FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 4 (2022), 829 - 838
https://doi.org/10.22190/FUMI220620057B
Original Scientific Paper

SOME GROWTH ANALYSIS OF ENTIRE FUNCTION ON THE BASIS OF THEIR MAXIMUM TERM AND GENERALIZED RELATIVE (α, β)-ORDER

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Abstract. The main aim of this paper is to study some growth properties of entire functions on the basis of their maximum terms and generalized relative (α, β)-order.
Keywords:Entire function, growth, composition, maximum terms, generalized relative-(α, β) order, generalized relative (α, β)-lower order.

1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ and the maximum term $\mu_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ are defined as $M_f = \max_{|z|=r} |f(z)|$ and $\mu_f(r) = \max_{n\geq 0} (|a_n| r^n)$ respectively. We use the standard notations and definitions of the theory of entire functions which are available in [9] and [10], and therefore we do not explain those in details.

Now let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$

Received June 20, 2022. accepted September 16, 2022.
Communicated by Dijana Mosić
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2010 *Mathematics Subject Classification.* Primary 30D35; Secondary 30D30, 30D20

and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \le x \to +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$. Moreover, we assume that throughout the present paper α and β always denote the functions belonging to L^0. The value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} (\alpha \in L, \beta \in L)$$

is called [8] generalized (α, β)-order of f.

Further, we introduce the definitions of the generalized (α, β)-order and generalized (α, β)-lower order of an entire function after giving a minor modification to the original definition of generalized (α, β)-order of an entire function (e.g. see, [8]).

Definition 1.1. The generalized (α, β)-order and generalized (α, β)-lower order of an entire function f are defined as:

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

Since for $0 \le r < R$,

$$\mu_f(r) \le M_f(r) \le \frac{R}{R-r}\mu_f(R) \{cf.\ [7]\},$$

it is easy to see that

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\mu_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\mu_f(r))}{\beta(r)}.$$

Mainly the growth investigation of entire functions has usually been done through their maximum moduli function in comparison to those of exponential function. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. Now in order to make some progress in the study of relative order, one may introduce the definitions of generalized relative (α, β)-order and generalized relative (α, β)-lower order of an entire function with respect to another entire function in the following way:

Definition 1.2. The generalized relative (α, β)-order and generalized relative (α, β)-lower order of an entire function f with respect to an entire function g are defined as:

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}.$$

In terms of maximum terms of entire functions, Definition 1.2 can be reformulated as:

Definition 1.3. The growth indicators $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to another entire function g are defined as:

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty}\frac{\alpha(\mu_g^{-1}(\mu_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty}\frac{\alpha(\mu_g^{-1}(\mu_f(r)))}{\beta(r)}.$$

In fact, the Definition 1.2 and Definition 1.3 are equivalent {cf. [4]}.

In this connection, we state the following notations which are used through out the paper unless otherwise specifically stated.

Notation 1: For any $\eta > 0$, A^* and A_* are defined as

$$A^* = \limsup_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_g(r)))}{(\beta(r))^\eta} \text{ and } A_* = \liminf_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_g(r)))}{(\beta(r))^\eta}.$$

Notation 2: For any $\gamma > 0$, B^* and B_* are defined as

$$B^* = \limsup_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_f(r)))}{\left(\alpha(\mu_h^{-1}(r))\right)^{\gamma+1}} \text{ and } B_* = \liminf_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_f(r)))}{\left(\alpha(\mu_h^{-1}(r))\right)^{\gamma+1}}.$$

Notation 3: For any $\gamma > 0$, C^* and C_* are defined as

$$C^* = \limsup_{r\to+\infty}\frac{\log\left[\frac{\alpha(\mu_h^{-1}(\mu_f(r)))}{\alpha(\mu_h^{-1}(r))}\right]}{\left[\alpha(\mu_h^{-1}(r))\right]^\gamma} \text{ and } C_* = \liminf\frac{\log\left[\frac{\alpha(\mu_h^{-1}(\mu_f(r)))}{\alpha(\mu_h^{-1}(r))}\right]}{\left[\alpha(\mu_h^{-1}(r))\right]^\gamma}.$$

The main aim of this paper is to establish some newly developed results related to the growth rates of maximum terms of composition of two entire functions on the basis of generalized relative (α,β)-order and generalized relative (α,β)-lower order of entire function with respect to another entire function which extend some earlier results (see, e.g., [3]).

