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LIGHTLIKE HYPERSURFACES OF ALMOST NORDEN GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we give some classifications about lightlike hypersurfaces of almost Norden Golden semi-Riemannian manifolds. We show that, there is no radical anti-invariant lightlike hypersurfaces of an almost Norden Golden semi-Riemannian manifold. Also, we define invariant and screen semi-invariant lightlike hypersurface of almost Norden Golden semi-Riemannian manifolds and give examples.
Keywords: Lightlike hypersurface, Norden Golden semi-Riemannian manifold, screen semi-invariant hypersurface, almost Norden Golden structure.

1. Introduction

Manifolds are used as a tool to solve many problems in some fields of natural and engineering sciences. It has also become a popular topic as it contributes to the development of these fields and finds new application areas.

The biggest shortcoming in differential geometry is that very little isometrics have been studied, except for positive definite manifolds. This is an important shortcoming, especially considering the applications in engineering and physics. Indeed, isometric immersions and Riemannian submersions are the most studied subjects, and their degeneracy cases have been less studied due to the difficulty posed by the metric. However, it is only possible to obtain more general and powerful

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results in terms of both mathematics and applications by switching from the non-degenaracy (or positive definiteness) condition to the more arbitrary degenaracy case. The degenerate version of isometric immersion has been studied by a large group of researchers under the name of lightlike submanifold, [1, 2, 3, 12, 14, 15].

Let M be a real m-dimensional differentiable manifold and g be the symmetric tensor field of order $(0, 2)$ on this manifold. If the index of the bilinear form g_x is the same for every point x of the manifold and g_x is non-degenerate on the tangent space of T_xM, then the bilinear form is called a semi-Riemannian (briefly, s-Riemannian) metric and in this case the manifold is called a s-Riemannian manifold. When the index of the metric is zero (one), the manifold is called a Riemann (Lorentz) manifold. These manifolds are considered in mathematical physics and especially in the configuration space of space-time models. On the other hand, the geometries of some manifolds with differentiable geometric structures are quite interesting. These manifolds and the maps between them have been extensively studied in differential geometry.

The golden ratio, which has been started to be studied in modern physics in recent years [5, 6], also has an important role in nuclear physics [7]. A close connection has emerged between the golden ratio and the transition from Newtonian physics to relativistic mechanics. Indeed, the golden rectangle is used to obtain the expansion of time intervals and the Lorentz contraction of lengths in the special theory of relativity [8]. At the same time, the golden ratio produces interesting and important results in Kantor space-time, conformal field theory, topology of $4-$manifolds, mathematical probability theory, Kantor fractal theory and El Naschie's field theory [9, 10, 13]. Almost complex golden structure was introduced by Crasmareanu and Hretcanu in [11]. This structure is the analogue of almost golden structure in the complex case. In [8], the authors also studied almost complex golden structure which admits a compatible s-Riemannian metric. Compatible metrics on almost complex golden manifolds are introduced in the same way that Norden metrics on almost complex manifolds. As well as the authors [8] studied holomorphic Norden golden manifolds, which are almost Norden golden manifolds such that the Levi-Civita connection of the s-Riemannian metric parallelizes the almost complex golden structure. They proved that (M, Φ, g) is a holomorphic Norden Golden manifold if and only if the Levi-Civita connection of g parallelizes the almost complex structure $J\Phi$, i.e., $(M, J\Phi, g)$ is a Kaehler Norden manifold (see [8] - Prop. 4.3). After the studies mentioned, the concept of adapted coupling in almost Norden Golden manifolds was introduced by Etoya et al. [4].

This paper is arranged as follows. First, we begin with preliminaries and basic facts related to lightlike hypersurfaces of a s-Riemannian manifold. Afterwards, we introduce lightlike hypersurfaces of an almost Norden Golden (briefly, ANG) s-Riemannian manifold and study two special types, namely invariant and screen semi-invariant lightlike hypersurfaces, in ANG s-Riemannian manifolds. We investigate several properties and derive some geometric results of these types hypersurfaces. We also present an example. Finally, we study screen conformal semi-invariant lightlike hypersurfaces of s-Riemann manifold.

2. Preliminaries

Let $\tilde{M}$ be a manifold and I be a identity tensor field on $\tilde{M}$. Then a polynomial structure $\tilde{\varphi}$ of degree 2 satisfying

$$\tilde{\varphi}^2 = \tilde{\varphi} - \frac{3}{2}I, \tag{2.1}$$

is called an almost complex golden structure. So, $(\tilde{M}, \tilde{\varphi})$ is an almost complex golden manifold [4].

