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INVESTIGATION OF QUASI BI-SLANT RIEMANNIAN MAPS

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Abstract. Riemannian maps are generalization of well-known notions of isometric immersions and Riemannian submersions. In this paper, we define and study a natural generalization of previously defined quasi bi-slant submersions [18] in the case of Riemannian maps. We mainly investigate fundamental results on quasi bi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds: the integrability of distributions, geometry of foliations, the condition for such maps to be totally geodesic, etc. At the end of the article, we give proper non-trivial examples for this notion.
Keywords: Riemannian maps, Quasi bi-slant Riemannian maps, Almost Hermitian manifolds.

1. Introductions

In differential geometry, initiating and utilising the idea of appropriate transformations to compare geometric properties between two manifolds is one of the main features. Immersions and submersions are the most used transformations in this sense. The study of Riemannian submersions was initiated by O′Neill [8] and Gray

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[2]. Watson [9] studied almost complex type of Riemannian submersions. Further, several kinds of Riemannian submersions were introduced and studied [3]. These maps have a wide range of applications in different branches of science and engineering, for example, the Yang-Mills theory [10], Kaluza-Klein theory [11], supergravity and superstring theories [13], [14], Euclidean super-symmetry [25] etc.

On the other side, the study of Riemannian maps have risen in popularity in recent geometric evaluations due to its envolvement in the mathematical physics. The basic properties of Riemannian maps were first given by Fischer [1]. More precisely, a differentiable map $\pi : (N_1, g_1) \to (N_2, g_2)$ between Riemannian manifolds (N_1, g_1) and (N_2, g_2) is called a Riemannian map ($0 < rank\pi_* < \min\{m, n\}$, where $\dim N_1 = m, \dim N_2 = n$) if it satisfies the equation:

$$g_2(\pi_* V_1, \pi_* V_2) = g_1(V_1, V_2), \tag{1.1}$$

for $V_1, V_2 \in \Gamma(\ker \pi_*)^\perp$.

Consequently, isometric immersions and Riemannian submersions are particular cases of Riemannian maps with $\ker \pi_* = \{0\}$ and $(range\pi_*)^\perp = 0$, respectively. In [1], the author has shown a conspicuous property of Riemannian map is that it satisfies the generalized eikonal equation $\| \pi_* \|^2 = rank\pi$ and also proposed an approach to build a quantum model using Riemannian maps that would provide an interesting relationship between Riemannian maps, harmonic maps, and Lagrangian field theory. Further, the notion of Riemannian map and related topics are being studied continuously from different perspectives, as Invariant and anti-invariant Riemannian map [4], semi-invariant Riemannian map [5], slant Riemannian map ([6], [15], [19]), semi-slant Riemannian map ([12], [16], [20], [22]) and hemi-slant Riemannian map ([7], [17]) quasi-hemi-slant Riemannian map [21] etc.

In this paper, we study the notion of quasi bi-slant Riemannian maps from an almost Hermitian manifold to a Riemannian manifold. The paper is organized as follows: In Section 2, we will recall some basic definitions related to quasi bi-slant Riemannian maps. In section 3, we will define quasi bi-slant Riemannian map from Kähler manifolds to Riemannian manifolds and study the geometry of leaves of distributions that are involved in the definition of such maps. We will provide necessary and sufficient conditions for quasi bi-slant Riemannian maps to be totally geodesic. In section 4, we will provide some non-trivial examples of such Riemannian maps.

2. Preliminaries

Let N_1 be an even-dimensional differentiable manifold. Let J be a $(1, 1)$ tensor field on N_1 such that $J^2 = -I$, where I is identity operator. Then J is called an almost complex structure on N_1. The manifold N_1 with an almost complex structure J is called an almost complex manifold [24]. It is well known that an almost complex manifold is necessarily orientable. Nijenhuis tensor N of an almost complex structure is defined as:

$$N(X_1, X_2) = [JX_1, JX_2] - [X_1, X_2] - J[JX_1, X_2] - J[X_1, JX_2], \tag{2.1}$$

for all $X_1, X_2 \in \Gamma(TN_1)$.

If Nijenhuis tensor field N on an almost complex manifold N_1 is zero, then the almost complex manifold N_1 is called a complex manifold.

Let g_1 be a Riemannian metric on N_1 such that

$$g_1(JX_1, JX_2) = g_1(X_1, X_2), \tag{2.2}$$

for all $X_1, X_2 \in \Gamma(TN_1)$.

Then g_1 is called an almost Hermitian metric on N_1 and manifold N_1 with Hermitian metric g_1 is called almost Hermitian manifold. The Riemannian connection ∇ of the almost Hermitian manifold N_1 can be extended to the whole tensor algebra on N_1. Tensor fields $(\nabla_{Y_1} J)$ is defined as

$$(\nabla_{Y_1} J)Y_2 = \nabla_{Y_1} JY_2 - J\nabla_{Y_1} Y_2 \tag{2.3}$$

for $Y_1, Y_2 \in \Gamma(TN_1)$.

An almost Hermitian manifold (N_1, g_1, J) is called a Kähler manifold if

$$(\nabla_{Y_1} J)Y_2 = 0 \tag{2.4}$$

for $Y_1, Y_2 \in \Gamma(TN_1)$.

Now, we recall following definitions for later use:

Definition 2.1. [3] Let π be a Riemannian map from an almost Hermitian manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). If for any non-zero vector $Y_1 \in (\ker \pi_*)_q$, $q \in N_1$, the angle $\theta(Y_1)$ between JY_1 and the space $(\ker \pi_*)_q$ is constant, i.e., it is independent of the choice of the point $q \in N_1$ and the tangent vector Y_1 in $\ker \pi_*$, then we say that π is a slant Riemannian map. In this case, the angle θ is called the slant angle of the Riemannian map. If the slant angle is $0 < \theta < \frac{\pi}{2}$, then the Riemannian map is called a proper slant Riemannian map.

Definition 2.2. [3] Let (N_1, g_1, J) be an almost Hermitian manifold and (N_2, g_2) a Riemannian manifold. A Riemannian map $\pi : (N_1, g_1, J) \to (N_2, g_2)$ is called a semi-slant Riemannian map if there is a distribution $\mathcal{D}_1 \subset \ker \pi_*$ such that

$$\ker \pi_* = \mathcal{D} \oplus \mathcal{D}_1, J(\mathcal{D}) = \mathcal{D}, \tag{2.5}$$

and the angle $\theta = \theta(Y_1)$ between JY_1 and the space $(\mathcal{D}_1)_q$ is constant for non-zero vector $Y_1 \in (\mathcal{D}_1)_q$ and $q \in N_1$, where $\mathcal{D}_1$ is the orthogonal complement of $\mathcal{D}$ in $\ker \pi_*$.

