FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 2 (2023), 429–435
https://doi.org/10.22190/FUMI220606028A
**Original Scientific Paper**

# ULAM TYPE STABILITY FOR A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

**Serkan Aslıyüce and Süleyman Öğrekçi**

**Amasya University, Science & Arts Faculty**
**Department of Mathematics, 05100 Amasya, Turkey**

**Abstract.** In this paper, we are concerned with the stability problem of a general class of second order nonlinear differential equations in the sense of Hyers-Ulam-Rassias and Hyers-Ulam. In our proofs, we show that some of the common restrictions widely used in well-known papers that deal with similar problems on bounded intervals are unnecessary. Therefore, we obtain stability results for second order differential equations with few assumptions on bounded intervals.
**Keywords**: Hyers-Ulam-Rassias stability, differential equations, generalized metric.

## 1. Introduction

Hyers-Ulam stability, initiated with a talk of S. M. Ulam [26] at Wisconsin University in 1940 and the response of D. H. Hyers [5] in 1941, is a subject that provides an approximate solution for the exact solution in a simple form for differential equations. In 1978, T. Rassias [21] provided a remarkable generalization, which known as Hyers-Ulam-Rassias stability today.

Ulam type stability problem of differential equations were initiated by the papers of M. Obloza [10, 11]. Later C. Alsina and R. Ger [1] established the Hyers-Ulam stability of the differential equation $y'(t) = y(t)$ on an open interval. These remarkable results were later extended by S. H. Takahasi, T. Miura and S. Miyajima [24] to the equation $y'(t) = \lambda y(t)$ in Banach spaces, and [8, 9] to higher order linear differential equations with constant coefficients. S. M. Jung [6] proved Hyers-Ulam

Received June 06, 2022, accepted: July 21, 2023
Communicated by Praveen Agarwal
Corresponding Author: Serkan Aslıyüce, Amasya University, Science & Arts Faculty, Department of Mathematics, 05100 Amasya, Turkey | E-mail: serkan.asliyuce@amasya.edu.tr
2010 *Mathematics Subject Classification.* 34A34, 34D20, 55M20

stability and Hyers-Ulam-Rassias stability of the equation $y' = f(t, y)$ which extends the theory to nonlinear case. After these investigations, a number of papers devoted to this subject have been published (see e.g. [3, 2, 7, 12, 13, 14, 22, 23] and references therein).

In this paper, we consider the stability problem of the second order initial value problem (IVP) in the form of

$$x''(t) + \mu^2 x(t) = f(t, x(t)),\ \mu > 0 \tag{1.1}$$

with initial conditions

$$x(0) = 0,\ x'(0) = 1 \tag{1.2}$$

in the sense of Hyers-Ulam-Rassias and Hyers-Ulam.

## 2. Preliminaries

Let $I$ be an open interval. For some $\varepsilon > 0$ and $y \in C^2(I)$ satisfying

$$\left| x''(t) + \mu^2 x(t) - f(t, x(t)) \right| \leq \varepsilon$$

and initial conditions (1.2), if there exists a solution y0 of the initial value problem (1.1)-(1.2) such that

$$|y(t) - y_0(t)| \leq K\varepsilon,$$

where $K$ is a constant which does not depend on $\varepsilon$ and $y$, then the differential equation (1.1) is said to be stable in the sense of Hyers-Ulam. If the above statement remains true after replacing the constants $\varepsilon$ and $K$ with the functions $\varphi, \Phi : I \to [0, \infty)$ respectively, where these functions does not depend on $y$ and $y_0$, then the differential equation (1.1) is said to be stable in the sense of Hyers-Ulam-Rassias. This definition may be applied to different classes of differential equations, we refer to Jung [6] and references cited therein for more detailed definitions of Hyers-Ulam stability and Hyers-Ulam-Rassias stability.

We now introduce the concept of generalized metric which will be employed in proofs of our main results. For a nonempty set $X$, a function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if satisfies

**(M1)** $d(x, y) = 0$ if and only if $x = y$,

**(M2)** $d(x, y) = d(y, x)$ for all $x, y \in X$,

**(M3)** $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

It should be remarked that the only difference of the generalized metric from the usual metric is that the range of the former is permitted to be an unbounded interval.

We will use the following fixed point result as a main tool in our proofs, we refer to [4] for the proof of this result. One can see [15, 16, 17, 18, 19, 20] for some examples of fixed-point techniques on differential equations.