Moreover, let $\tilde{g}$ be a s-Riemannian metric. Then $\tilde{g}$ is called a Norden golden metric on $\tilde{M}$ if it satisfies

$$\tilde{g}(\tilde{\varphi}U_1, U_2) = \tilde{g}(U_1, \tilde{\varphi}U_2), \tag{2.2}$$

$$\tilde{g}(\tilde{\varphi}U_1, \tilde{\varphi}U_2) = \tilde{g}(\tilde{\varphi}U_1, U_2) - \frac{3}{2}\tilde{g}(U_1, U_2), \tag{2.3}$$

for $U_1, U_2 \in \Gamma(T\tilde{M})$. In this case, $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ is called s-Riemannian manifold [4].

Example 2.1. Define a map by

$$\begin{array}{rcl} \tilde{\varphi} : \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \\ (x_1, x_2, x_3, x_4) & \longrightarrow & \tilde{\varphi}(x_1, x_2, x_3, x_4) = (\Phi_c x_1, \Phi_c x_2, \tilde{\Phi}_c x_3, \tilde{\Phi}_c x_4) \end{array}$$

where $\Phi_c = \dfrac{1+\sqrt{5}i}{2}$ and $\tilde{\Phi}_c = \dfrac{1-\sqrt{5}i}{2}$. Then it is easy to see that $\tilde{\varphi}$ satisfies (2.1). Therefore $(\mathbb{R}^4, \tilde{\varphi})$ is an almost complex golden manifold.

Let $\tilde{M}$ be a s-Riemannian manifold with index q, $0 < q < m+2$, and M be a hypersurface of $\tilde{M}$, with $g = \tilde{g}\mid_M$. Then M is a lightlike hypersurface of $\tilde{M}$, if the metric g is of rank $m+1$ and the orthogonal complement $TM^\perp$ of TM, given as

$$TM^\perp = \bigcup_{p\in M} \{V_p \in T_p\tilde{M} : g_p(U_p, V_p) = 0, \forall U_p \in \Gamma(T_pM)\},$$

is a distribution of rank 1 on M, [1]. $TM^\perp \subset TM$ and then it coincides with the radical distribution $Rad\,TM = TM \cap TM^\perp$.

A complementary bundle of $TM^\perp$ in TM is a non-degenerate distribution over M, which is known the screen distribution and denoted by $S(TM)$.

Theorem 2.1. *[1] Let $(M, g, S(TM))$ be a lightlike hypersurface of a s-Riemannian manifold $\tilde{M}$. Then there exists a unique rank 1 vector sub-bundle $ltr(TM)$ which is called the lightlike transversal vector bundle of TM, with base space N, such that for every non-zero section ξ of $Rad\,TM$, there exists a section of $ltr(TM)$ satisfying:*

$$\tilde{g}(N, N) = 0, \quad \tilde{g}(N, W) = 0, \quad \tilde{g}(N, \xi) = 1, \; for \;\; W \in \Gamma(S(TM)).$$

By the previous theorem, we can state:

$$TM = S(TM) \perp Rad\, TM \tag{2.4}$$

and

$$\begin{aligned} T\tilde{M} &= TM \oplus ltr(TM) \\ &= S(TM) \perp \{Rad\, TM \oplus ltr(TM)\}. \end{aligned} \tag{2.5}$$

Let $\omega : \Gamma(TM) \to \Gamma(S(TM))$ be the projection morphism. For $U, V \in \Gamma(TM)$, we have

$$\tilde{\nabla}_U V = \nabla_U V + B(U, V)N, \tag{2.6}$$

$$\tilde{\nabla}_U N = -A_N U + \tau(U)N, \tag{2.7}$$

$$\nabla_U \omega V = \nabla^*_U \omega V + C(U, \omega V)\xi, \tag{2.8}$$

$$\nabla_U \xi = -A^*_\xi U - \tau(U)\xi, \tag{2.9}$$

where ∇ and ∇^* are the linear connections on TM and $S(TM)$, respectively, A_N and A^*_ξ are called the shape operators on TM and $S(TM)$, respectively, τ is a 1-form on TM. In addition B and C are called local second fundamental forms on TM and $S(TM)$, respectively.

For the induced connection ∇, we have

$$(\nabla_U g)(V, Z) = B(U, Z)\theta(V) + B(U, V)\theta(Z), \tag{2.10}$$

where θ is a differential 1-form and

$$\theta(U) = \tilde{g}(N, U). \tag{2.11}$$

Also note that

$$B(U, \xi) = 0, \tag{2.12}$$

$$g(A^*_\xi U, \omega V) = B(U, \omega V), \quad g\ (A^*_\xi U, N) = 0, \tag{2.13}$$

$$g(A_N U, \omega V) = C(U, \omega V), \quad g(A_N U, N) = 0, \tag{2.14}$$

$$A^*_\xi \xi = 0. \tag{2.15}$$

3. Lightlike Hypersurfaces of Almost Norden Golden Semi-Riemannian Manifolds

Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be a $(m+2)$-dimensional ANG s-Riemannian manifold and (M, g) be a lightlike hypersurface of $\tilde{M}$. Then for any $U_1 \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$, we can write

