\right| dtds \\
& + \frac{(b-\varkappa)^2(\gamma-c)^2}{(b-a)(d-c)} \\
& \times \int_0^1\int_0^1 \left|t^\alpha s^\beta\right| \left|\frac{\partial^2 F}{\partial s\partial t}\left(t(a+b-\varkappa)+(1-t)a, s(c+d-\gamma)+(1-s)d\right)\right| dtds \\
& + \frac{(\varkappa-a)^2(d-\gamma)^2}{(b-a)(d-c)} \\
& \times \int_0^1\int_0^1 \left|t^\alpha s^\beta\right| \left|\frac{\partial^2 F}{\partial s\partial t}\left(t(a+b-\varkappa)+(1-t)b, s(c+d-\gamma)+(1-s)c\right)\right| dsdt.
\end{aligned}
$$

Since $\left|\frac{\partial^2 F}{\partial s\partial t}\right|$ is co-ordinated convex, we get

$$
\begin{aligned}
& \int_0^1\int_0^1 \left|t^\alpha s^\beta\right| \left|\frac{\partial^2 F}{\partial s\partial t}\left(t(a+b-\varkappa)+(1-t)a, s(c+d-\gamma)+(1-s)c\right)\right| dtds \\
\leq \; & \int_0^1\int_0^1 \Bigg[t^{\alpha+1}s^{\beta+1}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa, c+d-\gamma)\right| + t^{\alpha+1}s^\beta(1-s)\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa, c)\right| \\
& + t^\alpha(1-t)s^{\beta+1}\left|\frac{\partial^2 F}{\partial s\partial t}(a, c+d-\gamma)\right| + t^\alpha s^\beta(1-s)(1-t)\left|\frac{\partial^2 F}{\partial s\partial t}(a,c)\right|\Bigg] dtds \\
= \; & \frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa, c+d-\gamma)\right| + \frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa, c)\right| \\
& + \frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a, c+d-\gamma)\right| + \frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a,c)\right|.
\end{aligned} \tag{2.9}
$$

Similarly, one can establish

$$
\begin{aligned}
& \int_0^1\int_0^1 \left|t^\alpha s^\beta\right| \left|\frac{\partial^2 F}{\partial s\partial t}\left(t(a+b-\varkappa)+(1-t)b, s(c+d-\gamma)+(1-s)d\right)\right| dtds \\
\leq \; & \frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa, c+d-\gamma)\right| + \frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa, d)\right|
\end{aligned} \tag{2.10}
$$

$$+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(b,c+d-\gamma)\right|+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(b,d)\right|,$$

(2.11)

$$\begin{aligned}&\int_0^1\int_0^1\left|t^\alpha s^\beta\right|\left|\frac{\partial^2 F}{\partial s\partial t}(t(a+b-\varkappa)+(1-t)a,s(c+d-\gamma)+(1-s)d)\right|dtds\\ \leq\ &\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c+d-\gamma)\right|+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,d)\right|\\ &+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a,c+d-\gamma)\right|+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a,d)\right|,\end{aligned}$$

(2.12)

$$\begin{aligned}&\int_0^1\int_0^1\left|t^\alpha s^\beta\right|\left|\frac{\partial^2 F}{\partial s\partial t}(t(a+b-\varkappa)+(1-t)b,s(c+d-\gamma)+(1-s)c)\right|dtds\\ \leq\ &\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c+d-\gamma)\right|\\ &+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c)\right|\\ &+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(b,c+d-\gamma)\right|\\ &+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(b,c)\right|.\end{aligned}$$

If we substitute the inequalities (2.9)-(2.12) in (2.8), we obtain the required result (2.7). □

