FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 3 (2023), 473 - 486
https://doi.org/10.22190/FUMI220801031P
Original Scientific Paper

SOME RESULTS ON RICCI SOLITON ON CONTACT METRIC MANIFOLDS

Pankaj Pandey[1] and Kamakshi Sharma[1]

[1]**Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, India**

Abstract. The objective of this paper is to study the nature of Ricci soliton admitting various type of contact metric manifolds such as Kenmostu manifold, LP Sasakian manifold and LCS manifold. In this paper, it is proved that a Kemnotsu Ricci soliton is expanding, whereas the LP-Sasakian Ricci soliton is shrinking. Further, the conditions have been obtained on $(LCS)_n$ Ricci soliton to be expanding, shrinking and steady and the results are verified by suitable examples. It is also proved that the possible values of soliton constant is the set of all even integers $\mathbb{Z}^{2n}$ and the set of negative integres $\mathbb{Z}^{-}$ respectively for Kenmostu Ricci soliton and LP Sasakian Ricci Soliton and if the (LCS) Ricci soliton is expanding then soliton constant lies on the interval $(0, \infty)$ and for shrinking it lies on $(-\infty, 0)$. The Projectively flat cases for the above manifolds are also discussed to be expanding, shrinking and steady. Finally, we study these Ricci solitons admitting Ricci semi-symmetric condition $R.S = 0$ and prove that the soliton constant λ is an eigen value of metric tensor g with respect to associated vector field ξ.
Keywords: Ricci soliton, Contant metric manifolds, Kenmostu manifold.

1. Introduction

The Ricci flow was introduced by Richard S. Hamilton [10] in 1982 to study compact three dimensional manifolds with positive Ricci curvature and called the equation defined Ricci flow as a evolution equation. The concept of Ricci soliton also introduced by Hamilton [10] and identified as a generalization of an Einstein

Received August 01, 2022. accepted August 18, 2023.
Communicated by Ljubica Velimirović
Corresponding Author: Pankaj Pandey, Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara-144411, India | E-mail: pankaj.anvarat@gmail.com
2010 *Mathematics Subject Classification.* Primary 53C25, 53C50, 53D15; Secondary 35C08, 53C44

metric. In a Riemannian manifold (M, g), g is called a Ricci soliton [10] if it satisfies the following equation:

$$L_v g(F,G) + 2S(F,G) + 2\lambda g(F,G) = 0, \tag{1.1}$$

where L denotes the Lie derivative along the complete vector field v, S is the Ricci tensor, λ is a constant and called soliton constatnt, and X, Y are vector fields on $\chi(M)$. The value of soliton constant play an important role to study the nature of the Ricci soliton. A Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively.

Kenmotsu, in 1972 studied a class of Riemannian manifolds satisfying conditions which are known as Kenmotsu manifolds [18]. The notion of LP-Sasakian manifold was introduced by K. Matsumoto [15] in 1989. In [28] Yano and Sawaki defined and studied a tensor field W on a Riemannian manifold of dimension n which includes both the conformal curvature tensor C and the concircular curvature tensor C' as special cases.
The notion of Lorentzian concircular manifolds was introduced by Shaikh [27] in 2003. The $(LCS)_n$ manifold has been studied by several authors [12, 23, 27]. Pokhariyal [21] studied some properties of this curvature tensor in a Sasakian manifold. Matsumoto et al. [16] and, Yildiz and De [34] studied W_2-curvature tensor in LP-Sasakian and Kenmotsu manifolds, respectively. In the continuation of this study, Pandey and Chaturvedi also studied the constant and generalized quasi constant curvature tensor and established some examples of various Riemannian manifolds [3, 4, 19, 20].

Recently, Y.C. Mandal and S.K. Hui [14] studied Yambe soliton with torse forming vector field and obtained the conditions of existence for shrinking steady and expanding and contruct an example to prove thier results. Shaikh and S.K. Hui in their paper introduced the notion of generalized ϕ-recurrent β-Kenmotsu manifolds and obtained the necessary and sufficient condition for manifold to be generalized ricci recurrent manifold. Recently, the concepts of Ricci soliton are generalized and grew up by several authors [2, 5, 6, 7, 8, 11, 12, 18, 23, 24, 25, 30, 1].