**Theorem 2.1.** *Let $(X, d)$ be a generalized complete metric space. Assume that $T : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there is a nonnegative integer $k$ such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following are true:*

*(a) The sequence $\{T^n x\}$ converges to a fixed point $x^*$ of $T$,*

*(b) $x^*$ is the unique fixed point of $T$ in*

$$X^* = \{y \in X : d(T^k x, y) < \infty\},$$

*(c) If $y \in X^*$, then*

$$d(y, x^*) \leq \frac{1}{1-L} d(Ty, y).$$

## 3. Main Results

Throughout this section we define $I := [t_o, t_0 + r]$ for given real numbers $t_0$ and $r$ with $r > 0$. Further, we define the set $S$ of all continuous functions on $I$ by

$$S := \{f : I \to \mathbb{R} \,|\, \text{f is continuous}\} = C(I, \mathbb{R}). \tag{3.1}$$

We will use the following result in our proofs.

**Lemma 3.1.** [3] *Define the function $d : S \times S \to [0, \infty]$ with*

$$d(f, g) := \inf\{C \in [0, \infty] : |f(t) - g(t)|\, e^{-M(t-t_0)} \leq C\Phi(t),\ t \in I\} \tag{3.2}$$

*where $M > 0$ is a given constant and $\Phi : I \to (0, \infty)$ is a given continuous function. Then $(S, d)$ is a generalized complete metric space.*

We are now ready to study stability of differential equation (1.1) in the sense of Hyers-Ulam.

**Theorem 3.1.** *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition*

$$|f(t, y_1) - f(t, y_2)| \leq L\, |y_1 - y_2| \tag{3.3}$$

*for all $t \in I$ and all $y_1, y_2 \in \mathbb{R}$, and some Lipschitz constant $L$ with $L < \mu/r$. If a continuously differentiable function $x : I \to \mathbb{R}$ satisfies*

$$\left|x''(t) + \mu^2 x(t) - f(t, x(t))\right| \leq \varphi(t) \tag{3.4}$$

*and initial conditions (1.2) for all $t \in I$ and a nondecreasing function $\varphi : T \to (0, \infty)$, then there exists a unique solution $x_0$ of (1.1)–(1.2) satisfying*

$$|x(t) - x_0(t)| \leq \frac{\mu r}{\mu - Lr} \varphi(t) \tag{3.5}$$

*for all $t \in I$, that is, the equation (1.1) is stable in the sense of Hyers-Ulam-Rassias.*

*Proof.* Consider the set $S$ defined by (3.1) and introduce the generalized metric $d : S \times S \to [0, \infty]$ with

$$d(f, g) := \inf\{C \in [0, \infty] : |f(t) - g(t)| \leq C\varphi(t),\ t \in I\}$$

in view of Lemma 1, $(S, d)$ is a complete generalized metric space. Now let us define the operator $\Lambda : S \to S$ by

$$(\Lambda x)(t) = \frac{\sin \mu t}{\mu} + \int_{t_0}^{t} \frac{\sin \mu(t-s)}{\mu} f(s, x(s))ds. \tag{3.6}$$

Note that any fixed point of $\Lambda$ solves the IVP (1.1)-(1.2), one can see this correspondence by applying Laplace transform to (1.1) and then applying inverse Laplace transform and the Laplace convolution operator, or multiplying (1.1) with $\sin \mu(t-s)$ and then integrating from $s = t_0$ to $s = t$.

For any $u, v \in S$ let $C_{uv} \in [0, \infty]$ be an arbitrary constant satisfying $d(u, v) \leq C_{uv}$, that is,$|u(t) - v(t)| \leq C_{uv}\varphi(t)$ for all $t \in I$. Then we have,

$$\begin{aligned}
|(\Lambda u)(t) - (\Lambda v)(t)| &= \left| \int_{t_0}^{t} \frac{\sin \mu(t-s)}{\mu} f(s, u(s))ds - \int_{t_0}^{t} \frac{\sin \mu(t-s)}{\mu} f(s, v(s))ds \right| \\
&\leq \int_{t_0}^{t} \left| \frac{\sin \mu(t-s)}{\mu} \right| |f(s, u(s)) - f(s, v(s))|\, ds \\
&\leq \frac{L}{\mu} \int_{t_0}^{t} |u(s) - v(s)|\, ds \\
&\leq \frac{L}{\mu} C_{uv} \int_{t_0}^{t} \varphi(s)ds \\
&\leq \frac{L}{\mu} C_{uv}\varphi(t) \int_{t_0}^{t} ds \\
&\leq \frac{Lr}{\mu} C_{uv}\varphi(t)
\end{aligned}$$

for all $t \in I$, i.e. $d(\Lambda u, \Lambda v) \leq (Lr/\mu)C_{uv}\varphi(t)$. Therefore, for all $u, v \in S$, we conclude that

$$d(\Lambda u, \Lambda v) \leq \frac{Lr}{\mu} d(u, v),$$

that is, the operator $\Lambda : S \to S$ is strictly contractive.