We call the angle θ a semi-slant angle.

Definition 2.3. [7] Let N_1 be an almost Hermitian manifold with Hermitian metric g_1 and almost complex structure J, and N_2 be a Riemannian manifold with Riemannian metric g_2. A Riemannian map $\pi : (N_1, g_1, J) \to (N_2, g_2)$ is called a hemi-slant Riemannian map if the vertical distribution $\ker\pi_*$ of π admits two orthogonal complementary distributions D^θ and $D^\perp$ such that D^θ is slant with angle θ and $D^\perp$ is anti-invariant, i.e, we have

$$\ker\pi_* = D^\theta \oplus D^\perp. \tag{2.6}$$

In this case, the angle θ is called the hemi-slant angle of the Riemannian map.

Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by [8]

$$\mathcal{A}_{F_1}F_2 = \mathcal{H}\nabla_{\mathcal{H}F_1}\mathcal{V}F_2 + \mathcal{V}\nabla_{\mathcal{H}F_1}\mathcal{H}F_2, \tag{2.7}$$

$$\mathcal{T}_{F_1}F_2 = \mathcal{H}\nabla_{\mathcal{V}F_1}\mathcal{V}F_2 + \mathcal{V}\nabla_{\mathcal{V}F_1}\mathcal{H}F_2, \tag{2.8}$$

for any vector fields F_1, F_2 on N_1, where ∇ is the Levi-Civita connection of g_1. It is easy to see that $\mathcal{T}_{F_1}$ and $\mathcal{A}_{F_1}$ are skew-symmetric operators on the tangent bundle of N_1 reversing the vertical and the horizontal distributions.

From equations (2.7) and (2.8), we have

$$\nabla_{Z_1}Z_2 = \mathcal{T}_{Z_1}Z_2 + \mathcal{V}\nabla_{Z_1}Z_2, \tag{2.9}$$

$$\nabla_{Z_1}Y_1 = \mathcal{T}_{Z_1}Y_1 + \mathcal{H}\nabla_{Z_1}Y_1, \tag{2.10}$$

$$\nabla_{Y_1}Z_1 = \mathcal{A}_{Y_1}Z_1 + \mathcal{V}\nabla_{Y_1}Z_1, \tag{2.11}$$

$$\nabla_{Y_1}Y_2 = \mathcal{H}\nabla_{Y_1}Y_2 + \mathcal{A}_{Y_1}Y_2 \tag{2.12}$$

for $Z_1, Z_2 \in \Gamma(\ker\pi_*)$ and $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$, where $\mathcal{H}\nabla_{Z_1}Y_1 = \mathcal{A}_{Y_1}Z_1$, if Y_1 is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution [3].

It is seen that for $p \in N_1$, $Z_1 \in \mathcal{V}_p$ and $Y_1 \in \mathcal{H}_p$ the linear operators

$$\mathcal{A}_{Y_1},\ \mathcal{T}_{Z_1} : T_qN_1 \to T_qN_1, \tag{2.13}$$

are skew-symmetric, that is

$$g_1(\mathcal{A}_{Y_1}F_1, F_2) = -g_1(F_1, \mathcal{A}_{Y_1}F_2), g_1(\mathcal{T}_{Z_1}F_1, F_2) = -g_1(F_1, \mathcal{T}_{Z_1}F_2) \tag{2.14}$$

for $F_1, F_2 \in \Gamma(T_pN_1)$. Since $\mathcal{T}_{Y_1}$ is skew-symmetric, we observe that π has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$.

We recall that the notation of second fundamental form of a map between two Riemannian manifolds. Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds and $\pi : (N_1, g_1) \to (N_2, g_2)$ be a C^∞ map then the second fundamental form of π is given by

$$(\nabla\pi_*)(Z_1, Z_2) = \nabla^\pi_{Z_1}\pi_*Z_2 - \pi_*(\nabla^{N_1}_{Z_1}Z_2) \tag{2.15}$$

for $Z_1, Z_2 \in \Gamma(TN_1)$, where ∇^π is the pullback connection and we denote for convenience by ∇ the Riemannian connections of the metrics g_1 and g_2 [23].

Finally we also recall that a differentiable map π between two Riemannian manifolds is totally geodesic if

$$(\nabla\pi_*)(Z_1, Z_2) = 0, \tag{2.16}$$

for all $Z_1, Z_2 \in \Gamma(TN_1)$. A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

3. Quasi bi-slant Riemannian maps

Now, we introduce the notion of a quasi bi-slant Riemannian map as a natural generalization of hemi-slant Riemannian map and semi-slant Riemannian map from almost Hermitian manifolds to Riemannian manifolds.

Definition 3.1. Let (N_1, g_1, J) be an almost Hermitian manifold and (N_2, g_2) be a Riemannian manifold. A Riemannian map

$$\pi : (N_1, g_1, J) \to (N_2, g_2), \tag{3.1}$$

is called a quasi bi-slant Riemannian map if there exist three mutually orthogonal distribution D, D_1 and D_2 such that

(i) $\ker \pi_* = D \oplus D_1 \oplus D_2$,

(ii) $J(D) = D$ i.e., D is invariant,

(iii) $J(D_1) \perp D_2$ and $J(D_2) \perp D_1$,

(iv) for any non-zero vector field $Y_1 \in (D_1)_q$, $q \in N_1$, the angle θ_1 between JY_1 and $(D_1)_q$ is constant and independent of the choice of point q and Y_1 in $(D_1)_q$,

(v) for any non-zero vector field $Z_1 \in (D_2)_q$, $q \in N_1$, the angle θ_2 between JZ_1 and $(D_2)_q$ is constant and independent of the choice of point q and Z_1 in $(D_2)_q$,

These angles θ_1 and θ_2 are called slant angles of the Riemannian map.

We easily observe that

(a) If $\dim D = 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then π is a hemi-slant Riemannian map.

(b) If $\dim D = 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then π is a bi-slant Riemannian map.

(c) If $\dim D \neq 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then we may call π is an quasi-hemi-slant Riemannian map.

(d) If $\dim D \neq 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then π is proper quasi bi-slant Riemannian map.

Let π be quasi bi-slant Riemannian maps from an almost Hermitian manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, we have

$$TN_1 = \ker \pi_* \oplus (\ker \pi_*)^{\perp}. \tag{3.2}$$

Now, for any vector field $V_1 \in \Gamma(\ker \pi_*)$, we put

$$V_1 = PV_1 + QV_1 + RV_1, \tag{3.3}$$

where P, Q and R are projection morphisms [13] of $\ker \pi_*$ onto D, D_1 and D_2, respectively.