Corollary 2.1. *Under assumption of Theorem* 2.1 *with* $\varkappa=\frac{a+b}{2}$ *and* $\gamma=\frac{c+d}{2}$, *we have the following midpoint-type inequality,*

$$\begin{aligned}&\left|\mathcal{K}^{\alpha,\beta}(a,b,c,d)\right|\\ \leq\ &\frac{(b-a)(d-c)}{16(\alpha+2)(\beta+2)}\left[4\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right.\\ &+\frac{2}{(\beta+1)}\left(\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|\right)\\ &+\frac{2}{(\alpha+1)}\left(\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 F}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|\right)\\ &+\frac{1}{(\alpha+1)(\beta+1)}\left(\left|\frac{\partial^2 F}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 F}{\partial s\partial t}(b,d)\right|\right)\\ &\left.+\frac{1}{(\alpha+1)(\beta+1)}\left(\left|\frac{\partial^2 F}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 F}{\partial s\partial t}(b,c)\right|\right)\right]\end{aligned}$$

$$\leq \frac{(b-a)(d-c)}{16(\alpha+1)(\beta+1)}\left[\left|\frac{\partial^2 F}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 F}{\partial s\partial t}(b,d)\right|+\left|\frac{\partial^2 F}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 F}{\partial s\partial t}(b,c)\right|\right]$$

where

$$\begin{aligned}
&\mathcal{K}^{\alpha,\beta}(a,b,c,d)\\
= {} & F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&-\frac{\Gamma(\beta+1)2^{\beta-1}}{(d-c)^{\beta}}\left[\mathcal{J}^{\beta}_{\left(\frac{c+d}{2}\right)^-}F\left(\frac{a+b}{2},c\right)+\mathcal{J}^{\beta}_{\left(\frac{c+d}{2}\right)^+}F\left(\frac{a+b}{2},d\right)\right]\\
&-\frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b-a)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}F\left(a,\frac{c+d}{2}\right)+\mathcal{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}F\left(b,\frac{c+d}{2}\right)\right.\\
&+\frac{2^{2-(\alpha+\beta)}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{2-\alpha}(d-c)^{2-\beta}}\\
&\times\left[\mathcal{J}^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^-,\left(\frac{c+d}{2}\right)^-}F(a,c)+\mathcal{J}^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^+,\left(\frac{c+d}{2}\right)^+}F(b,d)\right.\\
&\left.+\mathcal{J}^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^-,\left(\frac{c+d}{2}\right)^+}F(a,d)+\mathcal{J}^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^+,\left(\frac{c+d}{2}\right)^-}F(b,c)\right].
\end{aligned}$$

Remark 2.1. If we choose $\alpha=\beta=1$ in Corollary 2.1, then Corollary 2.1 reduces to [11, Theorem 2].

Theorem 2.2. *Let $F:\Delta\to\mathbb{R}$ be twice partially differentiable mapping on Δ°. If $\frac{\partial^2 F}{\partial s\partial t}\in L(\Delta)$ and $\left|\frac{\partial^2 F}{\partial s\partial t}\right|^q$, $q\geq 1$, is a co-ordinated convex, then for all $(\varkappa,\gamma)\in\Delta$*

we have the following inequality for fractional integrals,

$$
\begin{aligned}
&\left|\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)\right| \\
&\leq \tfrac{1}{(b-a)(d-c)}\left(\tfrac{1}{(\alpha p+1)(\beta p+1)}\right)^{\frac{1}{p}}\Big[(b-\varkappa)^2(d-\gamma)^2 \\
&\times\left(\tfrac{\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c+d-\gamma)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(a,c+d-\gamma)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(a,c)\right|^q}{4}\right)^{\frac{1}{q}} \\
&+(\varkappa-a)^2(\gamma-c)^2 \\
&\times\left(\tfrac{\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c+d-\gamma)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,d)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(b,c+d-\gamma)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(b,d)\right|^q}{4}\right)^{\frac{1}{q}} \\
&+(b-\varkappa)^2(\gamma-c)^2 \\
&\times\left(\tfrac{\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c+d-\gamma)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,d)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(a,c+d-\gamma)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(a,d)\right|^q}{4}\right)^{\frac{1}{q}} \\
&+(\varkappa-a)^2(d-\gamma)^2 \\
&\times\left(\tfrac{\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c+d-\gamma)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(b,c+d-\gamma)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(b,c)\right|^q}{4}\right)^{\frac{1}{q}}\Big]
\end{aligned}
$$

where $\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)$ *is defined as in (2.2) and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Hölder's inequality, we have