$$\tilde{\varphi}U_1 = \varphi U_1 + \nu(U_1)N, \tag{3.1}$$

$$\tilde{\varphi}N = U + \nu(N)N, \tag{3.2}$$

where $\varphi U_1, U \in \Gamma(TM)$ and ν is a $1-$form defined as below,

$$\nu(U_1) = \tilde{g}(U_1, \tilde{\varphi}\xi). \tag{3.3}$$

Now for all $U_1, U_2 \in \Gamma(TM)$, if we apply $\tilde{\varphi}$ to both side of (3.1) and use (2.1) we get

$$\varphi U_1 + \nu(U_1)N - \frac{3}{2}U_1 = \varphi^2 U_1 + \nu(\varphi U_1)N + \nu(U_1)U + \nu(U_1)\nu(N)N.$$

If the tangential and transversal components on both sides of above equation are equalized, we obtain

$$\varphi^2 U_1 = \varphi U_1 - \frac{3}{2}U_1 - \nu(U_1)U,$$

and

$$\nu(U_1)N = \nu(\varphi U_1)N + \nu(U_1)\nu(N)N,$$

respectively. Similarly, if we apply $\tilde{\varphi}$ to both side of (3.2) and using (2.1), we get

$$U + \nu(N)N - \frac{3}{2}N = \varphi U + \nu(U)N + \nu(N)(U + \nu(N)N).$$

The following equations are obtained by equalizing the tangential and transversal components on both sides of the above equation:

$$\varphi U = U - \nu(N)U$$

and

$$(\nu(N))^2 N = \nu(N)N - \frac{3}{2}N - \nu(U)N.$$

If we use (3.1) in (2.2) and (2.3), then we get

$$g(\varphi U_1, U_2) + \nu(U_1)g(N, U_2) = g(U_1, \varphi U_2) + \nu(U_2)g(U_1, N)$$

and

$$\tilde{g}(\varphi U_1 + \nu(U_1)N, \varphi U_2 + \nu(U_2)N) = g(\varphi U_1 + \nu(U_1)N, U_2) - \frac{3}{2}g(U_1, U_2).$$

Now we can introduce the following lemma.

Lemma 3.1. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and (M, g) be a lightlike hypersurface of $\tilde{M}$. Then for all $U_1, U_2, U \in \Gamma(TM)$, we have*

$$\varphi^2 U_1 = (\varphi - \frac{3}{2} I - \nu \otimes U) U_1, \tag{3.4}$$

$$\nu(\varphi U_1) = (1 - \nu(N))\nu(U_1), \tag{3.5}$$

$$\varphi U = (I - \nu(N))U, \tag{3.6}$$

$$(\nu(N))^2 = \nu(N) - \frac{3}{2} I - \nu(U), \tag{3.7}$$

$$g(\varphi U_1, U_2) = g(U_1, \varphi U_2) + \nu(U_2)\eta(U_1) - \nu(U_1)\eta(U_2), \tag{3.8}$$

$$\begin{aligned} g(\varphi U_1, \varphi U_2) &= g(\varphi U_1, U_2) - \eta(\varphi U_1)\nu(U_2) - \eta(\varphi U_2)\nu(U_1) \\ &\quad + \nu(U_1)\eta(U_2) - \frac{3}{2} g(U_1, U_2). \end{aligned} \tag{3.9}$$

Definition 3.1. An ANG s-Riemannian structure $\tilde{\varphi}$ is called Norden Golden (briefly, NG) semi-Riemannian structure if $\tilde{\varphi}$ is parallel, i.e.,

$$\tilde{\nabla} \tilde{\varphi} = 0. \tag{3.10}$$

Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be a NG s-Riemannian manifold and (M, g) be a lightlike hypersurface of $\tilde{M}$. Then for all $U_1, U_2 \in \Gamma(TM)$ from (3.10), we get

$$\begin{pmatrix} \nabla_{U_1} \varphi U_2 + B(U_1, \varphi U_2) N \\ + \eta(U_1)\nu(U_2) N \end{pmatrix} = \begin{pmatrix} \varphi \nabla_{U_1} U_2 + \nu(\varphi \nabla_{U_1} U_2) N \\ -U_1(\nu(U_2)) N + B(U_1, U_2) U \\ + B(U_1, U_2)\nu(N) N + \nu(U_2) A_N U_1 \end{pmatrix} \tag{3.11}$$

and

$$\begin{pmatrix} \nabla_{U_1} U + B(U_1, U) N + U_1(\nu(N)) N \\ + \nu(N)(-A_N U_1 + \eta(U_1) N) \end{pmatrix} = \begin{pmatrix} -\varphi A_N U_1 - \nu(A_N U_1) N \\ + \eta(U_1) U + \nu(N)\eta(U_1) N \end{pmatrix}. \tag{3.12}$$