For $Z_1 \in (\Gamma \ker \pi_*)$, we set

$$JZ_1 = \phi Z_1 + \omega Z_1, \tag{3.4}$$

where $\phi Z_1 \in (\Gamma \ker \pi_*)$ and $\omega Z_1 \in (\Gamma \ker \pi_*)^{\perp}$.

From equations (3.3) and (3.4), we have

$$\begin{aligned} JZ_1 &= J(PZ_1) + J(QZ_1) + J(RZ_1), \\ &= \phi(PZ_1) + \omega(PZ_1) + \phi(QZ_1) + \omega(QZ_1) + \phi(RZ_1) + \omega(RZ_1), \end{aligned}$$

since $JD = D$, we get $\omega PZ_1 = 0$.

Hence above equation reduces to

$$JZ_1 = \phi(PZ_1) + \phi QZ_1 + \omega QZ_1 + \phi RZ_1 + \omega RZ_1. \tag{3.5}$$

Thus we have the following decomposition

$$J(\ker \pi_*) = D \oplus (\phi D_1 \oplus \phi D_2) \oplus (\omega D_1 \oplus \omega D_2), \tag{3.6}$$

where $\oplus$ denotes orthogonal direct sum.

Further, let $V_1 \in \Gamma(D_1)$ and $V_2 \in \Gamma(D_2)$. Then

$$g_1(V_1, V_2) = 0. \tag{3.7}$$

From definition 3.1(iii), we have

$$g_1(JV_1, V_2) = g_1(V_1, JV_2) = 0. \tag{3.8}$$

Now, consider

$$\begin{aligned} g_1(\phi V_1, V_2) &= g_1(JV_1 - \omega V_1, V_2), \\ &= g_1(JV_1, V_2), \\ &= 0. \end{aligned}$$

Similarly, we have

$$g_1(V_1, \phi V_2) = 0. \tag{3.9}$$

Let $U_1 \in \Gamma(D)$ and $U_2 \in \Gamma(D_1)$. Then we have

$$\begin{aligned} g_1(\phi U_2, U_1) &= g_1(JU_2 - \omega U_2, U_1), \\ &= g_1(JU_2, U_1), \\ &= -g_1(U_2, JU_1), \\ &= 0, \end{aligned}$$

as D is invariant i.e., $JU_1 \in \Gamma(D)$.

Similarly, for $U_1 \in \Gamma(D)$ and $U_3 \in \Gamma(D_2)$, we obtain

$$g_1(\phi U_3, U_1) = 0, \tag{3.10}$$

From above equations, we have

$$g_1(\phi W_1, \phi W_2) = 0, \tag{3.11}$$

and

$$g_1(\omega W_1, \omega W_2) = 0, \tag{3.12}$$

for all $W_1 \in \Gamma(D_1)$ and $W_2 \in \Gamma(D_2)$.

So, we can write

$$\phi D_1 \cap \phi D_2 = \{0\}, \omega D_1 \cap \omega D_2 = \{0\}. \tag{3.13}$$

If $\theta_2 = \frac{\pi}{2}$, then $\phi R = 0$ and D_2 is anti-invariant, i.e., $J(D_2) \subseteq (\ker \pi_*)^\perp$. In this case we denote D_2 by $D^\perp$.

We also have

$$J(\ker \pi_*) = D \oplus \phi D_1 \oplus \omega D_1 \oplus JD^\perp. \tag{3.14}$$

Since $\omega D_1 \subseteq (\ker \pi_*)^\perp$, $\omega D_2 \subseteq (\ker \pi_*)^\perp$. So we can write

$$(\ker \pi_*)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mu, \tag{3.15}$$

where μ is orthogonal complement of $(\omega D_1 \oplus \omega D_2)$ in $(\ker \pi_*)^\perp$.

Also for any non-zero vector field $Y_1 \in \Gamma(\ker \pi_*)^\perp$, we have

$$JY_1 = BY_1 + CY_1, \tag{3.16}$$

where $BY_1 \in \Gamma(\ker \pi_*)$ and $CY_1 \in \Gamma(\ker \pi_*)^\perp$.

Lemma 3.1. *Let π be a quasi bi-slant Riemannian map from an almost Hermitian manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, we have*

$$\phi^2 W_1 + B\omega W_1 = -W_1, \omega\phi W_1 + C\omega W_1 = 0, \tag{3.17}$$

$$\omega B W_2 + C^2 W_2 = -W_2, \phi B W_2 + BC W_2 = 0, \tag{3.18}$$

for all $W_1 \in \Gamma(\ker \pi_*)$ and $W_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using equations (3.4), (3.16) and $J^2 = -I$, we have Lemma 3.1. □

Lemma 3.2. *Let π be a quasi bi-slant Riemannian map from an almost Hermitian manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, we have*

(*i*) $\phi^2 Z_i = -(\cos^2 \theta_1) Z_i$

(*ii*) $g_1(\phi Z_i, \phi V_i) = \cos^2 \theta_1 g_1(Z_i, V_i)$,

(*iii*) $g_1(\omega Z_i, \omega V_i) = \sin^2 \theta_1 g_1(Z_i, V_i)$,

for all $Z_i, V_i \in \Gamma(D_i)$, where $i = 1, 2$.

Proof. By Lemma (3.2) in [18], we obtain Lemma 3.2. □

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Lemma 3.3. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, we have*

$$\mathcal{V}\nabla_{Y_1}\phi Y_2 + \mathcal{T}_{Y_1}\omega Y_2 = \phi\mathcal{V}\nabla_{Y_1}Y_2 + B\mathcal{T}_{Y_1}Y_2, \tag{3.19}$$

$$\mathcal{T}_{Y_1}\phi Y_2 + \mathcal{H}\nabla_{Y_1}\omega Y_2 = \omega\mathcal{V}\nabla_{Y_1}Y_2 + C\mathcal{T}_{Y_1}Y_2, \tag{3.20}$$

$$\mathcal{V}\nabla_{Z_1}BZ_2 + \mathcal{A}_{Z_1}CZ_2 = \phi\mathcal{A}_{Z_1}Z_2 + B\mathcal{H}\nabla_{Z_1}Z_2, \tag{3.21}$$

$$\mathcal{A}_{Z_1}BZ_2 + \mathcal{H}\nabla_{Z_1}CZ_2 = \omega\mathcal{A}_{Z_1}Z_2 + C\mathcal{H}\nabla_{Z_1}Z_2, \tag{3.22}$$