$$
\begin{aligned}
(2.13)\ &\left|\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)\right| \\
\leq\ & \frac{(b-\varkappa)^2(d-\gamma)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|t^\alpha s^\beta\right|^p dtds\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}(t(a+b-\varkappa)+(1-t)a, s(c+d-\gamma)+(1-s)c)\right|^q dtds\right)^{\frac{1}{q}} \\
&+\frac{(\varkappa-a)^2(\gamma-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|t^\alpha s^\beta\right|^p dtds\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}(t(a+b-\varkappa)+(1-t)b, s(c+d-\gamma)+(1-s)d)\right|^q dtds\right)^{\frac{1}{q}} \\
&+\frac{(b-\varkappa)^2(\gamma-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|t^\alpha s^\beta\right|^p dtds\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}(t(a+b-\varkappa)+(1-t)b, s(c+d-\gamma)+(1-s)d)\right|^q dtds\right)^{\frac{1}{q}} \\
&+\frac{(\varkappa-a)^2(d-\gamma)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|t^\alpha s^\beta\right|^p dtds\right)^{\frac{1}{p}}
\end{aligned}
$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)b,s\left(c+d-\gamma\right)+\left(1-s\right)c\right)\right|^q dtds\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^2 F}{\partial s\partial t}\right|^q$ is a co-ordinated convexity on Δ, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)a,s\left(c+d-\gamma\right)+\left(1-s\right)c\right)\right|^q dtds \\
\leq\ &\int_0^1\int_0^1 ts\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa,c+d-\gamma\right)\right|^q dtds \\
&+\int_0^1\int_0^1 t\left(1-s\right)\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa,c\right)\right|^q dtds \\
&+\int_0^1\int_0^1 \left(1-t\right)s\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,c+d-\gamma\right)\right|^q dtds \\
&+\int_0^1\int_0^1 \left(1-t\right)\left(1-s\right)\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,c\right)\right|^q dtds \\
=\ &\frac{\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa,c+d-\gamma\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa,c\right)\right|^q}{4} \\
&+\frac{\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,c+d-\gamma\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,c\right)\right|^q}{4}.
\end{aligned}\tag{2.14}$$

Similarly we get

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)b,s\left(c+d-\gamma\right)+\left(1-s\right)d\right)\right|^q dtds \\
\leq\ &\frac{\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa,c+d-\gamma\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa,d\right)\right|^q}{4}, \\
&+\frac{\left|\frac{\partial^2 F}{\partial s\partial t}\left(b,c+d-\gamma\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(b,d\right)\right|^q}{4}
\end{aligned}\tag{2.15}$$

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)b,s\left(c+d-\gamma\right)+\left(1-s\right)d\right)\right|^q dtds \\
\leq\ &\frac{\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa,c+d-\gamma\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa,d\right)\right|^q}{4} \\
&+\frac{\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,c+d-\gamma\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,d\right)\right|^q}{4},
\end{aligned}\tag{2.16}$$

and

$$\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)b,s\left(c+d-\gamma\right)+\left(1-s\right)c\right)\right|^q dtds\tag{2.17}$$

$$\leq \frac{\left|\frac{\partial^2 F}{\partial s \partial t}(a+b-\varkappa, c+d-\gamma)\right|^q + \left|\frac{\partial^2 F}{\partial s \partial t}(a+b-\varkappa, c)\right|^q}{4} + \frac{\left|\frac{\partial^2 F}{\partial s \partial t}(b, c+d-\gamma)\right|^q + \left|\frac{\partial^2 F}{\partial s \partial t}(b, c)\right|^q}{4}.$$

If we substitute the inequalities (2.14)-(2.17) in (2.13), then we obtain the desired result. □

Corollary 2.2. *Under assumption of Theorem* 2.2 *with* $\varkappa = \frac{a+b}{2}$ *and* $\gamma = \frac{c+d}{2}$, *we have the following midpoint type inequality,*

$$\begin{aligned} &\left|\mathcal{K}^{\alpha,\beta}(a,b,c,d)\right| \\ &\leq \tfrac{(b-a)(d-c)}{16}\left(\tfrac{1}{(\alpha p+1)(\beta p+1)}\right)^{\frac{1}{p}} \\ &\times\left[\left(\tfrac{\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(a,c)\right|^q}{4}\right)^{\frac{1}{q}}\right. \\ &+\left(\tfrac{\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(b,d)\right|^q}{4}\right)^{\frac{1}{q}} \\ &+\left(\tfrac{\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(a,d)\right|^q}{4}\right)^{\frac{1}{q}} \\ &\left.+\left(\tfrac{\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 F}{\partial s\partial t}(b,c)\right|^q}{4}\right)^{\frac{1}{q}}\right] \end{aligned} \tag{2.18}$$