2. Known Results

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *[6] Let f and g be entire functions. Then for every $\delta > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \le \frac{\delta}{\delta-1}\mu_f\left(\frac{\delta R}{R-r}\mu_g(R)\right).$$

Lemma 2.2. *[6] If f and g are any two entire functions. Then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \ge \frac{1}{2}\mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right).$$

Lemma 2.3. *[2] Suppose f is an entire function and $A>1$, $0<B<A$. Then for all sufficiently large r,*

$$M_f(Ar) \geq BM_f(r).$$

Lemma 2.4. *[5] If f be an entire and $A>1$, $0<B<A$, then for all sufficiently large r,*

$$\mu_f(Ar) \geq B\mu_f(r).$$

3. Main Results

In this section we state the main results of the paper.

Theorem 3.1. *Let f, g and h be any three entire functions such that for real numbers $A(>0)$ and $B(>0)$,*

$$A^* = A \tag{3.1}$$

and

$$B_* = B, \tag{3.2}$$

where η and γ are used in the Notation 1 and Notation 2 satisfying $\eta<1$ and $\eta(\gamma+1)>1$. Then

$$\rho_{(\alpha,\beta)}[f\circ g]_h = +\infty.$$

Proof. In view of (3.1) and Notation 1, we get for a sequence of values of r tending to infinity that

$$\alpha(\mu_h^{-1}(\mu_g(r))) \geq (A-\varepsilon)(\beta(r))^{\eta} \tag{3.3}$$

and by (3.2) and Notation 2, we get for all sufficiently large values of r that

$$\alpha(\mu_h^{-1}(\mu_f(r))) \geq (B-\varepsilon)\left(\alpha(\mu_h^{-1}(r))\right)^{\gamma+1}.$$

As $\mu_g(r)$ is continuous, increasing and unbounded function of r, we obtain from above for all sufficiently large values of r that

$$\alpha(\mu_h^{-1}(\mu_f(\mu_g(r)))) \geq (B-\varepsilon)\left(\alpha(\mu_h^{-1}(\mu_g(r)))\right)^{\gamma+1}. \tag{3.4}$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, we get from Lemma 2.2, Lemma 2.4, (3.3) and (3.4) for a sequence of values of r tending to infinity that

$$\begin{aligned} \alpha(\mu_h^{-1}(\mu_{f\circ g}(r))) &\geq \alpha\left(\mu_h^{-1}\left(\mu_f\left(\mu_g\left(\frac{r}{196}\right)\right)\right)\right) \\ \textit{i.e.,}\ \alpha(\mu_h^{-1}(\mu_{f\circ g}(r))) &\geq (B-\varepsilon)\left(\alpha\left(\mu_h^{-1}\left(\mu_g\left(\frac{r}{196}\right)\right)\right)\right)^{\gamma+1} \end{aligned} \tag{3.5}$$

$$i.e.,\ \alpha(\mu_h^{-1}(\mu_{f\circ g}(r))) \geq (B-\varepsilon)\left[(A-\varepsilon)\left(\beta\left(\frac{r}{196}\right)\right)^{\eta}\right]^{\gamma+1}$$

$$i.e.,\ \frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \geq \frac{(B-\varepsilon)(A-\varepsilon)^{\gamma+1}\left[\beta\left(\frac{r}{196}\right)\right]^{\eta(\gamma+1)}}{\beta(r)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)}$$

$$\geq \liminf_{r\to+\infty}\frac{(B-\varepsilon)(A-\varepsilon)^{\gamma+1}\left[(1+o(1))\beta(r)\right]^{\eta(\gamma+1)}}{\beta(r)}.$$

As $\varepsilon\,(>0)$ is arbitrary and $\eta\,(\gamma+1)>1$, therefore in view of Definition 1.3, it follows from above that$\rho_{(\alpha,\beta)}[f\circ g]_h=+\infty$.

Thus the theorem follows. □

In the line of Theorem 3.1, one may state the following two theorems without their proofs :

Theorem 3.2. *Let f, g and h be any three entire functions such that for real numbers $A(>0)$ and $B(>0)$,*

$$A_* = A \text{ and } B^* = B,$$

where η and γ are used in the Notation 1 and Notation 2 satisfying $\eta<1$ and $\eta\,(\gamma+1)>1$. Then

$$\rho_{(\alpha,\beta)}[f\circ g]_h=+\infty.$$

Theorem 3.3. *Let f, g and h be any three entire functions such that for real numbers $A(>0)$ and $B(>0)$,*

$$A_* = A \text{ and } B_* = B,$$

where η and γ are used in the Notation 1 and Notation 2 satisfying $\eta<1$ and $\eta\,(\gamma+1)>1$. Then