$$\mathcal{V}\nabla_{Y_1}BZ_1 + \mathcal{T}_{Y_1}CZ_1 = \phi\mathcal{T}_{Y_1}Z_1 + B\mathcal{H}\nabla_{Y_1}Z_1, \tag{3.23}$$

$$\mathcal{T}_{Y_1}BZ_1 + \mathcal{H}\nabla_{Y_1}CZ_1 = \omega\mathcal{T}_{Y_1}Z_1 + C\mathcal{H}\nabla_{Y_1}Z_1, \tag{3.24}$$

$$\mathcal{V}\nabla_{Z_1}\phi Y_1 + \mathcal{A}_{Z_1}\omega Y_1 = B\mathcal{A}_{Z_1}Y_1 + \phi\mathcal{V}\nabla_{Z_1}Y_1, \tag{3.25}$$

$$\mathcal{A}_{Z_1}\phi Y_1 + \mathcal{H}\nabla_{Z_1}\omega Y_1 = C\mathcal{A}_{Z_1}Y_1 + \omega\mathcal{V}\nabla_{Z_1}Y_1 \tag{3.26}$$

for any $Y_1, Y_2 \in \Gamma(\ker \pi_*)$ and $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using equations (2.9), (2.10), (2.11), (2.12), (3.4) and (3.16), we get equations (3.19)-(3.26). □

Now, we define

$$(\nabla_{V_1}\phi)V_2 = \mathcal{V}\nabla_{V_1}\phi V_2 - \phi\mathcal{V}\nabla_{V_1}V_2, \tag{3.27}$$

$$(\nabla_{V_1}\omega)V_2 = \mathcal{H}\nabla_{V_1}\omega V_2 - \omega\mathcal{V}\nabla_{V_1}V_2, \tag{3.28}$$

$$(\nabla_{Z_1}C)Z_2 = \mathcal{H}\nabla_{Z_1}CZ_2 - C\mathcal{H}\nabla_{Z_1}Z_2, \tag{3.29}$$

$$(\nabla_{Z_1}B)Z_2 = \mathcal{V}\nabla_{Z_1}BZ_2 - B\mathcal{H}\nabla_{Z_1}Z_2 \tag{3.30}$$

for $V_1, V_2 \in \Gamma(\ker \pi_*)$ and $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$.

Lemma 3.4. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, we have*

$$(\nabla_{V_1}\phi)V_2 = B\mathcal{T}_{V_1}V_2 - \mathcal{T}_{V_1}\omega V_2, \tag{3.31}$$

$$(\nabla_{V_1}\omega)V_2 = C\mathcal{T}_{V_1}V_2 - \mathcal{T}_{V_1}\phi V_2, \tag{3.32}$$

$$(\nabla_{Z_1}C)Z_2 = \omega\mathcal{A}_{Z_1}Z_2 - \mathcal{A}_{Z_1}BZ_2, \tag{3.33}$$

$$(\nabla_{Z_1}B)Z_2 = \phi\mathcal{A}_{Z_1}Z_2 - \mathcal{A}_{Z_1}CZ_2, \tag{3.34}$$

for $V_1, V_2 \in \Gamma(\ker \pi_*)$ and $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using equations (3.19), (3.20), (3.21), (3.22), (3.27), (3.28), (3.29) and (3.30), we get all equations of Lemma 3.4. □

If the tensors ϕ and ω are parallel with respect to the linear connection ∇ on N_1, respectively, then

$$B\mathcal{T}_{U_1}U_2 = \mathcal{T}_{U_1}\omega U_2, C\mathcal{T}_{U_1}U_2 = \mathcal{T}_{U_1}\phi U_2, \tag{3.35}$$

for $U_1, U_2 \in \Gamma(TN_1)$.

Theorem 3.1. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, the invariant distribution D is integrable if and only if*

$$g_1(\mathcal{T}_{Z_1}JZ_2 - \mathcal{T}_{Z_2}JZ_1, \omega QV_1 + \omega RV_1) = g_1(\mathcal{V}\nabla_{Z_1}JZ_2 - \mathcal{V}\nabla_{Z_2}JZ_1, \phi QV_1 + \phi RV_1) \tag{3.36}$$

for $Z_1, Z_2 \in \Gamma(D)$ and $V_1 \in \Gamma(D_1 \oplus D_2)$.

Proof. For $Z_1, Z_2 \in \Gamma(D)$, and $V_1 \in \Gamma(D_1 \oplus D_2)$, using equations (2.2), (2.4), (2.9), (3.3) and (3.4), we have

$$\begin{aligned} & g_1([Z_1, Z_2], V_1) \\ = \;& g_1(\nabla_{Z_1}JZ_2, JV_1) - g_1(\nabla_{Z_2}JZ_1, JV_1), \\ = \;& g_1(\mathcal{T}_{Z_1}JZ_2 - \mathcal{T}_{Z_2}JZ_1, \omega QV_1 + \omega RV_1) - g_1(\mathcal{V}\nabla_{Z_1}JZ_2 - \mathcal{V}\nabla_{Z_2}JZ_1, \phi QV_1 + \phi RV_1), \end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, the slant distribution D_1 is integrable if and only if*

$$\begin{aligned} & g_1(\mathcal{T}_{X_1}\omega\phi X_2 - \mathcal{T}_{X_2}\omega\phi X_1, Z_1) \\ = \;& g_1(\mathcal{T}_{X_1}\omega X_2 - \mathcal{T}_{X_2}\omega X_1, JPZ_1 + \phi RZ_1) + g_1(\mathcal{H}\nabla_{X_1}\omega X_2 - \mathcal{H}\nabla_{X_2}\omega X_1, \omega RZ_1) \end{aligned}$$

for $X_1, X_2 \in \Gamma(D_1)$ and $Z_1 \in \Gamma(D \oplus D_2)$.