It follows from (3.6) that for arbitrary $u_0 \in S$, we can find a constant $C_0 < \infty$ such that

$$|(\Lambda u_0)(t) - u_0(t)| = \left| \frac{\sin \mu t}{\mu} + \int_{t_0}^{t} \frac{\sin \mu(t-s)}{\mu} f(s, u_0(s))ds - u_0 \right| \leq C_0 \varphi(t)$$

for all $t \in I$, since $f(t, u_0(t))$ and $u_0(t)$ are bounded on $I$ and $\min_{t \in I} \varphi(t) > 0$. For any $u \in S$, since $u_0$ and $u$ are bounded on $I$, there exists a constant $C_u < \infty$ such that $|u(t) - u_o(t)| \leq C_u \varphi(t)$ for all $t \in I$. Hence, we have $d(u_0, u) < \infty$ for all $u \in S$, and therefore we obtained $\{u \in S : d(u_0, u) < \infty\} = S$. Thus, in view of Theorem 1, there exists a unique solution $x_0 : I \to \mathbb{R}$ of the IVP (1.1)-(1.2).

In the other hand, from the inequality (3.4), we have

$$-\varphi(t) \leq x''(t) + \mu^2 x(t) - f(t, x(t)) \leq \varphi(t)$$

for all $t \in I$. Multiplying this inequality with $\sin \mu(t-s)$ and integrating from $s = t_0$ to $s = t$, after some calculation we obtain

$$|(\Lambda x)(t) - x(t)| \leq \left| \int_{t_0}^{t} \sin \mu(t-s) \varphi(s) ds \right| \leq \varphi(t) \int_{t_0}^{t} ds \leq \varphi(t) r$$

for all $t \in I$, which implies that

$$d(\Lambda x, x) \leq r.$$

From Theorem 1 and above inequality, we conclude that

$$d(x, x_0) \leq \frac{1}{1 - Lr/\mu} d(\Lambda x, x) \leq \frac{r}{1 - Lr/\mu}$$

which implies (3.5) and completes the proof. □

By taking $\varphi(t) = \varepsilon$, we obtain Hyers-Ulam stability of the IVP (1.1)-(1.2). We state this result as

a corollary of Theorem 2.

**Corollary 3.1.** *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition* (3.3) *for all $t \in I$, all $y_1, y_2 \in \mathbb{R}$, and some Lipschitz constant $L$ with $L < \mu/r$. If a continuously differentiable function $x : I \to \mathbb{R}$ satisfies*

$$\left| x''(t) + \mu^2 x(t) - f(t, x(t)) \right| \leq \varepsilon$$

*and initial conditions* (1.2) *for all $t \in I$ and some $\varepsilon > 0$, then there exists a unique solution $x_0$ of* (1.1)–(1.2) *satisfying*

$$|x(t) - x_0(t)| \leq \frac{\mu r}{\mu - Lr} \varepsilon$$

*for all $t \in I$, that is, the equation* (1.1) *is stable in the sense of Hyers-Ulam.*

**Remark 3.1.** We note that, in the proof of Theorem 2, we do not impose any integral restriction on the function $\varphi : I \to \mathbb{R}$ such as

$$\left| \int_{t_0}^{t} \varphi(s)ds \right| \leq K\varphi(t) \tag{3.7}$$

for all $t \in I$ and some $K > 0$. This kind of restriction is widely assumed in similar problems in the literature, see for example [3, 6, 12, 25].

## 4. Conclusions

In this study, we are concerned with the stability problem of the second order nonlinear differential equations in the form of (1.1) subject to initial condition (1.2) in the sense of Hyers-Ulam-Rassias and Hyers-Ulam. By Theorem 2, we prove the Hyers-Ulam-Rassias stability of this problem with very few assumptions. As a corollary of this theorem, by Corollary 3.3, we provide a result on Hyers-Ulam stability of (1.1)–(1.2).

Stability problem of higher order differential equations is planned to be investigated in our future works.