where $\mathcal{K}^{\alpha,\beta}(a,b,c,d)$ *is defined as in (2.18).*

Theorem 2.3. *Let* $F : \Delta \to \mathbb{R}$ *be twice partially differentiable mapping on* Δ°. *If* $\frac{\partial^2 F}{\partial s\partial t} \in L(\Delta)$ *and* $\left|\frac{\partial^2 F}{\partial s\partial t}\right|^q$, $q \geq 1$, *is a co-ordinated convex then for all* $(\varkappa, \gamma) \in \Delta$ *we have the following inequality for fractional integrals,*

$$\begin{aligned} &\left|\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)\right| \\ \leq\ & \frac{1}{(b-a)(d-c)}\left(\frac{1}{(\alpha+1)(\beta+1)}\right)^{1-\frac{1}{q}} \times \left\{(b-\varkappa)^2(d-\gamma)^2\right. \\ &\times\left(\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa, c+d-\gamma)\right|^q\right. \\ &+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa, c)\right|^q \\ &+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a, c+d-\gamma)\right|^q \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a,c)\right|^q\Bigg)^{\frac{1}{q}}\\
&+(\varkappa-a)^2(\gamma-c)^2\\
&\times\Bigg(\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c+d-\gamma)\right|^q\\
&+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,d)\right|^q\\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(b,c+d-\gamma)\right|^q\\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(b,d)\right|^q\Bigg)^{\frac{1}{q}}\\
&+(b-\varkappa)^2(\gamma-c)^2\\
&\times\Bigg(\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c+d-\gamma)\right|^q\\
&+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,d)\right|^q\\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a,c+d-\gamma)\right|^q\\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a,d)\right|^q\Bigg)^{\frac{1}{q}}\\
&+(\varkappa-a)^2(d-\gamma)^2\\
&\times\Bigg(\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c+d-\gamma)\right|^q\\
&+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a+b-\varkappa,c)\right|^q\\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(b,c+d-\gamma)\right|^q\\
&+\ \frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(b,c)\right|^q\Bigg)^{\frac{1}{q}}\Bigg\}
\end{aligned}
$$

where $\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)$ *is defined as in (2.2).*

Proof. By using power-mean inequality, we have

$$
\begin{aligned}
&(2.19)\\
&\left|\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)\right|\\
\leq\ & \frac{(b-\varkappa)^2(d-\gamma)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|t^\alpha s^\beta\right|dtds\right)^{1-\frac{1}{q}}
\end{aligned}
$$

$$\times\left(\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)a, s\left(c+d-\gamma\right)+\left(1-s\right)c\right)\right|^q dtds\right)^{\frac{1}{q}}$$
$$+\frac{\left(\varkappa-a\right)^2\left(\gamma-c\right)^2}{\left(b-a\right)\left(d-c\right)}\left(\int_0^1\int_0^1\left|t^{\alpha}s^{\beta}\right|dtds\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)b, s\left(c+d-\gamma\right)+\left(1-s\right)d\right)\right|^q dtds\right)^{\frac{1}{q}}$$
$$+\frac{\left(b-\varkappa\right)^2\left(\gamma-c\right)^2}{\left(b-a\right)\left(d-c\right)}\left(\int_0^1\int_0^1\left|t^{\alpha}s^{\beta}\right|dtds\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)a, s\left(c+d-\gamma\right)+\left(1-s\right)d\right)\right|^q dtds\right)^{\frac{1}{q}}$$
$$+\frac{\left(\varkappa-a\right)^2\left(d-\gamma\right)^2}{\left(b-a\right)\left(d-c\right)}\left(\int_0^1\int_0^1\left|t^{\alpha}s^{\beta}\right|dtds\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)b, s\left(c+d-\gamma\right)+\left(1-s\right)c\right)\right|^q dsdt\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^2 F}{\partial s\partial t}\right|^q$, $q\geq 1$, is a co-ordinated convexity on Δ, we obtain