2. Preliminaries

Definition 2.1. Kenmotsu manifold:
Let M^{2n+1} be an almost contact Riemannian manifold, where ϕ is a $(1,1)$ tensor field, η is a 1-form and g is the riemannian metric which satisfy [24]

$$\begin{aligned} &\phi\xi = 0, \quad \eta(\phi F) = 0, \quad \eta(\xi) = 1, \quad g(F,\xi) = \eta(F), \\ &\phi^2 F = -F + \eta(F)\xi, \quad g(\phi F, \phi G) = g(F,G) - \eta(F)\eta(G), \end{aligned} \tag{2.1}$$

for any vector fields F, G on M and

$$\nabla_F \xi = F - \eta(F)\xi, \qquad (\nabla_F \phi)G = -\eta(G)\phi(F) - g(X, \phi Y)\xi, \tag{2.2}$$

where ∇ denotes the Riemannian connection of g.

Definition 2.2. LP-Sasakian manifold:
An n-dimensional differentiable manifold M is called an LP-Sasakian manifold ([28], [34]) if it admits a (1, 1) tensor field ϕ , a contravariant vector field ξ , a 1-form η and a Lorentzian metric g which satisfy [32]

$$\begin{aligned} \eta(\xi) = -1, \quad \phi^2 F = F + \eta(F)\xi, \\ g(\phi F, \phi G) = g(F, G) + \eta(F)\eta(G), \\ g(F, \xi) = \eta(F), \end{aligned} \tag{2.3}$$

$$(\nabla_F \phi)F = g(F, G)\xi + \eta(G)F + 2\eta(F)\eta(G)\xi, \qquad \nabla_F \xi = \phi F, \tag{2.4}$$

where ∇ denotes the operator of covarient differentiation with respect to the Lorentzian metric g.

Definition 2.3. $(LCS)_n$-manifold:
Let (M, g) be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ. More general the Lorentzian manifold M together with the unit timelike concircular vector field ξ, an 1-form η, and an $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold $(M, g, \xi, \eta, \phi, \alpha)$ (briefly,$(LCS)_n$-manifold), which was introduced by A. A. Shaikh [26].

$$\begin{aligned} \eta(\xi) = -1, \phi\xi - 0, \phi^2 F = X + \eta(F)\xi, \\ g(\phi F, \phi G) = g(F, G) + \eta(F)\eta(G), \\ g(F, \xi) = \eta(F), \end{aligned} \tag{2.5}$$

for any vector fields F, G on M.

$$(\nabla_F \phi)G = -\eta(G)\phi(F) - g(F, \phi G)\xi, \qquad (\nabla_F \xi) = \alpha[X + \eta(X)\xi] \tag{2.6}$$

where ∇ denotes the Riemannian connection of g.

3. Ricci Soliton on Kenmotsu Manifold

Let M^{2n+1} be a Kenmotsu manifold with structure (ϕ, ξ, η, g) then we have [24],

$$\eta(R(F, G)H) = g(F, H)\eta(G) - g(G, H)\eta(F), \tag{3.1}$$

$$R(\xi, F)G = \eta(G)F - g(F, G)\xi, \tag{3.2}$$

$$R(F, G)\xi = \eta(F)G - \eta(G)F, \tag{3.3}$$

$$S(F,\xi) = -2n\eta(F), \tag{3.4}$$

$$S(\phi F, \phi G) = S(F,G) + 2n\eta(F)\eta(G), \tag{3.5}$$

$$(\nabla_F \eta)G = g(F,G) - \eta(F)\eta(G), \tag{3.6}$$

for the vector fields F, G, H, where R denotes the Riemannian curvature tensor. Let (g, ξ, λ) be a Ricci soliton on Kenmotsu manifold M. Then, from (2.2) we get,

$$L_v g(F,G) = 2(g(F,G) - \eta(F)\eta(G)). \tag{3.7}$$

Taking $V = \xi$ in (3.7) and using (1.1) and (2.2), we obtain:

$$S(F,G) = -(\lambda+1)g(F,G) + \eta(F)\eta(G), \tag{3.8}$$

$$QF = -(\lambda+1)F + \eta(F), \tag{3.9}$$

$$S(F,\xi) = -\lambda\eta(F), \tag{3.10}$$

$$r = -\lambda(2n+1) - 2n. \tag{3.11}$$

Let M^{2n+1} be a Kenmotsu manifold with structure (ϕ, ξ, η, g) then using (3.7) in (1.1), we get

$$(\lambda+1)g(F,G) - \eta(F)\eta(G) + S(F,G) = 0, \tag{3.12}$$

Taking $G = \xi$ in (3.12) and using (3.4), we get

$$(\lambda - 2n)\eta(F) = 0, \tag{3.13}$$

As $\eta(F) \neq 0$

$$\lambda = 2n. \tag{3.14}$$

So, we can state the following theorem:

Theorem 3.1. *A $(2n+1)$-dimensional Kemnotsu Ricci soliton is always expanding and the set of all possible values of soliton constant λ is a discrete set of even integers $\mathbb{Z}^{2n}$.*

Also, from the equation (3.8), we have the result:

Corollary 3.1. *If a $(2n+1)$-dimensional Kenmotsu manifold admits a Ricci Soliton, then the manifold is η-Einstein manifold.*

Here, we present an example to verify the above result stated in **Theorem 3.1**.

Example 3.1. Let us consider a 3-dimensional manifold $M = (f, g, h) : h \neq 0$. Let e_1, e_2, e_3 be be a linearly independent global frame on M given by

$$e_1 = h^2 \frac{d}{df}, \quad e_2 = h^2 \frac{d}{dg}, \quad e_3 = \frac{d}{dh} \tag{3.15}$$

Let g be the Riemannian metric defined by $g(e_i, e_j) = 1, \;\; if \;\; i = j \;\; and \;\; 0 \;\; otherwise$, then Koszul formula yields

$$\begin{gathered} \nabla_{e_1} e_1 = \frac{2}{z} e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\frac{2}{z} e_1, \\ \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = \frac{2}{z} e_3, \quad \nabla_{e_2} e_3 = -\frac{2}{z} e_2, \\ \nabla_{e_3} e_1 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_3} e_3 = 0, \end{gathered} \tag{3.16}$$

The scalar curvature of this manifold is also computed and it is $r = \frac{32}{h^2}$. Since (e_1, e_2, e_3) forms a basis, any vector field $F, G, K \in \chi(M)$ can be written as

$$\begin{aligned} F &= a_1 e_1 + b_1 e_2 + c_1 e_3, \\ G &= a_2 e_1 + b_2 e_2 + c_2 e_3, \\ K &= a_3 e_1 + b_3 e_2 + c_3 e_3, \end{aligned}$$

where $a_i, b_i, c_i \in R^+$ for $i = 1, 2, 3$.
If we choose the 1-form η by $\eta(W) = g(W, e_2)$ for any $W \in \chi(M)$, then we have the relation

$$\nabla_F G = F - \eta(F) G, \tag{3.17}$$

that is G is a torse-forming vector field.
Now from $(***)$ we get

$$L_v g(F, G) = g(\nabla_F G, K) + g(F, \nabla_K G). \tag{3.18}$$

which yields

$$L_v g(F, G) = 2(g(F, G) - \eta(F)\eta(G)), \tag{3.19}$$

Also we can calculate

$$\begin{aligned} g(F, K) &= a_1 a_3 + b_1 b_3 + c_1 c_3, \\ g(G, K) &= a_2 a_3 + b_2 b_3 + c_2 c_3, \\ g(K, H) &= a_1 a_2 + b_1 b_2 + c_1 c_2, \end{aligned} \tag{3.20}$$

$$\eta(F) = b_1, \quad \eta(G) = b_2, \quad \eta(K) = b_3, \tag{3.21}$$

Using equation (3.20) and (3.21) in (3.13) we get

$$(a_1 a_3 + b_1 b_3 + c_1 c_3 - b_1 b_3) + S(F, K) + \lambda(a_1 a_3 + b_1 b_3 + c_1 c_3) = 0. \tag{3.22}$$

After simplification, we have

$$S(F, K) + (\lambda + 1)(a_1 a_3 + b_1 b_3 + c_1 c_3) - b_1 b_3, \tag{3.23}$$

which yields

$$S(F,K) + (\lambda+1)g(F,K) - \eta(F)\eta(K). \tag{3.24}$$

Put $K = \xi$ and using (3.4), the equation (3.24) gives the result

$$(\lambda - 2)\eta(F) = 0, \tag{3.25}$$

As $\eta(F) \neq 0$

$$\lambda = 2, \tag{3.26}$$

which verifies the result of theorem 3.1 for 3-dimensional Kenmostu manifolds that is for $n = 1$.