By equating the tangential and transversal components of (3.11) and (3.12) respectively, we state

Lemma 3.2. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be a NG s-Riemannian manifold and (M, g) be a lightlike hypersurface of $\tilde{M}$. Then for all $U_1, U_2 \in \Gamma(TM)$, we have*

$$(\nabla_{U_1} \varphi) U_2 = g(A^*_\xi U_1, U_2) U + \nu(U_2) A_N U_1, \tag{3.13}$$

$$(\nabla_{U_1}\nu)U_2 = B(U_1, U_2)\nu(N) - B(U_1, \varphi U_2) - \nu(U_2)\eta(U_1), \tag{3.14}$$

$$\nabla_{U_1}U = -\varphi A_N U_1 + \eta(U_1)U + \nu(N)A_N U_1, \tag{3.15}$$

$$U_1(\nu(N)) = -B(U_1, U) - \nu(A_N U_1). \tag{3.16}$$

Definition 3.2. Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and (M, g) be a lightlike hypersurface of $\tilde{M}$. If

i) $\tilde{\varphi}(TM) \subset TM$ then M is called an invariant lightlike hypersurface.

ii) $\tilde{\varphi}(Rad(TM)) \subset S(TM)$ and $\tilde{\varphi}(ltr(TM)) \subset S(TM)$ then M is called a screen semi-invariant lightlike hypersurface.

iii) $\tilde{\varphi}(Rad(TM)) \subset ltr(TM)$ then M is called a radical anti-invariant lightlike hypersurface.

Theorem 3.1. *$(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and (M, g) be a lightlike hypersurface of $\tilde{M}$. Then the following three statements are equivalent:*

i) M is $\varphi-$invariant.

ii) ν vanishes on M.

iii) φ is an ANG structure on M.

Proof. Let M be an invariant hypersurface. In this case for all $U_1 \in \Gamma(TM)$, we have $\varphi U_1 = U_1$, which implies (ii). On the other hand, if ν is equal to zero on M then we get

$$\varphi^2 U_1 = \tilde{\varphi}^2 U_1 = \varphi U_1 - \frac{3}{2}U_1$$

and

$$g(\varphi U_1, U_2) = \tilde{g}(\tilde{\varphi}U_1, U_2) = \tilde{g}(U_1, \tilde{\varphi}U_2) = g(U_1, \varphi U_2),$$

for all $U_1 \in \Gamma(TM)$. By the way φ is an ANG structure on M and so (ii) implies (iii). It is easy to see that if φ is an ANG structure on M, then M is $\varphi-$invariant. □

Theorem 3.2. *There is no radical anti-invariant lightlike hypersurface of an ANG s-Riemannian manifold.*

Proof. Let an ANG s-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ have a radical anti-invariant lightlike hypersurface. Then for $\xi \in \Gamma(Rad\,TM)$, we have $\tilde{\varphi}\xi \in \Gamma(ltr(TM))$. By using (2.2) and (2.3), we have

$$\tilde{g}(\tilde{\varphi}\xi, \xi) = 0,$$

which implies $\tilde{\varphi}\xi \notin \Gamma(ltr(TM))$. This is a contradiction with our assumption. □

4. Screen Semi-Invariant Lightlike Hypersurfaces of Almost Norden Golden Semi-Riemannian Manifolds

Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be a $(m+2)$-dimensional ANG s-Riemannian manifold and $(M, g, S(TM))$ be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. If

$$D_1 = \tilde{\varphi} Rad\, TM, \quad D_2 = \tilde{\varphi} ltrTM$$

and

$$D = D_\circ \perp Rad\, TM \perp \tilde{\varphi} Rad\, TM,$$

then we get

$$S(TM) = D_\circ \perp (D_1 \oplus D_2), \tag{4.1}$$

$$TM = D \oplus D_2, \tag{4.2}$$

$$T\tilde{M} = D \oplus D_2 \oplus ltrTM, \tag{4.3}$$

where $D_\circ$ is a $(m-2)$-dimensional distribution. We denote vector fields U and V as below,

$$U = \tilde{\varphi} N, \quad V = \tilde{\varphi} \xi. \tag{4.4}$$

Lemma 4.1. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be a $(m+2)$-dimensional ANG s-Riemannian manifold and $(M, g, S(TM))$ be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. Then, for all $U_1, U_2 \in \Gamma(TM)$, $U \in \Gamma(D_2)$ and $V \in \Gamma(D_1)$, we have*