$$\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)a, s\left(c+d-\gamma\right)+\left(1-s\right)c\right)\right|^q dtds \tag{2.20}$$
$$\begin{aligned}
\leq\ & \int_0^1\int_0^1 t^{\alpha+1}s^{\beta+1}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa, c+d-\gamma\right)\right|^q dtds \\
& +\int_0^1\int_0^1 t^{\alpha+1}s^{\beta}\left(1-s\right)\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa, c\right)\right|^q dtds \\
& +\int_0^1\int_0^1 t^{\alpha}\left(1-t\right)s^{\beta+1}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a, c+d-\gamma\right)\right|^q dtds \\
& +\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left(1-t\right)\left(1-s\right)\left|\frac{\partial^2 F}{\partial s\partial t}\left(a, c\right)\right|^q dtds \\
=\ & \frac{1}{\left(\alpha+2\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa, c+d-\gamma\right)\right|^q \\
& +\frac{1}{\left(\alpha+2\right)\left(\beta+1\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa, c\right)\right|^q \\
& +\frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a, c+d-\gamma\right)\right|^q \\
& +\frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta+1\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a, c\right)\right|^q.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}(2.21)\quad &\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)b, s\left(c+d-\gamma\right)+\left(1-s\right)d\right)\right|^q dtds\\
\leq\ & \frac{1}{\left(\alpha+2\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa, c+d-\gamma\right)\right|^q\\
&+\frac{1}{\left(\alpha+2\right)\left(\beta+1\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa, d\right)\right|^q\\
&+\frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(b, c+d-\gamma\right)\right|^q\\
&+\frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta+1\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(b, d\right)\right|^q,\end{aligned}$$

$$\begin{aligned}(2.22)\quad &\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)a, s\left(c+d-\gamma\right)+\left(1-s\right)d\right)\right|^q dtds\\
\leq\ & \frac{1}{\left(\alpha+2\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa, c+d-\gamma\right)\right|^q\\
&+\frac{1}{\left(\alpha+2\right)\left(\beta+1\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa, d\right)\right|^q\\
&+\frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a, c+d-\gamma\right)\right|^q\\
&+\frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta+1\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a, d\right)\right|^q,\end{aligned}$$

and

$$\begin{aligned}(2.23)\quad &\int_0^1\int_0^1 t^{\alpha}s^{\beta}\left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+\left(1-t\right)b, s\left(c+d-\gamma\right)+\left(1-s\right)c\right)\right|^q dtds\\
\leq\ & \frac{1}{\left(\alpha+2\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa, c+d-\gamma\right)\right|^q\\
&+\frac{1}{\left(\alpha+2\right)\left(\beta+1\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a+b-\varkappa, c\right)\right|^q\\
&+\frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(b, c+d-\gamma\right)\right|^q\\
&+\frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta+1\right)\left(\beta+2\right)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(b, c\right)\right|^q.\end{aligned}$$

If we substitue the inequalities (2.20)-(2.23) in (2.19), then we obtain the desired result. □

Corollary 2.3. *Under assumption of Theorem 2.3 with $\varkappa = \frac{a+b}{2}$ and $\gamma = \frac{c+d}{2}$, we have the following midpoint type inequality,*

$$
\begin{aligned}
&\left|\mathcal{K}^{\alpha,\beta}(a,b,c,d)\right| \\
\leq\ & \frac{1}{(b-a)(d-c)}\left(\frac{1}{(\alpha+1)(\beta+1)}\right)^{1-\frac{1}{q}} \\
&\times\left\{(b-\varkappa)^2(d-\gamma)^2\left(\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right.\right. \\
&+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q \\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q \\
&\left.+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a,c)\right|^q\right)^{\frac{1}{q}} \\
&+(\varkappa-a)^2(\gamma-c)^2\left(\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right. \\
&+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q \\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q \\
&\left.+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(b,d)\right|^q\right)^{\frac{1}{q}} \\
&+(b-\varkappa)^2(\gamma-c)^2\left(\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right. \\
&+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q \\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q \\
&\left.+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(a,d)\right|^q\right)^{\frac{1}{q}} \\
&+(\varkappa-a)^2(d-\gamma)^2\left(\frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right. \\
&+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q \\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q
\end{aligned}
$$

$$+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 F}{\partial s\partial t}(b,c)\right|^q\Bigg)^{\frac{1}{q}}\Bigg\}$$

where $\mathcal{K}^{\alpha,\beta}(a,b,c,d)$ *is defined as in (2.18).*

3. Generalized Midpoint-type Inequalities for Co-ordinated Concave Functions

In this section, we prove some midpoint-type inequalities for co-ordinated cancave functions by utulizing Jensen inequality.