4. Ricci Soliton on LP-Sasakian Manifold

In an LP-Sasakian manifold, we know that the following conditions hold [32]:

$$\phi\xi = 0, \qquad \eta(\phi F) = 0, \qquad rank \;\; \phi = n - 1. \tag{4.1}$$

Let M be an n-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g) then we have the following results [32]:

$$g(R(F,G)H, \xi) = \eta(R(F,G)H) = g(G,H)\eta(F) - g(F,H)\eta(G), \tag{4.2}$$

$$R(\xi, F)G = g(F,G)\xi - \eta(G)F, \tag{4.3}$$

$$R(F,G)\xi = \eta(G)F - \eta(F)G, \tag{4.4}$$

$$R(\xi, F)\xi = F + \eta(F)\xi, \tag{4.5}$$

$$S(F,\xi) = (n-1)\eta(F), \tag{4.6}$$

$$S(\phi F, \phi G) = S(F,G) + (n-1)\eta(F)\eta(G), \tag{4.7}$$

$$(\nabla_F \eta)G = g(\phi F, G), \tag{4.8}$$

for the vector fields F, G, H, where R denotes the Riemannian curvature tensor. Let (g, ξ, λ) be a Ricci soliton on LP-Sasakian manifold on M. Then, from (2.4) we get,

$$L_v g(F,G) = 0. \tag{4.9}$$

Taking $V = \xi$ in (4.9) and using (1.1) and (2.4), we obtain:

$$S(F,G) = -\lambda g(F,G). \tag{4.10}$$

$$QF = -\lambda F, \tag{4.11}$$

$$S(F,\xi) = -\lambda\eta(F), \tag{4.12}$$

$$r = -\lambda n. \tag{4.13}$$

Let M^n be an LP-Sasakian manifold with structure (ϕ, ξ, η, g), then using (4.9) in (1.1), we get

$$\lambda g(F,G)) + S(F,G) = 0, \tag{4.14}$$

Taking $G = \xi$ in (4.14) and using (3.20), we have

$$(\lambda + (n-1))\eta(F) = 0 \tag{4.15}$$

As $\eta(F) \neq 0$

$$\lambda = -(n-1), \tag{4.16}$$

So, we have the following theorem:

Theorem 4.1. *An n-dimensional LP-Sasakian Ricci soliton is always shrinking and the set of all possible values of soliton constant λ is a discrete set of negative integers $\mathbb{Z}^-$.*

The equation (4.10) gives the following result:

Corollary 4.1. *If an n-dimensional LP-Sasakian manifold admits a Ricci Soliton, then the manifold is Einstein manifold.*

Now, we construct an example to verify the above result.

Example 4.1. Let us consider the case of **example 3.1** and choose the 1-form η by $\eta(W) = g(W, e_2)$ for any $W \in \chi(M)$, then one can esily verify that the following relation holds:

$$\nabla_F G = \phi F. \tag{4.17}$$

Now from (4.17) we get

$$L_v g(F,G) = g(\nabla_F G, K) + g(F, \nabla_K G) L_v g(F,G) = 0 \tag{4.18}$$

Using (3.20) and (3.21), equation (4.14) yields

$$0 + S(F, K) + \lambda(a_1a_3 + b_1b_3 + c_1c_3) = 0, \tag{4.19}$$

which simplifies

$$S(F, K) + \lambda(a_1a_3 + b_1b_3 + c_1c_3). \tag{4.20}$$

From above we have

$$S(F, K) + \lambda g(F, K). \tag{4.21}$$

Put $K = \xi$ and using (4.6), we obtain

$$(\lambda + 2)\eta(F) = 0 \tag{4.22}$$

As $\eta(F) \neq 0$

$$\lambda = -2, \tag{4.23}$$

which verifies the theorem for dimension 3.

5. Ricci Soliton On $(LCS)_n$-manifold

In an $(LCS)_n$ manifold, $n > 2$, we know the following relation holds [12]:

$$\eta(R(F, G)H) = (\alpha^2 - \rho)g(G, H)\eta(F) - g(F, H)\eta(G), \tag{5.1}$$

$$(R(F, G)\xi) = (\alpha^2 - \rho)\eta(G)F - \eta(F)G, \tag{5.2}$$

$$(R(\xi, G)H) = (\alpha^2 - \rho)g(G, H)\xi - \eta(H)G, \tag{5.3}$$

$$S(F, \xi) = (n - 1)(\alpha^2 - \rho)\eta(F) \tag{5.4}$$

$$S(\phi F, \phi G) = S(F, G) + (\alpha^2 - \rho)\eta(F)\eta(G), \tag{5.5}$$

$$\nabla_F\xi = \alpha[F + \eta(F)\xi], \tag{5.6}$$

Let (g, ξ, λ) be a Ricci soliton on (LCS_n) manifold on M then from (2.2) we get,