$$\lambda_{(\alpha,\beta)}[f\circ g]_h=+\infty.$$

Theorem 3.4. *Let f, g and h be any three entire functions such that for real numbers $A(>0)$ and $C(>0)$,*

$$A^* = A \tag{3.6}$$

and

$$C_* = C, \tag{3.7}$$

where η and γ are used in the Notation 1 and Notation 3 satisfying $\eta>1$, $\gamma<1$ and $\eta\gamma>1$. Then

$$\rho_{(\alpha,\beta)}[f\circ g]_h=+\infty.$$

Proof. In view of (3.6) and Notation 1, we obtain for a sequence of values of r tending to infinity that

$$\alpha(\mu_h^{-1}(\mu_g(r)) \geq (A-\varepsilon)(\beta(r))^{\eta} \tag{3.8}$$

Again from (3.7)and Notation 3, we get for all sufficiently large values of r that

$$\log\left[\frac{\alpha(\mu_h^{-1}(\mu_f(r)))}{\alpha(\mu_h^{-1}(r))}\right] \geq (C-\varepsilon)\left[\alpha(\mu_h^{-1}(r))\right]^{\gamma}$$

$$i.e., \ \frac{\alpha(\mu_h^{-1}(\mu_f(r)))}{\alpha(\mu_h^{-1}(r))} \geq \exp\left[(C-\varepsilon)\left[\alpha(\mu_h^{-1}(r))\right]^{\gamma}\right].$$

As $\mu_g(r)$ is continuous, increasing and unbounded function of r, we have from above for all sufficiently large values of r that

$$\frac{\alpha(\mu_h^{-1}(\mu_f(\mu_g(r))))}{\alpha(\mu_h^{-1}(\mu_g(r)))} \geq \exp\left[(C-\varepsilon)\left[\alpha(\mu_h^{-1}(\mu_g(r)))\right]^{\gamma}\right]. \tag{3.9}$$

Further it follows from (3.5), (3.8) and (3.9) for a sequence of values of r tending to infinity that

$$\frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \geq \frac{\alpha\left(\mu_h^{-1}\left(\mu_f\left(\mu_g\left(\frac{r}{196}\right)\right)\right)\right)}{\beta(r)}$$

$$i.e., \ \frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \geq \frac{\alpha\left(\mu_h^{-1}\left(\mu_f\left(\mu_g\left(\frac{r}{196}\right)\right)\right)\right)}{\alpha\left(\mu_h^{-1}\left(\mu_g\left(\frac{r}{196}\right)\right)\right)} \cdot \frac{\alpha\left(\mu_h^{-1}\left(\mu_g\left(\frac{r}{196}\right)\right)\right)}{\beta(r)}$$

$$i.e., \ \frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)}$$
$$\geq \exp\left[(C-\varepsilon)\left[\alpha\left(\mu_h^{-1}\left(\mu_g\left(\frac{r}{196}\right)\right)\right)\right]^{\gamma}\right] \cdot \frac{(A-\varepsilon)\left(\beta\left(\frac{r}{196}\right)\right)^{\eta}}{\beta(r)}$$

$$i.e., \ \frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)}$$
$$\geq \exp\left[(C-\varepsilon)(A-\varepsilon)^{\gamma}\left(\beta\left(\frac{r}{196}\right)\right)^{\gamma\eta}\right] \cdot \frac{(A-\varepsilon)\left(\beta\left(\frac{r}{196}\right)\right)^{\eta}}{\beta(r)}$$

$$i.e., \ \frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \geq$$
$$\exp\left[(C-\varepsilon)(A-\varepsilon)^{\gamma}\left(\beta\left(\frac{r}{196}\right)\right)^{\gamma\eta-1}\beta\left(\frac{r}{196}\right)\right] \cdot \frac{(A-\varepsilon)\left(\beta\left(\frac{r}{196}\right)\right)^{\eta}}{\beta(r)}$$

$$i.e., \ \frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \geq$$

$$\left(\exp\left(\beta\left(\frac{r}{196}\right)\right)\right)^{(C-\varepsilon)(A-\varepsilon)^{\gamma}\left(\beta\left(\frac{r}{196}\right)\right)^{\gamma\eta-1}} \cdot \frac{(A-\varepsilon)\left(\beta\left(\frac{r}{196}\right)\right)^{\eta}}{\beta(r)}$$

$$i.e., \ \limsup_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \geq$$

$$\liminf_{r\to+\infty}\left(\exp\left(\beta\left(\frac{r}{196}\right)\right)\right)^{(C-\varepsilon)(A-\varepsilon)^{\gamma}\left(\beta\left(\frac{r}{196}\right)\right)^{\gamma\eta-1}} \cdot \frac{(A-\varepsilon)\left(\beta\left(\frac{r}{196}\right)\right)^{\eta}}{\beta(r)}.$$