Theorem 3.1. *Let* $F:\Delta\to\mathbb{R}$ *be twice partially differentiable mapping on* Δ°*.If* $\frac{\partial^2 F}{\partial s\partial t}\in L(\Delta)$ *and* $\left|\frac{\partial^2 F}{\partial s\partial t}\right|^q$, $q\geq 1$, *is a co-ordinated concave then for all* $(\varkappa,\gamma)\in\Delta$ *we have the following inequality for fractional integrals,*

$$\begin{aligned}
&\left|\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)\right|\\
\leq\;&\left(\frac{1}{(\alpha p+1)(\beta p+1)}\right)^{\frac{1}{p}}\\
&\times\left[\frac{(b-\varkappa)^2(d-\gamma)^2}{(b-a)(d-c)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{2a+b-\varkappa}{2},\frac{2c+d-\gamma}{2}\right)\right|\right.\\
&+\frac{(\varkappa-a)^2(\gamma-c)^2}{(b-a)(d-c)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+2b-\varkappa}{2},\frac{c+2d-\gamma}{2}\right)\right|\\
&+\frac{(b-\varkappa)^2(\gamma-c)^2}{(b-a)(d-c)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{2a+b-\varkappa}{2},\frac{c+2d-\gamma}{2}\right)\right|\\
&\left.+\frac{(\varkappa-a)^2(d-\gamma)^2}{(b-a)(d-c)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+2b-\varkappa}{2},\frac{2c+d-\gamma}{2}\right)\right|\right]
\end{aligned}$$

where $\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)$ *is defined as in (2.2) and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $\left|\frac{\partial^2 F}{\partial s\partial t}\right|^q$, $q\geq 1$, is a co-ordinated concave on Δ, by using Jensen integral inequality, we obtain

$$\begin{aligned}
(3.1)\quad&\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}(t(a+b-\varkappa)+(1-t)a,s(c+d-\gamma)+(1-s)c)\right|^q dtds\\
=\;&\int_0^1\int_0^1 t^0s^0\left|\frac{\partial^2 F}{\partial s\partial t}(t(a+b-\varkappa)+(1-t)a,s(c+d-\gamma)+(1-s)c)\right|^q dtds\\
\leq\;&\left(\int_0^1\int_0^1 t^0s^0dtds\right)
\end{aligned}$$

$$\times\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{1}{\int_0^1\int_0^1 t^0 s^0 dtds}\int_0^1\int_0^1 (t(a+b-\varkappa)+(1-t)a)\,dtds\right.\right.$$
$$\left.\left.,\frac{1}{\int_0^1\int_0^1 t^0 s^0 dtds}\int_0^1\int_0^1 (s(c+d-\gamma)+(1-s)c)\,dsdt\right)\right|^q$$
$$=\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{2a+b-\varkappa}{2},\frac{2c+d-\gamma}{2}\right)\right|^q.$$

Similarly, we have

$$\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}(t(a+b-\varkappa)+(1-t)b,s(c+d-\gamma)+(1-s)d)\right|^q dtds \tag{3.2}$$
$$\leq \left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+2b-\varkappa}{2},\frac{c+2d-\gamma}{2}\right)\right|^q$$

$$\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}(t(a+b-\varkappa)+(1-t)a,s(c+d-\gamma)+(1-s)d)\right|^q dtds \tag{3.3}$$
$$\leq \left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{2a+b-\varkappa}{2},\frac{c+2d-\gamma}{2}\right)\right|^q$$

$$\int_0^1\int_0^1\left|\frac{\partial^2 F}{\partial s\partial t}(t(a+b-\varkappa)+(1-t)b,s(c+d-\gamma)+(1-s)c)\right|^q dsdt \tag{3.4}$$
$$\leq \left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+2b-\varkappa}{2},\frac{2c+d-\gamma}{2}\right)\right|^q.$$