$$\varphi^2 U_1 = (\varphi - \frac{3}{2} I - \nu \otimes U) U_1, \tag{4.5}$$

$$\nu(\varphi U_1) = \nu(U_1), \quad \varphi U = U, \quad \nu(U) = -\frac{3}{2}, \tag{4.6}$$

$$g(\varphi U_1, U_2) = g(U_1, \varphi U_2) + (\eta \otimes \nu - \nu \otimes \eta)(U_1, U_2), \tag{4.7}$$

$$\begin{aligned} g(\varphi U_1, \varphi U_2) = {} & g(\varphi U_1, U_2) + g(U_1, U_2) + \nu(U_1)\eta(U_2) \\ & -\eta(\varphi U_1)\nu(U_2) - \nu(U_1)\eta(\varphi U_2), \end{aligned} \tag{4.8}$$

$$(\nabla_{U_1} \varphi) U_2 = \nu(U_2) A_N U_1 + B(U_1, U_2) U, \tag{4.9}$$

$$(\nabla_{U_1} \nu) U_2 = -B(U_1, \varphi U_2), \tag{4.10}$$

$$(\nabla_{U_1} U) = -\varphi A_N U_1 + \eta(U_1) U. \tag{4.11}$$

Proof. (4.5) is obvious from Lemma 3.1. Since $\tilde{\varphi}\xi \in \Gamma(ltr(TM))$, it is easy to see that

$$\nu(N) = \tilde{g}(N, \tilde{\varphi}\xi) = 0, \tag{4.12}$$

and from (3.5), we get

$$\nu(\varphi U_1) = \nu(U_1).$$

Then using (3.6) and with the help of (4.12), we get

$$\varphi U = U.$$

Finally via (3.3), we obtain

$$\nu(U) = g(\tilde{\varphi}N, \tilde{\varphi}\xi)$$

and

$$\tilde{g}(\tilde{\varphi}N, \tilde{\varphi}\xi) = \tilde{g}(N, \varphi\xi) - \frac{3}{2}\tilde{g}(N, \xi) = -\frac{3}{2}.$$

It's obvious that (4.7), (4.8) and (4.9) are obtained from Lemma 3.1 and (4.10) follows from Lemma 3.2. With the help of Lemma 3.2, we get $\nu(U_2) = g(U_2, \tilde{\varphi}\xi) = g(\tilde{\varphi}U_2, \xi) = 0$, via (4.12). Thus we obtain (4.11). Finally if we use (4.12) in (3.15), (4.11) is easily obtained. □

Additionally since $\nu(N) = 0$, (3.16) becomes

$$B(U_1, U) = -\nu(A_N U_1) = -g(A_N U_1, \tilde{\varphi}\xi) = -g(A_N U_1, V).$$

Then by using (2.14) we get

$$B(U_1, U) = -C(U_1, V). \tag{4.13}$$

Moreover, for $\tilde{\nabla}_{U_1} V = \tilde{\nabla}_{U_1}\tilde{\varphi}\xi = \tilde{\varphi}\tilde{\nabla}_{U_1}\xi$, we obtain

$$\nabla_{U_1} V + B(U_1, V)N = -\varphi A^*_\xi U_1 - \eta(U_1)V - \nu(A^*_\xi U_1)N. \tag{4.14}$$

Then by using tangent and transversal components of (4.14) following equalities are obtained,

$$\nabla_{U_1} V = -\varphi A^*_\xi U_1 - \eta(U_1)V,$$

and

$$B(U_1, V) = -\nu(A^*_\xi U_1).$$

Via these equalities we get

$$B(U_1, V) = -\nu(A^*_\xi U_1) = -g(A^*_\xi U_1, \tilde{\varphi}\xi) = -g(A^*_\xi U_1, V) = -B(U_1, V)$$

and so $B(U_1, V) = 0$. On the other hand since $\tilde{\nabla}$ is a metric connection then by using $\tilde{g}(U, N) = \tilde{g}(\tilde{\varphi}N, N) = 0$, we get

$$C(U_1, U) = 0.$$

Thus we can state:

Corollary 4.1. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. In this case for $U = \tilde{\varphi}N$ on M, we have $B(U_1, U) = 0$. Namely, the vector field U makes the second fundamental form of lightlike hypersurface degenerate.*

Corollary 4.2. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. In this case we have $B(U_1, U) = g(A^*_\xi U_1, U) = g(A^*_\xi U_1, \tilde{\varphi}N) = 0$, that is, there is no component of A^*_ξ in D_2.*

Corollary 4.3. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. In this case we have $C(U_1, V) = g(A_N U_1, V) = g(A_N U_1, \tilde{\varphi}\xi) = 0$, that is, there is no component of $A_N U_1$ in D_1.*

Proposition 4.1. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. Then $D_\circ$ is a $\tilde{\varphi}-$invariant distribution.*