Since $\varepsilon\,(>0)$ is arbitrary and $\eta > 1$, $\gamma\eta > 1$, therefore in view of Definition 1.3, the conclusion of the theorem follows from above. □

In the line of Theorem 3.4, one may also state the following two theorems without their proofs :

Theorem 3.5. *Let f, g and h be any three entire functions such that for real numbers $A(>0)$ and $C(>0)$,*

$$A_* = A \text{ and } C^* = C,$$

where η and γ are used in the Notation 1 and Notation 3 satisfying $\eta > 1$, $\gamma < 1$ and $\eta\gamma > 1$. Then

$$\rho_{(\alpha,\beta)}[f\circ g]_h = +\infty.$$

Theorem 3.6. *Let f, g and h be any three entire functions such that for real numbers $A(>0)$ and $C(>0)$,*

$$A_* = A \text{ and } C_* = C,$$

where η and γ are used in the Notation 1 and Notation 3 satisfying $\eta > 1$, $\gamma < 1$ and $\eta\gamma > 1$. Then

$$\lambda_{(\alpha,\beta)}[f\circ g]_h = +\infty.$$

Theorem 3.7. *Let f, g and h be any three entire functions such that $0 < \lambda_{(\alpha,\beta)}[g]_h \leq \rho_{(\alpha,\beta)}[g]_h < +\infty$ and*

$$\limsup_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_f(r)))}{\alpha(\mu_h^{-1}(r))} = A, \text{ a real number } < +\infty.$$

Then

$$\lambda_{(\alpha,\beta)}[f\circ g]_h \leq A\cdot\lambda_{(\alpha,\beta)}[g]_h \leq \rho_{(\alpha,\beta)}[f\circ g]_h \leq A\cdot\rho_{(\alpha,\beta)}[g]_h.$$

Proof. Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1 and Lemma 2.4 for all sufficiently large values of r that

$$\mu_h^{-1}(\mu_{f\circ g}(r)) \leq \mu_h^{-1}(\mu_f(\mu_g(26r))). \tag{3.10}$$

Now from (3.5) we get for all sufficiently large values of r that

$$\frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \geq \frac{\alpha\left(\mu_h^{-1}\left(\mu_f\left(\mu_g\left(\frac{r}{196}\right)\right)\right)\right)}{\alpha\left(\mu_h^{-1}\left(\mu_g\left(\frac{r}{196}\right)\right)\right)} \cdot \frac{\alpha\left(\mu_h^{-1}\left(\mu_g\left(\frac{r}{196}\right)\right)\right)}{\frac{1}{(1+o(1))}\beta\left(\frac{r}{196}\right)}$$

$$i.e., \limsup_{r\to+\infty} \frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \geq$$

$$\limsup_{r\to+\infty}\left[\frac{\alpha\left(\mu_h^{-1}\left(\mu_f\left(\mu_g\left(\frac{r}{196}\right)\right)\right)\right)}{\alpha\left(\mu_h^{-1}\left(\mu_g\left(\frac{r}{196}\right)\right)\right)} \cdot \frac{\alpha\left(\mu_h^{-1}\left(\mu_g\left(\frac{r}{196}\right)\right)\right)}{\frac{1}{(1+o(1))}\beta\left(\frac{r}{196}\right)}\right]$$

$$i.e., \limsup_{r\to+\infty} \frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \geq$$

$$\limsup_{r\to+\infty}\frac{\alpha\left(\mu_h^{-1}\left(\mu_f\left(\mu_g\left(\frac{r}{196}\right)\right)\right)\right)}{\alpha\left(\mu_h^{-1}\left(\mu_g\left(\frac{r}{196}\right)\right)\right)} \cdot \liminf_{r\to+\infty}\frac{\alpha\left(\mu_h^{-1}\left(\mu_g\left(\frac{r}{196}\right)\right)\right)}{\frac{1}{(1+o(1))}\beta\left(\frac{r}{196}\right)}.$$

Now in view of Definition 1.3, we obtain from above that

$$\rho_{(\alpha,\beta)}[f\circ g]_h \geq A \cdot \lambda_{(\alpha,\beta)}[g]_h. \tag{3.11}$$