If we substitue the inequalities (3.1)-(3.4) in (2.13) , we obtain the desired result. □

Corollary 3.1. *Under assumption of Theorem* 3.1 *with* $\varkappa=\frac{a+b}{2}$ *and* $\gamma=\frac{c+d}{2}$, *we have the following midpoint type inequality*

$$\begin{aligned}&\left|\mathcal{K}^{\alpha,\beta}(a,b,c,d)\right|\\ \leq\ &\frac{(b-a)(d-c)}{16}\left(\frac{1}{(\alpha p+1)(\beta p+1)}\right)^{\frac{1}{p}}\\ &\times\left[\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{3a+b}{2},\frac{3c+d}{2}\right)\right|+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+3b}{2},\frac{c+3d}{2}\right)\right|\right.\\ &\left.+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{3a+b}{2},\frac{c+3d}{2}\right)\right|+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{a+3b}{2},\frac{3c+d}{2}\right)\right|\right]\end{aligned}$$

where $\mathcal{K}^{\alpha,\beta}(a,b,c,d)$ *is defined as in (2.18).*

Theorem 3.2. *Let $\digamma : \Delta \to \mathbb{R}$ be twice partially differentiable mapping on Δ°. If $\frac{\partial^2 \digamma}{\partial s \partial t} \in L(\Delta)$ and $\left|\frac{\partial^2 \digamma}{\partial s \partial t}\right|^q$, $q \geq 1$, is a co-ordinated concave then for all $(\varkappa, \gamma) \in \Delta$ we have the following inequality for fractional integrals,*

$$
\begin{aligned}
&\left|\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)\right| \\
\leq\ & \left(\frac{1}{(\alpha+1)(\beta+1)}\right)^{2-\frac{1}{q}} \\
&\times\left[\frac{(b-\varkappa)^2(d-\gamma)^2}{(b-a)(d-c)}\left|\frac{\partial^2 \digamma}{\partial s \partial t}\left(\frac{\alpha+1}{\alpha+2}(b-\varkappa)+a, \frac{\beta+1}{\beta+2}(d-\gamma)+c\right)\right|\right. \\
&+\frac{(\varkappa-a)^2(\gamma-c)^2}{(b-a)(d-c)}\left|\frac{\partial^2 \digamma}{\partial s \partial t}\left(\frac{\alpha+1}{\alpha+2}(a-\varkappa)+b, \frac{\beta+1}{\beta+2}(c-\gamma)+d\right)\right| \\
&+\frac{(b-\varkappa)^2(\gamma-c)^2}{(b-a)(d-c)}\left|\frac{\partial^2 \digamma}{\partial s \partial t}\left(\frac{\alpha+1}{\alpha+2}(b-\varkappa)+a, \frac{\beta+1}{\beta+2}(c-\gamma)+d\right)\right| \\
&\left.+\frac{(\varkappa-a)^2(d-\gamma)^2}{(b-a)(d-c)}\left|\frac{\partial^2 \digamma}{\partial s \partial t}\left(\frac{\alpha+1}{\alpha+2}(a-\varkappa)+b, \frac{\beta+1}{\beta+2}(d-\gamma)+c\right)\right|\right]
\end{aligned}
$$

where $\mathcal{H}^{\alpha,\beta}(a,b,c,d;\varkappa,\gamma)$ is defined as in (2.2).

Proof. Since $\left|\frac{\partial^2 \digamma}{\partial s \partial t}\right|^q$ is a co-ordinated concave on Δ, by using Jensen integral inequality, we obtain