Proof. For all $U_1 \in \Gamma(D_\circ)$, $\xi \in \Gamma(Rad\,TM)$ and $N \in \Gamma(ltr(TM))$, we have

$$\begin{aligned} g(\tilde{\varphi}U_1, \xi) &= g(U_1, \tilde{\varphi}\xi) = 0, \\ g(\tilde{\varphi}U_1, N) &= g(U_1, \tilde{\varphi}N) = 0. \end{aligned}$$

Therefore there is no component of $\tilde{\varphi}U_1$ in $Rad\,TM$ and $ltr(TM)$. On the other hand for all $U_1 \in \Gamma(D_\circ)$, $U \in \Gamma(D_1)$ and $V \in \Gamma(D_2)$, we get

$$\begin{aligned} \tilde{g}(\tilde{\varphi}U_1, U) &= \tilde{g}(U_1, \tilde{\varphi}U) = 0, \\ \tilde{g}(\tilde{\varphi}U_1, V) &= \tilde{g}(U_1, \tilde{\varphi}V) = 0, \end{aligned}$$

which imply that there is no component of $\tilde{\varphi}U_1$ in D_1 and D_2. So the proof is completed. □

Corollary 4.4. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. Then D is a $\tilde{\varphi}-$invariant distribution.*

Theorem 4.1. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. Then V is a parallel vector field on M if and only if $\eta = 0$ and M is totally geodesic on $\tilde{M}$.*

Proof. Let V be a parallel vector field on M. For all $U_1 \in \Gamma(TM)$, if we use (3.1), (3.4) and (2.1), we get

$$-\varphi A^*_\xi U_1 - \eta(U_1)V = -\tilde{\varphi} A^*_\xi U_1 - \eta(U_1)V = 0. \tag{4.15}$$

Then if we apply $\tilde{\varphi}$ to both sides of (4.15) and via (3.1), (3.4) with (2.1), we obtain

$$-\varphi A^*_\xi U_1 + \frac{3}{2} A^*_\xi U_1 - \eta(U_1)V + \frac{3}{2}\eta(U_1)\xi = 0. \tag{4.16}$$

After that by using (4.15) and (4.16), we obtain

$$\frac{3}{2}A^*_\xi U_1 + \frac{3}{2}\eta(U_1)\xi = 0. \tag{4.17}$$

Because of $A^*_\xi U_1 \in \Gamma(S(TM))$, (4.17) is satisfied if and only if $A^*_\xi U_1 = 0$ and $\eta(U_1) = 0$. Hence $A^*_\xi U_1 = 0$ if and only if $B = 0$. The proof is completed. □

Theorem 4.2. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold, M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$ and V be a parallel vector field on M. Then, either φ or U is parallel on M if and only if both M and $S(TM)$ are totally geodesic on $\tilde{M}$.*

Proof. Let φ is parallel on M. For all $U_1, U_2 \in \Gamma(TM)$, if we use (4.9) we get

$$B(U_1, U_2)U = -\nu(U_2)A_N U_1. \tag{4.18}$$

Since V is a parallel vector field on M via Theorem 4.1, $B = 0$ and via (4.18), we have $\nu(U_2)A_N U_1 = 0$. Here if we substitute $U = \tilde{\varphi}N$ instead of U_2, we write

$$\nu(U) = g(\tilde{\varphi}N, \tilde{\varphi}\xi) = g(N, \tilde{\varphi}\xi) - \frac{3}{2}g(N, \xi) \neq 0.$$

Here for all $U_1, U_2 \in \Gamma(TM)$, we obtain $A_N U_1 = 0$ hence $C = 0$.
Similarly let U is a parallel vector field on M, from (3.1) and (4.11) we get

$$-\tilde{\varphi}A_N U_1 + \nu(A_N U_1)N + \eta(U_1)U = 0. \tag{4.19}$$

Then if we apply $\tilde{\varphi}$ to (4.19), we obtain

$$\varphi A_N U_1 + \big(\nu(A_N U_1) + \frac{3}{2}\eta(U_1)\big)N = \frac{3}{2}A_N U_1 + \big(\nu(A_N U_1) + \eta(U_1)\big)U. \tag{4.20}$$

By subtracting (4.19) from (4.20) we get

$$\frac{3}{2}A_N U_1 + \nu(A_N U_1)U - \nu(A_N U_1)N - \frac{3}{2}\eta(U_1)N = 0, \tag{4.21}$$

and from tangential and normal components of (4.21), we obtain

$$\frac{3}{2}A_N U_1 = -\nu(A_N U_1)U, \quad \nu(A_N U_1) = -\frac{3}{2}\eta(U_1). \tag{4.22}$$

Since V is a parallel vector field from Theorem 4.1 we know that $\eta(U_1) = 0$, therefore $\nu(A_N U_1) = 0$ and $A_N U_1 = 0$. So the proof is complete. □

Definition 4.1. Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a lightlike hypersurface of $\tilde{M}$. If $B(U_1, U_2) = 0$, for all $U_1, U_2 \in \Gamma(D_2)$, then M is called as D_2-totally geodesic lightlike hypersurface.