## REFERENCES

1. C. Alsine and R. Ger: *On Some Inequalities and Stability Results Related to the Exponential Function.* J. Inequal. Appl. **2** (1998), 373–380.
2. Y. Başci S. Öğrekçi and A. Misir: *Hyers-Ulam-Rassias Stability for Abel-Riccati Type First-Order Differential Equations.* In: GU. J. Sci. **32(4)** (2019), pp. 1238–1252.
3. Y. Başci A. Misir and S. Öğrekçi: *On the Stability Problem of Differential Equations in the Sense of Ulam.* Results Math.**75(6)** (2020), doi:10.1007/s00025-019-1132-6.
4. J. B. Diaz and B. Margolis: *A fixed point theorem of alternative, for contractions on a generalized complete metric space.* Bull. Amer. Math. Soc. **74** (1968), 305–309.
5. D. H. Hyers: *On the Stability of the Linear Functional Equation.* Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
6. E. B. Saff and R. S. Varga: Jung, S. M. *A fixed point approach to the stability of differential equations $y' = F(x; y)$.* Bull. Malays. Math. Sci. Soc. **33(1)** (2010), 47–56.
7. S. M. Jung: *Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients.* Journal of Mathematical Analysis and Applications **320** (2006), 549–561.
8. T. Miura, S. Miyajima and S. H. Takahasi: *A Characterization of Hyers-Ulam Stability of First Order Linear Differential Operators.* J. Math. Anal. Appl. **286** (2003), pp. 136-146.
9. T. Miura, S. Miyajima and S. H. Takahasi: *Hyers-Ulam Stability of Linear Differential Operator with Constant Coefficients.* Math. Nacr. **258** (2003), 90–96.
10. M. Obloza: *Connections Between Hyers and Lyapunov Stability of the Ordinary Differential Equations.* Rocznik. Nauk. Dydakt. Prace. Mat. **14** (1997), 141–146.

11. M. Obloza: *Hyers-Ulam Stability of the Linear Differential Equations.* Rocznik. Nauk. Dydakt. Prace. Mat. **13** (1993), 259–270.
12. S. Öğrekçi: *Stability of delay differential equations in the sense of Ulam on unbounded intervals.* An International Journal of Optimization and Control: Theories & Applications (IJOCTA) **9(2)** (2019), 125–131.
13. D. Popa and G. Pugna: *Hyers-Ulam Stability of Euler's Differential Equation.* Results in Mathematics **69** (2016), 317–325.
14. D. Popa and I. Raşa: *On the Hyers-Ulam stability of the linear differential equation.* Journal of Mathematical Analysis and Applications **381** (2011), 530–537.
15. T. Rasham and et. al.: *Common fixed point results for new Ciric-type rational multivalued F-contraction with an application.* Journal of Fixed Point Theory and Applications **20** (2018), 45.
16. T. Rasham and et. al.: *Fixed point results for a pair of fuzzy mappings and related applications in b-metric like spaces.* Advances in Difference Equations 2021.1 (2021), p. 259.
17. T. Rasham and et. al.: *Multivalued Fixed Point Results in Dislocated b-Metric Spaces with Application to the System of Nonlinear Integral Equations*"". Symmetry **11** (2019), 40. (2019).
18. T. Rasham and et. al.: *On a pair of fuzzy dominated mappings on closed ball in the multiplicative metric space with applications.* Fuzzy Sets and Systems **437** (2022), 81–96.
19. T. Rasham and et. al.: *On pairs of fuzzy dominated mappings and applications.* Advances in Difference Equations **2021** (2021), 417.
20. T. Rasham and et. al.: *Sufficient conditions to solve two systems of integral equations via fixed point results.* Journal of Inequalities and Applications **2019.1** (2019), 182.
21. T. M. Rassias: *On the Stability of Linear Mappings in Banach Spaces.* Proc. Amer. Math. Soc. **72** (1978), 297–300.
22. H. Rezai, S. M. Jung and T. M. Rassias: *Laplace transform and Hyers-Ulam stability of linear differential equations.* Journal of Mathematical Analysis and Applications **403(1)** (2013), 244–251.
23. Y. Shen: *The Ulam Stability of First Order Linear Dynamic Equations on Time Scales.* Results in Mathematics **72(4)** (2017), 1881–1895.
24. S. H. Takahasi, T. Miura, and S. Miyajima: *The Hyers-Ulam Stability Constants of First Order Linear Differential Operators.* Bull. Korean Math. Soc. **39** (2002), 309–315.
25. C. Tunç and E. Biçer: *Hyers-Ulam-Rassias Stability for a First Order Functional Differential Equation.* J. Math. Fund. Sci. **47(2)** (2015), 143–153.
26. S. M. Ulam: *A Collection of Mathematical Problems.* Interscience, New York, 1960.