Proof. For $X_1, X_2 \in \Gamma(D_1)$ and $Z_1 \in \Gamma(D \oplus D_2)$, we have

$$g_1([X_1, X_2], Z_1) = g_1(\nabla_{X_1}X_2, Z_1) - g_1(\nabla_{X_2}X_1, Z_1). \tag{3.37}$$

Using equations (2.2), (2.4), (2.9), (2.10), (3.4) and Lemma 3.2, we have

$$\begin{aligned} & g_1([X_1, X_2], Z_1) \\ = \;& g_1(\nabla_{X_1}JX_2, JZ_1) - g_1(\nabla_{X_2}JX_1, JZ_1), \\ = \;& g_1(\nabla_{X_1}\phi X_2, JZ_1) + g_1(\nabla_{X_1}\omega X_2, JZ_1) - g_1(\nabla_{X_2}\phi X_1, JZ_1) - g_1(\nabla_{X_2}\omega X_1, JZ_1), \\ = \;& \cos^2\theta_1 g_1(\nabla_{X_1}X_2, Z_1) - \cos^2\theta_1 g_1(\nabla_{X_2}X_1, Z_1) - g_1(\mathcal{T}_{X_1}\omega\phi X_2 - \mathcal{T}_{X_2}\omega\phi X_1, Z_1) \\ & + g_1(\mathcal{H}\nabla_{X_1}\omega X_2 + \mathcal{T}_{X_1}\omega X_2, JPZ_1 + \phi RZ_1 + \omega RZ_1) \\ & - g_1(\mathcal{H}\nabla_{X_2}\omega X_1 + \mathcal{T}_{X_2}\omega X_1, JPZ_1 + \phi RZ_1 + \omega RZ_1). \end{aligned}$$

Now, we have

$$\begin{aligned}&\sin^2\theta_1 g_1([X_1,X_2],Z_1)\\ =\;& g_1(\mathcal{T}_{X_1}\omega X_2-\mathcal{T}_{X_2}\omega X_1, JPZ_1+\phi RZ_1)+g_1(\mathcal{H}\nabla_{X_1}\omega X_2-\mathcal{H}\nabla_{X_2}\omega X_1,\omega RZ_1)\\ &-g_1(\mathcal{T}_{X_1}\omega\phi X_2-\mathcal{T}_{X_2}\omega\phi X_1, Z_1),\end{aligned}$$

which completes the proof. □

The proof of the following theorem is similar as the Theorem 3.2.

Theorem 3.3. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, the slant distribution D_2 is integrable if and only if*

$$\begin{aligned}&g_1(\mathcal{T}_{Z_1}\omega\phi Z_2-\mathcal{T}_{Z_2}\omega\phi Z_1, X_1)\\ =\;& g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2-\mathcal{H}\nabla_{Z_2}\omega Z_1,\omega X_1)+g_1(\mathcal{T}_{Z_1}\omega Z_2-\mathcal{T}_{Z_2}\omega Z_1,\phi X_1)\end{aligned}\tag{3.38}$$

for $Z_1, Z_2 \in \Gamma(D_2)$ and $X_1 \in \Gamma(D \oplus D_1)$.

Theorem 3.4. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then the horizontal distribution $(\ker\pi_*)^\perp$ defines a totally geodesic foliation on N_1 if and only if*

$$\begin{aligned}&g_1(\mathcal{A}_{V_1}V_2, PW_1+\cos^2\theta_1 QW_1+\cos^2\theta_2 RW_1)\\ =\;& g_1(\mathcal{H}\nabla_{V_1}V_2,\omega\phi PW_1+\omega\phi QW_1+\omega\phi RW_1)\\ &+g_1(\mathcal{A}_{V_1}BV_2+\mathcal{H}\nabla_{V_1}CV_2,\omega W_1)\end{aligned}\tag{3.39}$$

for $V_1, V_2 \in \Gamma(\ker\pi_*)^\perp$ and $W_1 \in \Gamma(\ker\pi_*)$.

Proof. For $V_1, V_2 \in \Gamma(\ker\pi_*)^\perp$ and $W_1 \in \Gamma(\ker\pi_*)$, we have

$$g_1(\nabla_{V_1}V_2, W_1) = g_1(\nabla_{V_1}V_2, PW_1+QW_1+RW_1).\tag{3.40}$$

Using equations (2.2), (2.4), (2.11), (2.12), (3.3), (3.4), (3.16) and 3.2, we have

$$\begin{aligned}g_1(\nabla_{V_1}V_2, W_1) &= g_1(\nabla_{V_1}JV_2, JPW_1)+g_1(\nabla_{V_1}JV_2, JQW_1)+g_1(\nabla_{V_1}JV_2, JRW_1),\\ &= g_1(\mathcal{A}_{V_1}V_2, PW_1+\cos^2\theta_1 QW_1+\cos^2\theta_2 RW_1)\\ &\quad -g_1(\mathcal{H}\nabla_{V_1}V_2,\omega\phi PW_1+\omega\phi QW_1+\omega\phi RW_1)\\ &\quad +g_1(\mathcal{A}_{V_1}BV_2+\mathcal{H}\nabla_{V_1}CV_2,\omega PW_1+\omega QW_1+\omega RW_1).\end{aligned}$$

Now, since $\omega PW_1+\omega QW_1+\omega RW_1=\omega W_1$ and $\omega PW_1=0$, one obtains

$$\begin{aligned}g_1(\nabla_{V_1}V_2, W_1) &= g_1(\mathcal{A}_{V_1}V_2, PW_1+\cos^2\theta_1 QW_1+\cos^2\theta_2 RW_1)\\ &\quad -g_1(\mathcal{H}\nabla_{V_1}V_2,\omega\phi PW_1+\omega\phi QW_1+\omega\phi RW_1)\\ &\quad +g_1(\mathcal{A}_{V_1}BV_2+\mathcal{H}\nabla_{V_1}CV_2,\omega W_1).\end{aligned}$$

□

Theorem 3.5. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then the vertical distribution $(\ker \pi_*)$ defines a totally geodesic foliation on N_1 if and only if*

$$\begin{aligned} & g_1(\mathcal{T}_{X_1}X_2, Z_1) + \cos^2\theta_1 g_1(\mathcal{T}_{X_1}QX_2, Z_1) + \cos^2\theta_2 g_1(\mathcal{T}_{X_1}RX_2, Z_1) \\ = \; & g_1(\mathcal{H}\nabla_{X_1}\omega\phi PX_2 + \mathcal{H}\nabla_{X_1}\omega\phi QX_2 + \mathcal{H}\nabla_{X_1}\omega\phi RX_2, Z_1) \\ & + g_1(\mathcal{T}_{X_1}\omega X_2, BZ_1) + g_1(\mathcal{H}\nabla_{X_1}\omega X_2, CZ_1) \end{aligned} \tag{3.41}$$

for $X_1, X_2 \in \Gamma(\ker \pi_*)$ and $Z_1 \in \Gamma(\ker \pi_*)^\perp$.