Definition 4.2. Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. If $B(U_1, U_2) = 0$, for all $U_1 \in \Gamma(D)$ and $U_2 \in \Gamma(D_2)$, then M is called as mixed geodesic lightlike hypersurface.

Theorem 4.3. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. Then the following three statements are equivalent:*

i) M is a mixed geodesic lightlike hypersurface.

ii) There is no component of A_N in D_2.

iii) There is no component of A^*_ξ in D_1.

Proof. Let M be a mixed geodesic lightlike hypersurface and ω be a projective projection on $S(TM)$. Hence if we use (4.13), we have

$$C(U_1, \omega U_2) = g(A_N U_1, \omega U_2) \tag{4.23}$$

and

$$g(A_N U_1, N) = 0. \tag{4.24}$$

We obtain

$$B(U_1, U) = -C(U_1, V) = -g(A_N U_1, V) = 0. \tag{4.25}$$

Therefore since $V \in \Gamma(D_2)$ there is no component of A_N in D_2. On the other hand if we use (4.23), (4.24), (4.25), we get

$$g(A^*_\xi U_1, U) = -g(A_N U_1, V),$$

which implies there is no component of A^*_ξ in D_1. □

Theorem 4.4. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. Then distribution D is integrable on M if and only if*

$$B(\tilde{\varphi}U_1, \tilde{\varphi}U_2) = B(\tilde{\varphi}U_1, U_2) - \frac{3}{2}B(U_1, U_2). \tag{4.26}$$

Proof. Let D be an invariant distribution. For $U_1 \in \Gamma(D)$ we have $\tilde{\varphi}U_1 \in \Gamma(D)$. Thus for all $U_1, U_2 \in \Gamma(D)$ and $V \in \Gamma(D_1)$, the distribution D is integrable if and only if

$$\nu([\tilde{\varphi}U_1, U_2]) = 0.$$

Therefore we get

$$\begin{aligned} \nu([\tilde{\varphi}U_1, U_2]) &= \tilde{g}([\tilde{\varphi}U_1, U_2], V) \\ &= B(\tilde{\varphi}U_1, \tilde{\varphi}U_2) - B(U_2, \tilde{\varphi}U_1) + \frac{3}{2}B(U_2, U_1) = 0. \end{aligned}$$

Since B is symmetric as a result we get,

$$B(\tilde{\varphi}U_1, \tilde{\varphi}U_2) = B(\tilde{\varphi}U_1, U_2,) - \frac{3}{2}B(U_1, U_2).$$

□

Theorem 4.5. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. Then the following three statements are equivalent:*

i) D is a parallel distribution.

ii) D is totally geodesic.

iii) $(\nabla_{U_1}\varphi)U_2 = 0$, where $U_1, U_2 \in \Gamma(D)$.

Proof. For all $U_1, U_2 \in \Gamma(D)$ and $V \in \Gamma(D_1)$, D is parallel on M if and only if

$$\nu(\nabla_{U_1}U_2) = 0.$$

Here we get

$$\nu(\nabla_{U_1}U_2) = g(\nabla_{U_1}U_2, V) = B(U_1, \tilde{\varphi}U_2),$$

namely D is totally geodesic. On the other hand for all $U_2 \in \Gamma(D)$, we have $\nu(U_2) = 0$. Thus by using

$$(\nabla_{U_1}\varphi)U_2 = g(A^*_\xi U_1, U_2)U + \nu(U_2)A_N U_1,$$

we obtain

$$\begin{aligned}(\nabla_{U_1}\varphi)U_2 &= g(A^*_\xi U_1, U_2)U \\ &= B(U_1, U_2)U,\end{aligned}$$

which implies $(\nabla_{U_1}\varphi)U_2 = 0$. □

Theorem 4.6. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. Then distribution D is totally geodesic on M if and only if*

$$(\nabla_{U_1}\varphi)U_2 = 0, \tag{4.27}$$

$$(\nabla_{U_1}\omega)U = A_N U_1, \tag{4.28}$$

where $U_1 \in \Gamma(TM)$, $U_2 \in \Gamma(D)$ and $U \in \Gamma(D_2)$.