Similarly from (3.10) it follows for all sufficiently large values of r that

$$\frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \leq \frac{\alpha(\mu_h^{-1}(\mu_f(\mu_g(26r))))}{\beta(r)}$$

$$i.e., \frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \leq \frac{\alpha(\mu_h^{-1}(\mu_f(\mu_g(26r))))}{\alpha(\mu_h^{-1}(\mu_g(26r)))} \cdot \frac{\alpha(\mu_h^{-1}(\mu_g(26r)))}{\beta(r)}$$

$$i.e., \liminf_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)}$$
$$\leq \liminf_{r\to+\infty}\left[\frac{\alpha(\mu_h^{-1}(\mu_f(\mu_g(26r))))}{\alpha(\mu_h^{-1}(\mu_g(26r)))} \cdot \frac{\alpha(\mu_h^{-1}(\mu_g(26r)))}{\beta(r)}\right]$$

$$i.e., \liminf_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)}$$
$$\leq \limsup_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_f(\mu_g(26r))))}{\alpha(\mu_h^{-1}(\mu_g(26r)))} \cdot \liminf_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_g(26r)))}{\beta(r)}.$$

Now in view of Definition 1.3, it follows from above that

$$\lambda_{(\alpha,\beta)}[f\circ g]_h \leq A\cdot \lambda_{(\alpha,\beta)}[g]_h. \tag{3.12}$$

Also from (**??**) we obtain for all sufficiently large values of r that

$$\limsup_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \leq$$

$$\limsup_{r\to+\infty}\left[\frac{\alpha(\mu_h^{-1}(\mu_f(\mu_g(26r))))}{\alpha(\mu_h^{-1}(\mu_g(26r)))}\cdot\frac{\alpha(\mu_h^{-1}(\mu_g(26r)))}{\beta(r)}\right]$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_{f\circ g}(r)))}{\beta(r)} \leq$$

$$\limsup_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_f(\mu_g(26r))))}{\alpha(\mu_h^{-1}(\mu_g(26r)))}\cdot\limsup_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_g(26r)))}{\beta(r)}$$

$$i.e.,\ \rho_{(\alpha,\beta)}[f\circ g]_h \leq A\cdot \rho_{(\alpha,\beta)}[g]_h. \tag{3.13}$$

Therefore the theorem follows from (3.11), (3.12) and (3.13). □

Theorem 3.8. *Let f, g and h be any three entire functions such that $0<\rho_{(\alpha,\beta)}[g]_h<+\infty$ and*

$$\liminf_{r\to+\infty}\frac{\alpha(\mu_h^{-1}(\mu_f(r)))}{\alpha(\mu_h^{-1}(r))} = A,\ \text{a real number } <+\infty.$$

Then

$$\lambda_{(\alpha,\beta)}[f\circ g]_h \leq A\rho_{(\alpha,\beta)}[g]_h \leq \rho_{(\alpha,\beta)}[f\circ g]_h.$$

The proof of Theorem 3.8 is omitted because it can be carried out in the line of Theorem 3.7.

Acknowledgement

The authors are very much thankful to the reviewers for their valuable suggestions for delivering the paper in its present form.

REFERENCES

1. L. Bernal-González: *Crecimiento relativo de funciones enteras. Aportaciones al estudio de las funciones enteras con índice exponencial finito*. Doctoral Dissertation, University of Sevilla, Sevilla, 1984.
2. L. Bernal: *Orden relative de crecimiento de funciones enteras*. Collect. Math. **39** (1988), 209–229.
3. T. Biswas: *On some growth analysis of entire and meromorphic functions in the light of their relative $(p,q,t)L$-th order with respect to another entire function*. Annals of Oradea University Mathematics Fascicola, **XXVI**, Issue No.1 (2019), 59–80.
4. T. Biswas and C. Biswas: *On some growth properties of composite entire functions on the basis of their generalized relative order (α,β)*. J. Ramanujan Soc. Math. Math. Sci., **9**(1) (2021), 11–28.
5. S. K. Datta and A. R. Maji: *Relative order of entire functions in terms of their maximum terms*. Int. Journal of Math. Analysis, **5**(43) (2011), 2119–2126.
6. A. P. Singh: *On maximum term of composition of entire functions*. Proc. Nat. Acad. Sci. India, **59(A)**, Part I(1989), 103–115.
7. A. P. Singh and M. S. Baloria: *On the maximum modulus and maximum term of composition of entire functions*. Indian J. Pure Appl. Math. **22**(12) (1991), 1019–1026.
8. M. N. Sheremeta: *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion*. Izv. Vyssh. Uchebn. Zaved Mat. **2** (1967), 100–108 (in Russian).
9. G. Valiron: *Lectures on the general theory of integral functions*. Chelsea Publishing Company, 1949.
10. L. Yang: *Value distribution theory*. Springer-Verlag, Berlin, 1993.