Proof. For $X_1, X_2 \in \Gamma(\ker \pi_*)$ and $Z_1 \in \Gamma(\ker \pi_*)^\perp$, using equations (2.2), (2.4) and (3.3), we have

$$\begin{aligned} & g_1(\nabla_{X_1}X_2, Z_1) \\ = \; & g_1(\nabla_{X_1}JPX_2, JZ_1) + g_1(\nabla_{X_1}JQX_2, JZ_1) + g_1(\nabla_{X_1}JRX_2, JZ_1). \end{aligned}$$

Now, using equations (2.9), (2.10), (3.4), (3.16) and Lemma 3.2, we have

$$\begin{aligned} & g_1(\nabla_{X_1}X_2, Z_1) \\ = \; & g_1(\mathcal{T}_{X_1}X_2, Z_1) + \cos^2\theta_1 g_1(\mathcal{T}_{X_1}QX_2, Z_1) + \cos^2\theta_2 g_1(\mathcal{T}_{X_1}RX_2, Z_1) \\ & - g_1(\mathcal{H}\nabla_{X_1}\omega\phi PX_2 + \mathcal{H}\nabla_{X_1}\omega\phi QX_2 + \mathcal{H}\nabla_{X_1}\omega\phi RX_2, Z_1) \\ & + g_1(\nabla_{X_1}\omega PX_2 + \nabla_{X_1}\omega QX_2 + \nabla_{X_1}\omega RX_2, JZ_1). \end{aligned}$$

Since $\omega PX_2 + \omega QX_2 + \omega RX_2 = \omega X_2$ and $\omega PX_2 = 0$, we have

$$\begin{aligned} & g_1(\nabla_{X_1}X_2, Z_1) \\ = \; & g_1(\mathcal{T}_{X_1}X_2, Z_1) + \cos^2\theta_1 g_1(\mathcal{T}_{X_1}QX_2, Z_1) + \cos^2\theta_2 g_1(\mathcal{T}_{X_1}RX_2, Z_1) \\ & - g_1(\mathcal{H}\nabla_{X_1}\omega\phi PX_2 + \mathcal{H}\nabla_{X_1}\omega\phi QX_2 + \mathcal{H}\nabla_{X_1}\omega\phi RX_2, Z_1) \\ & + g_1(\mathcal{T}_{X_1}\omega X_2, BZ_1) + g_1(\mathcal{H}\nabla_{X_1}\omega X_2, CZ_1). \end{aligned}$$

□

Theorem 3.6. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, the invariant distribution D defines a totally geodesic foliation on N_1 if and only if*

$$g_1(\mathcal{T}_{U_1}JPU_2, \omega QW_1 + \omega RW_1) = -g_1(\mathcal{V}\nabla_{U_1}JPU_2, \phi QW_1 + \phi RW_1), \tag{3.42}$$

and

$$g_1(\mathcal{T}_{U_1}JPU_2, CW_2) = -g_1(\mathcal{V}\nabla_{U_1}JPU_2, BW_2) \tag{3.43}$$

for $U_1, U_2 \in \Gamma(D), W_1 \in \Gamma(D_1 \oplus D_2)$ and $W_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. For $U_1, U_1 \in \Gamma(D), W_1 \in \Gamma(D_1 \oplus D_2)$ and $W_2 \in \Gamma(\ker \pi_*)^\perp$, using equations (2.2), (2.4), (2.9), (3.3) and (3.4), we have

$$\begin{aligned} g_1(\nabla_{U_1}U_2, W_1) &= g_1(\nabla_{U_1}JU_2, JW_1), \\ &= g_1(\nabla_{U_1}JPU_2, JQW_1 + JRW_1), \\ &= g_1(\mathcal{T}_{U_1}JPU_2, \omega QW_1 + \omega RW_1) + g_1(\mathcal{V}\nabla_{U_1}JPU_2, \phi QW_1 + \phi RW_1). \end{aligned}$$

Using equations (2.2), (2.4), (2.9), (3.3) and (3.16), we have

$$\begin{aligned} g_1(\nabla_{U_1}U_2, W_2) &= g_1(\nabla_{U_1}JU_2, JW_2), \\ &= g_1(\nabla_{U_1}JPU_2, BW_2 + CW_2), \\ &= g_1(\mathcal{V}\nabla_{U_1}JPU_2, BW_2) + g_1(\mathcal{T}_{U_1}JPU_2, CW_2), \end{aligned}$$

which completes the proof. □

Theorem 3.7. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, the slant distribution D_1 defines a totally geodesic foliation on N_1 if and only if*

$$g_1(\mathcal{T}_{V_1}\omega\phi V_2, Z_1) = g_1(\mathcal{T}_{V_1}\omega QV_2, JPZ_1 + \phi RZ_1) + g_1(\mathcal{H}\nabla_{V_1}\omega QV_2, \omega RZ_1), \quad (3.44)$$

and

$$g_1(\mathcal{H}\nabla_{V_1}\omega\phi V_2, Z_2) = g_1(\mathcal{H}\nabla_{V_1}\omega V_2, CZ_2) + g_1(\mathcal{T}_{V_1}\omega V_2, BZ_2) \quad (3.45)$$

for $V_1, V_2 \in \Gamma(D_1), Z_1 \in \Gamma(D \oplus D_2)$ and $Z_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. For $V_1, V_2 \in \Gamma(D_1), Z_1 \in \Gamma(D \oplus D_2)$ and $Z_2 \in \Gamma(\ker \pi_*)^\perp$, using equations (2.2), (2.4), (2.10), (3.3), (3.4) and Lemma 3.2, we have

$$\begin{aligned} & g_1(\nabla_{V_1}V_2, Z_1) \\ = \ & g_1(\nabla_{V_1}\phi V_2, JZ_1) + g_1(\nabla_{V_1}\omega V_2, JZ_1), \\ = \ & \cos^2\theta_1 g_1(\nabla_{V_1}V_2, Z_1) - g_1(\mathcal{T}_{V_1}\omega\phi V_2, Z_1) \\ & + g_1(\mathcal{T}_{V_1}\omega QV_2, JPZ_1 + \phi RZ_1) + g_1(\mathcal{H}\nabla_{V_1}\omega QV_2, \omega RZ_1). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_1 g_1(\nabla_{V_1}V_2, Z_1) \\ = \ & -g_1(\mathcal{T}_{V_1}\omega\phi V_2, Z_1) + g_1(\mathcal{T}_{V_1}\omega QV_2, JPZ_1 + \phi RZ_1) \\ & + g_1(\mathcal{H}\nabla_{V_1}\omega QV_2, \omega RZ_1) \end{aligned}$$

Next, from equations (2.2), (2.4), (2.10), (3.3), (3.16) and Lemma 3.2, we have

$$\begin{aligned} g_1(\nabla_{V_1}V_2, Z_2) &= g_1(\nabla_{V_1}\phi V_2, JZ_2) + g_1(\nabla_{V_1}\omega V_2, JZ_2), \\ &= \cos^2\theta_1 g_1(\nabla_{V_1}V_2, Z_2) - g_1(\mathcal{H}\nabla_{V_1}\omega\phi V_2, Z_2) \\ &\quad + g_1(\mathcal{H}\nabla_{V_1}\omega V_2, CZ_2) + g_1(\mathcal{T}_{V_1}\omega V_2, BZ_2). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_1 g_1(\nabla_{V_1}V_2, Z_2) \\ = \ & -g_1(\mathcal{H}\nabla_{V_1}\omega\phi V_2, Z_2) + g_1(\mathcal{H}\nabla_{V_1}\omega V_2, CZ_2) + g_1(\mathcal{T}_{V_1}\omega V_2, BZ_2). \end{aligned}$$

□

The proof of the following theorem is similar as the Theorem 3.7.