Proof. Let M be a totally geodesic hypersurface. For all $U_2 \in \Gamma(D)$, $\nu(U_2) = 0$ and

$$(\nabla_{U_1}\varphi)U_2 = B(U_1, U_2)U + \nu(U_2)A_N U_1. \tag{4.29}$$

From (4.29), we find $(\nabla_{U_1}\varphi)U_2 = 0$. Similarly, for all $U \in \Gamma(D_2)$, since $\nu(U) = 1$ then by substutiting U instead of U_2 in (4.29), we obtain

$$(\nabla_{U_1}\varphi)U = B(U_1, U)U + A_N U_1. \tag{4.30}$$

Then from (4.30), we get $(\nabla_{U_1}\varphi)U = A_N U_1$.

Conversely, let us assume that (4.27) and (4.28) are satisfied. Let $U_2 \in \Gamma(TM)$ from (4.2), we can write $U_2 = (U_2)^D + fU$, where $(U_2)^D \in \Gamma(D)$ and f is a function. Therefore we have

$$B(U_1, U_2) = B(U_1, (U_2)^D) + fB(U_1, U)$$

and

$$(\nabla_{U_1}\omega)U = B(U_1, U_2)U + \nu(U_2)A_N U_1. \tag{4.31}$$

Then in (4.31) if we write $(U_2)^D$ instead of U_2 and via (4.27) we get

$$B(U_1, (U_2)^D) = -\nu((U_2)^D)A_N U_1 = -g((U_2)^D, \tilde{\varphi}\xi) = 0.$$

Similarly, in (4.31) if we substitute U instead of U_2 and via (4.28) we have

$$(\nabla_{U_1}\omega)U = B(U_1, U)U + \nu(U)A_N U_1 = A_N U_1.$$

Therefore we obtain $B(U_1, U) = 0$ and $B(U_1, U_2) = 0$. So the proof is complete. □

Theorem 4.7. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. If M is totally umbilical then M is totally geodesic in $\tilde{M}$.*

Proof. Let M be a totally umbilical screen semi-invariant lightlike hypersurface of $\tilde{M}$. Then

$$B(U_1, V) = \lambda g(U_1, V) = 0.$$

Here if we substitute U instead of U_1, we get $\lambda = 0$. Hence $B = 0$. So the proof is complete. □

Theorem 4.8. *Let $(\tilde{M}, \tilde{g}, \tilde{\varphi})$ be an ANG s-Riemannian manifold and M be a screen semi-invariant lightlike hypersurface of $\tilde{M}$. If screen distribution $S(TM)$ is totally umbilical then it is also totally geodesic.*

Proof. Let screen distribution $S(TM)$ be totally umbilical. Then for all $U_1 \in \Gamma(TM)$ we have

$$C(U_1, U) = \mu g(U_1, U) = 0.$$

If we substitute V instead of U_1, we get $\mu = 0$. Hence $C = 0$, which completes the proof. □

Example 4.1. Let $(\mathbb{R}_2^5, \tilde{g})$ be a semi-Riemannian manifold with signature $(-,+,-,+,+)$ and $(x_1, x_2, ..., x_5)$ be a standard coordinate system of $\mathbb{R}_2^5$. If we take

$$\tilde{\varphi}(x_1, x_2, ..., x_5) = (\tilde{\Phi}x_1, \tilde{\Phi}x_2, \Phi x_3, \Phi x_4, \Phi x_5),$$

thus $\tilde{\varphi}$ is an ANG structure on $\mathbb{R}_2^5$. Let M be a hypersurface in $\mathbb{R}_2^5$ defined by $X = \tilde{\varphi}x_1 + \tilde{\varphi}x_2 + x_3$. Then $TM = Span\{U_1, U_2, U_3, U_4\}$, where

$$\begin{aligned} U_1 &= \partial x_1 + \tilde{\varphi}\partial x_5, \\ U_2 &= \partial x_2 + \tilde{\varphi}\partial x_5, \\ U_3 &= \partial x_3 + \partial x_5, \\ U_4 &= \partial x_4. \end{aligned}$$

It is easy to check that M is a lightlike hypersurface. Thus

$$Rad\,TM = \{\xi = \Phi U_1 - \Phi U_2 + U_3\}$$

and

$$S(TM) = \{W_1, W_2, W_3\}$$

where

$$W_1 = \partial x_4$$
$$W_2 = -\partial x_1 + \partial x_2 + \Phi\partial x_3 + \Phi\partial x_5$$

$$W_3 = \frac{-1}{2(2+\Phi)}(-\partial x_1 - \partial x_2 + \Phi\partial x_3 - \Phi\partial x_5)$$

and

$$ltr(TM) = \left\{N = \frac{-1}{2(2+\Phi)}(\Phi\partial x_1 + \Phi\partial x_2 + \partial x_3 - \partial x_5)\right\}.$$

Also, $W_2 = \tilde{\varphi}\xi, W_3 = \tilde{\varphi}N$, thus M is a screen semi invariant lightlike hypersurface.

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