$$
\begin{aligned}
(3.5)\quad & \int_0^1\int_0^1 \left|t^\alpha s^\beta\right| \left|\frac{\partial^2 \digamma}{\partial s \partial t}\left(t(a+b-\varkappa)+(1-t)a, s(c+d-\gamma)+(1-s)c\right)\right|^q dtds \\
\leq\ & \left(\int_0^1\int_0^1 t^\alpha s^\beta dsdt\right) \\
&\times\left|\frac{\partial^2 \digamma}{\partial s \partial t}\left(\frac{1}{\left(\int_0^1\int_0^1 t^\alpha s^\beta dsdt\right)}\int_0^1\int_0^1 t^\alpha\left(t(a+b-\varkappa)+(1-t)a\right)dtds\right.\right. \\
&\left.\left., \frac{1}{\left(\int_0^1\int_0^1 t^\alpha s^\beta dsdt\right)}\int_0^1\int_0^1 s^\beta\left(s(c+d-\gamma)+(1-s)c\right)dtds\right)\right|^q \\
=\ & \frac{1}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 \digamma}{\partial s \partial t}\left(\frac{\alpha+1}{\alpha+2}(b-\varkappa)+a, \frac{\beta+1}{\beta+2}(d-\gamma)+c\right)\right|^q.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
(3.6)\quad & \int_0^1\int_0^1 \left|t^\alpha s^\beta\right| \left|\frac{\partial^2 \digamma}{\partial s \partial t}\left(t(a+b-\varkappa)+(1-t)b, s(c+d-\gamma)+(1-s)d\right)\right|^q dtds \\
\leq\ & \frac{1}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 \digamma}{\partial s \partial t}\left(\frac{\alpha+1}{\alpha+2}(a-\varkappa)+b, \frac{\beta+1}{\beta+2}(c-\gamma)+d\right)\right|^q,
\end{aligned}
$$

$$\int_0^1\int_0^1 \left|t^\alpha s^\beta\right| \left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+(1-t)\,a, s\left(c+d-\gamma\right)+(1-s)\,d\right)\right|^q dtds \tag{3.7}$$
$$\leq \frac{1}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{\alpha+1}{\alpha+2}(b-\varkappa)+a, \frac{\beta+1}{\beta+2}(c-\gamma)+d\right)\right|^q,$$

$$\int_0^1\int_0^1 \left|t^\alpha s^\beta\right| \left|\frac{\partial^2 F}{\partial s\partial t}\left(t\left(a+b-\varkappa\right)+(1-t)\,b, s\left(c+d-\gamma\right)+(1-s)\,c\right)\right|^q dtds \tag{3.8}$$
$$\leq \frac{1}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{\alpha+1}{\alpha+2}(a-\varkappa)+b, \frac{\beta+1}{\beta+2}(d-\gamma)+c\right)\right|^q.$$

If we substitute the inequalities (3.5)-(3.8) in (2.19), then we obtain the desired result. □

Corollary 3.2. *Under assumption of Theorem* 3.2 *with* $\varkappa=\frac{a+b}{2}$ *and* $\gamma=\frac{c+d}{2}$, *we have the following midpoint type inequality*

$$\begin{aligned}
&\left|\mathcal{K}^{\alpha,\beta}(a,b,c,d)\right| \\
\leq\ & \left(\frac{1}{(\alpha+1)(\beta+1)}\right)^{2-\frac{1}{q}}\frac{(b-a)(d-c)}{16} \\
&\times\left[\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{(\alpha+3)\,a+(\alpha+1)\,b}{2(\alpha+2)}, \frac{\beta+1}{\beta+2}(d-\gamma)+c\right)\right|\right. \\
&+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{(\alpha+3)\,b+(\alpha+1)\,a}{2(\alpha+2)}, \frac{\beta+1}{\beta+2}(c-\gamma)+d\right)\right| \\
&+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{(\alpha+3)\,a+(\alpha+1)\,b}{2(\alpha+2)}, \frac{\beta+1}{\beta+2}(c-\gamma)+d\right)\right| \\
&\left.+\left|\frac{\partial^2 F}{\partial s\partial t}\left(\frac{(\alpha+3)\,b+(\alpha+1)\,a}{2(\alpha+2)}, \frac{\beta+1}{\beta+2}(d-\gamma)+c\right)\right|\right]
\end{aligned}$$

where $\mathcal{K}^{\alpha,\beta}(a,b,c,d)$ *is defined as in (2.18).*

4. Conclusion

We prove some new midpoint-type integral inequalities for twice partially differentiable co-ordinated convex functions. Furthermore, by considering the special cases of newly proven inequalities, we were able to obtain some new midpoint-type integral inequalities. It is an exciting and new problem and researchers have proven similar inequalities for different types of convexity or different kinds of fractional integrals in their future works.

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