Theorem 3.8. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, the slant distribution D_2 defines a totally geodesic foliation on N_1 if and only if*

$$g_1(\mathcal{T}_{U_1}\omega\phi U_2, Y_1) = g_1(\mathcal{T}_{U_1}\omega QU_2, JPY_1 + \phi RY_1) + g_1(\mathcal{H}\nabla_{U_1}\omega QU_2, \omega RY_1), \quad (3.46)$$

and

$$g_1(\mathcal{H}\nabla_{U_1}\omega\phi U_2, Y_2) = g_1(\mathcal{H}\nabla_{U_1}\omega U_2, CY_2) + g_1(\mathcal{T}_{U_1}\omega U_2, BY_2) \quad (3.47)$$

for $U_1, U_2 \in \Gamma(D_2), Y_1 \in \Gamma(D \oplus D_1)$ and $Y_2 \in \Gamma(\ker \pi_*)^\perp$.

Theorem 3.9. *Let π be a quasi bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, π is a totally geodesic map if and only if*

$$\begin{aligned} & g_1(\mathcal{H}\nabla_{V_1}\omega\phi QV_2 + \mathcal{H}\nabla_{V_1}\omega\phi RV_2 - \cos^2\theta_1\nabla_{V_1}QV_2 - \cos^2\theta_2\nabla_{V_1}RV_2, U_1) \\ = \; & g_1(\mathcal{V}\nabla_{V_1}JPV_2 + \mathcal{T}_{V_1}\omega QV_2 + \mathcal{T}_{V_1}\omega RV_2, BU_1) \\ & + g_1(\mathcal{T}_{V_1}JPV_2 + \mathcal{H}\nabla_{V_1}\omega QV_2 + \mathcal{H}\nabla_{V_1}\omega RV_2, CU_1), \end{aligned}$$

and

$$\begin{aligned} & g_1(\mathcal{H}\nabla_{U_1}\omega\phi QV_1 + \mathcal{H}\nabla_{U_1}\omega\phi RV_1 - \cos^2\theta_1\nabla_{U_1}QV_1 - \cos^2\theta_2\nabla_{U_1}RV_1, U_2) \\ = \; & g_1(\mathcal{V}\nabla_{U_1}JPV_1 + \mathcal{A}_{U_1}\omega QV_1 + \mathcal{A}_{U_1}\omega RV_1, BU_2) \\ & + g_1(\mathcal{A}_{U_1}JPV_1 + \mathcal{H}\nabla_{U_1}\omega QV_1 + \mathcal{H}\nabla_{U_1}\omega RV_1, CU_2) \end{aligned}$$

for $V_1, V_2 \in \Gamma(\ker \pi_*)$ and $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Since π is a Riemannian map, we have

$$(\nabla\pi_*)(U_1, U_2) = 0, \quad (3.48)$$

for $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$.

For $V_1, V_2 \in \Gamma(\ker \pi_*)$ and $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$, using equations (2.2), (2.4), (2.9), (2.10), (2.15), (3.3), (3.4) and Lemma 3.2, we have

$$\begin{aligned} & g_2((\nabla\pi_*)(V_1, V_2), \pi_* U_1) \\ = \; & -g_1(\nabla_{V_1}V_2, U_1) \\ = \; & -g_1(\nabla_{V_1}JV_2, JU_1) \\ = \; & -g_1(\nabla_{V_1}JPV_2, JU_1) - g_1(\nabla_{V_1}JQV_2, JU_1) - g_1(\nabla_{V_1}JRV_2, JU_1), \\ = \; & -g_1(\nabla_{V_1}JPV_2, JU_1) - g_1(\nabla_{V_1}\phi QV_2, JU_1) - g_1(\nabla_{V_1}\phi RV_2, JU_1) \\ & -g_1(\nabla_{V_1}\omega QV_2, JU_1) - g_1(\nabla_{V_1}\omega RV_2, JU_1), \end{aligned}$$

$$\begin{aligned} & g_2((\nabla\pi_*)(V_1,V_2),\pi_*U_1) \qquad (3.49)\\ = & -g_1(\mathcal{V}\nabla_{V_1}JPV_2+\mathcal{T}_{V_1}\omega QV_2+\mathcal{T}_{V_1}\omega RV_2,U_1)\\ & -g_1(\mathcal{T}_{V_1}JPV_2+\mathcal{H}\nabla_{V_1}\omega QV_2+\mathcal{H}\nabla_{V_1}\omega RV_2,CU_1)\\ & -g_1(\cos^2\theta_1\nabla_{V_1}QV_2+\cos^2\theta_2\nabla_{V_1}RV_2-\mathcal{H}\nabla_{V_1}\omega\phi QV_2-\mathcal{H}\nabla_{V_1}\omega\phi RV_2,U_1). \end{aligned}$$

Next, using equations (2.2), (2.4), (2.9), (2.10), (2.15), (3.3), (3.4), (3.16) and Lemma 3.2, we have

$$\begin{aligned} & g_2((\nabla\pi_*)(U_1,V_1),\pi_*U_2)\\ = & -g_1(\nabla_{U_1}V_1,U_2)\\ = & -g_1(\nabla_{U_1}JV_1,JU_2)\\ = & -g_1(\nabla_{U_1}JPV_1,JU_2)-g_1(\nabla_{U_1}JQV_1,JU_2)-g_1(\nabla_{U_1}JRV_1,JU_2),\\ = & -g_1(\nabla_{U_1}JPV_1,JU_2)-g_1(\nabla_{U_1}\phi QV_1,JU_2)-g_1(\nabla_{U_1}\phi RV_1,JU_2)\\ & -g_1(\nabla_{U_1}\omega QV_1,JU_2)-g_1(\nabla_{U_1}\omega RV_1,JU_2), \end{aligned}$$

$$\begin{aligned} & g_2((\nabla\pi_*)(U_1,V_1),\pi_*U_2) \qquad (3.50)\\ = & -g_1(\mathcal{V}\nabla_{U_1}JPV_1+\mathcal{A}_{U_1}\omega QV_1+\mathcal{A}_{U_1}\omega RV_1,BU_2)\\ & -g_1(\mathcal{A}_{U_1}JPV_1+\mathcal{H}\nabla_{U_1}\omega QV_1+\mathcal{H}\nabla_{U_1}\omega RV_1,CU_2)\\ & -g_1(\cos^2\theta_1\nabla_{U_1}QV_1+\cos^2\theta_2\nabla_{U_1}RV_1-\mathcal{H}\nabla_{U_1}\omega\phi QV_1-\mathcal{H}\nabla_{U_1}\omega\phi RV_1,U_2). \end{aligned}$$

The proof follows in view of equations (3.49) and (3.50). □

4. Example

Note that given an Euclidean space R^{2s} with coordinates $(y_1, y_2, ..., y_{2s-1,}y_{2s})$ we can canonically choose an almost complex structure J on R^{2s} as follows:

$$\begin{aligned} & J(a_1\frac{\partial}{\partial y_1}+a_2\frac{\partial}{\partial y_2}+..........+a_{2s-1}\frac{\partial}{\partial y_{2s-1}}+a_{2s}\frac{\partial}{\partial y_{2s}})\\ = & -a_2\frac{\partial}{\partial y_1}+a_1\frac{\partial}{\partial y_2}+............-a_{2s}\frac{\partial}{\partial y_{2s-1}}+a_{2s-1}\frac{\partial}{\partial y_{2s}}, \end{aligned}$$

where $a_1, a_2, ..., a_{2s}$ are C^∞ functions defined on R^{2s}. Throughout this section, we will use this notation.

Example 4.1. Define a map $\pi : R^{16} \to R^8$ by

$$\pi(y_1,y_2,.........,y_{15},y_{16})=(y_3\sin\alpha-y_5\cos\alpha,2021,y_6,y_7\sin\beta-y_9\cos\beta,2022,y_{10},y_{13},y_{14}), \qquad (4.1)$$

which is a quasi bi-slant Riemannian map such that

$$\begin{aligned} V_1 = & \frac{\partial}{\partial y_1},V_2=\frac{\partial}{\partial y_2},V_3=\cos\alpha\frac{\partial}{\partial y_3}+\sin\alpha\frac{\partial}{\partial y_5},V_4=\frac{\partial}{\partial y_4},\\ V_5 = & \cos\beta\frac{\partial}{\partial y_7}+\sin\beta\frac{\partial}{\partial y_9},V_6=\frac{\partial}{\partial y_8},V_7=\frac{\partial}{\partial y_{11}},V_8=\frac{\partial}{\partial y_{12}},V_9=\frac{\partial}{\partial y_{15}},V_{10}=\frac{\partial}{\partial y_{16}}, \end{aligned}$$

$$\ker \pi_* = D \oplus D_1 \oplus D_2, \tag{4.2}$$

where

$$\begin{aligned} D &= < V_1 = \frac{\partial}{\partial y_1}, V_2 = \frac{\partial}{\partial y_2}, V_7 = \frac{\partial}{\partial y_{11}}, V_8 = \frac{\partial}{\partial y_{12}}, V_9 = \frac{\partial}{\partial y_{15}}, V_{10} = \frac{\partial}{\partial y_{16}} >, \\ D_1 &= < V_3 = \cos\alpha \frac{\partial}{\partial y_3} + \sin\alpha \frac{\partial}{\partial y_5}, V_4 = \frac{\partial}{\partial y_4} >, \\ D_2 &= < V_5 = \cos\beta \frac{\partial}{\partial y_7} + \sin\beta \frac{\partial}{\partial y_9}, V_6 = \frac{\partial}{\partial y_8} >, \end{aligned}$$

$$\begin{aligned} & (\ker \pi_*)^{\perp} \\ = & < \frac{\partial}{\partial y_6}, \sin\alpha \frac{\partial}{\partial y_3} - \cos\alpha \frac{\partial}{\partial y_5}, \sin\beta \frac{\partial}{\partial y_7} - \cos\beta \frac{\partial}{\partial y_9}, \frac{\partial}{\partial y_{10}}, \frac{\partial}{\partial y_{13}}, \frac{\partial}{\partial y_{14}} > \end{aligned}$$

with bi-slant angles α and β.

Example 4.2. Define a map $\pi : R^{14} \to R^8$ by

$$\pi(y_1, y_2, ..., y_{13}, y_{14}) = (\frac{y_1 - y_3}{\sqrt{2}}, 101, y_2, \frac{y_7 - \sqrt{3}y_9}{2}, 202, y_{10}, y_{13}, y_{14}), \tag{4.3}$$

which is a quasi bi-slant Riemannian map such that

$$\begin{aligned} V_1 &= \frac{1}{\sqrt{2}}(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3}), V_2 = \frac{\partial}{\partial y_4}, V_3 = \frac{\partial}{\partial y_5}, V_4 = \frac{\partial}{\partial y_6}, \\ V_5 &= \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial y_7} + \frac{\partial}{\partial y_9}), V_6 = \frac{\partial}{\partial y_8}, V_7 = \frac{\partial}{\partial y_{11}}, V_8 = \frac{\partial}{\partial y_{12}}, \end{aligned}$$

$$\ker \pi_* = D \oplus D_1 \oplus D_2, \tag{4.4}$$

$$\begin{aligned} D &= < V_3 = \frac{\partial}{\partial y_5}, V_4 = \frac{\partial}{\partial y_6}, V_7 = \frac{\partial}{\partial y_{11}}, V_8 = \frac{\partial}{\partial y_{12}} >, \\ D_1 &= < V_1 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3}), V_2 = \frac{\partial}{\partial y_4} >, \\ D_2 &= < V_5 = \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial y_7} + \frac{\partial}{\partial y_9}), V_6 = \frac{\partial}{\partial y_8} >, \end{aligned}$$

$$\begin{aligned} & (\ker \pi_*)^{\perp} \\ = & < \frac{\partial}{\partial y_2}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_3}), \frac{1}{2}(\frac{\partial}{\partial y_7} - \sqrt{3}\frac{\partial}{\partial y_9}), \frac{\partial}{\partial y_{10}}, \frac{\partial}{\partial y_{13}}, \frac{\partial}{\partial y_{14}} > \end{aligned}$$

with bi-slant angles $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{6}$.

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