$$L_v g(F, G) = 2(g(F, G) - \eta(F)\eta(G)) \tag{5.7}$$

Taking $V = \xi$ in (5.7) and using (1.1) and (2.6), we obtain

$$S(F, G) = -(\alpha + \lambda)g(F, G) - \alpha\eta(F)\eta(G), \tag{5.8}$$

$$S(F, \xi) = -\lambda\eta(F), \tag{5.9}$$

$$QF = -(\alpha+\lambda)F - \alpha\eta(F)\xi, \tag{5.10}$$

$$r = -n\lambda - (n-1)\alpha \tag{5.11}$$

Let M be an $(LCS)_n$ manifold with structure (ϕ, ξ, η, g).
Using (5.7) in (1.1), we get

$$(\alpha+\lambda)g(F,G) + \alpha\eta(F)\eta(G) + S(F,G) = 0, \tag{5.12}$$

Put $Y = \xi$ in (5.12) and using (5.4), we obtain

$$[\lambda + (n-1)(\alpha^2 - \rho)]\eta(F) = 0 \tag{5.13}$$

As $\eta(F) \neq 0$, so we have

$$\lambda = -(n-1)(\alpha^2 - \rho) \tag{5.14}$$

Theorem 5.1. *An n-dimensional $(LCS)_n$ Ricci soliton with vector field ξ is:*
(i) expanding if $(\alpha^2 < \rho)$ and the soliton constant λ lies on the interval $(0, \infty)$,
(ii) shrinking if $(\alpha^2 > \rho)$ the soliton constant λ lies on the interval $(-\infty, 0)$,
(iii) steady if $\alpha^2 = \rho$.

The equation (5.8) yields the following:

Theorem 5.2. *If an n-dimensional (LCS)- manifold admits a Ricci Soliton, then the manifold is η-Einstein manifold.*

Here we construct an example to verify the above result.

Example 5.1. Let us consider the case of **example 3.1** and choose the 1-form η by $\eta(W) = g(W, e_2)$ for any $W \in \chi(M)$, then one can esily verify that the following relation holds:

$$\nabla_F G = \alpha[F + \eta(F)\xi]. \tag{5.15}$$

Now from (5.15) we get

$$L_v g(F,G) = g(\nabla_F G, K) + g(F, \nabla_K G) \tag{5.16}$$

$$L_v g(F,G) = 2(g(F,G) - \eta(F)\eta(G)). \tag{5.17}$$

Using (3.20) and (3.21), equation (5.12) gives the following

$$\alpha(a_1a_3 + b_1b_3 + c_1c_3 + b_1b_3) + S(F,K) + \lambda(a_1a_3 + b_1b_3 + c_1c_3) = 0. \tag{5.18}$$

We can easily find from above equation

$$(\alpha+\lambda)((a_1a_3 + b_1b_3 + c_1c_3) + S(F,K) + \alpha(b_1b_3) = 0, \tag{5.19}$$

which yields the following relation

$$S(F,K)+(\alpha+\lambda)g(F,K)+\alpha\eta(F)\eta(K). \tag{5.20}$$

Put $K=\xi$ and using (5.4), we obtain

$$(\lambda+2(\alpha^2-\rho))\eta(F)=0 \tag{5.21}$$

As $\eta(F)\neq 0$

$$\lambda=-2(\alpha^2-\rho). \tag{5.22}$$

Hence, theorem 5.1 has been verified.

6. Ricci Soliton on Projectively flat Kenmotsu manifold

The Projective curvature tensor of (M,g) is given by [9]

$$P(F,G)H=R(F,G)H-\frac{1}{n-1}[S(G,H)F-S(F,H)G]. \tag{6.1}$$

Since, for Projectively flat

$$P(F,G)H=0, \tag{6.2}$$

and hence the equation (6.1) reduced to

$$R(F,G)H=\frac{1}{n-1}[S(G,H)F-S(F,H)G], \tag{6.3}$$

Replace H by ξ in (6.3), we get

$$R(F,G)\xi=\frac{1}{n-1}[S(G,\xi)F-S(F,\xi)G], \tag{6.4}$$

Substituting (3.3) in (6.4), we have

$$\eta(F)G-\eta(G)F=-\frac{\lambda}{n-1}[\eta(F)G-\eta(G)F], \tag{6.5}$$

which yields

$$\lambda=-(n-1). \tag{6.6}$$

This leads to the following statement:

Theorem 6.1. *A Projectively flat Kenmotsu Ricci soliton (M,g) is always shrinking and the value of soliton constant is $\lambda=(n-1)$, that is the set of all positive integers $\mathbb{Z}^+$.*

Similarly, we can calculate the results for projectively flat LP-Sasakian Ricci soliton and projectively flat $(LCS)_n$-Ricci soliton and we have the following statements:

Theorem 6.2. *Projectively flat LP-Sasakian manifold (M, g), admitting a Ricci soliton, with ξ being the vector field of the contact metric structure is always shrinking and the value of soliton constant $\lambda = -(n-1)$, that is the set of all negative integers $\mathbb{Z}^-$.*

Theorem 6.3. *Projectively flat $(LCS)_n$ manifold (M,g), admitting a Ricci soliton, with ξ being the vector field of the contact metric structure is (i) expanding if $(\alpha^2 - \rho)$ is negative and soliton constant lies on the interval $(0, \infty)$,*
(ii) shrinking if $(\alpha^2 - \rho)$ is positive and soliton constant lies on the interval $(-\infty, 0)$ and
(iii) steady if $\alpha^2 = \rho$.

7. Ricci Soliton admitting the semi-symmetric condition $R.S = 0$

Definition 7.1. A Kenmotsu manifold M is said to be Ricci semi-symmetric if the condition

$$R(F, G).S = 0 \tag{7.1}$$

holds for all vector fields F, G.

From , we have

$$(R(F, G).S)(H, K) = -S(R(F, G)H, K) - S(H, R(F, G)K) \tag{7.2}$$

Considering $R.S = 0$ and substituting $F = \xi$ in (7.2), we obtain

$$S(R(\xi, G)H, K) + S(H, R(\xi, G)K) = 0. \tag{7.3}$$

Using (3.2) in (7.3), we get

$$\eta(H)S(G, K) - g(G, H)S(\xi, K) + \eta(K)S(H, G) - g(G, K)S(H, \xi) = 0. \tag{7.4}$$

Put $H = \xi$ in (7.4) and using (2.1) and (3.10) and on simplification we get

$$S(G, K) - \lambda\eta(G)\eta(K) + \lambda\eta(K)\eta(G) + 2ng(G, K) = 0. \tag{7.5}$$

Above equation implies

$$S(G, K) = -\lambda g(G, K). \tag{7.6}$$

After substituting $K = \xi$, we have

$$S(G, \xi) = -\lambda g(G, \xi), \tag{7.7}$$

which is the case of eigen value and eigen vector and therefore, we can state the following theorem:

Theorem 7.1. *If a Kenmotsu Ricci soliton admits the Ricci semi-symmetric property $R.S = 0$ then the Ricci soliton constatnt λ is eigen value of associated vector field ξ.*

After operating $H = \xi$ in (7.4) and using (2.1) and (3.4) and we get

$$S(G, K) = -2ng(G, K). \tag{7.8}$$

Therefore, we can state here a corellary as follows:

Corollary 7.1. *If a Kenmotsu manifold admits the Ricci semi-symmetric property $R.S = 0$ then the it is an Einstein Manifold.*

Now, if we consider the LP-Sasakian manifold and (LCS)-manifold and apply the similar process we get the following result respectively:

Theorem 7.2. *If a LP Sasakian Ricci soliton admits the Ricci semi-symmetric property $R.S = 0$ then the Ricci soliton constatnt λ is eigen value of associated vector field ξ.*

Theorem 7.3. *If a $(LCS)_n$ Ricci soliton admits the Ricci semi-symmetric property $R.S = 0$ then the Ricci soliton constatnt λ is eigen value of associated vector field ξ.*

Corollary 7.2. *A Ricci semi-symmetric n-dimensional LP-Sasakian manifold is an Einstein manifold.*

Corollary 7.3. *A Ricci semi-symmetric n-dimensional (LCS)-manifold is an Einstein manifold.*

REFERENCES

1. Z. Ahsan: *Ricci Solitons and the Spacetime of General Relativity*, J.T.S., **12** (2018), 49–64.
2. M. Ali, and Z. Ahsan: *Ricci Solitons and symmetries of spacetime manifold of general relativity*, Global Journal of Advanced Research on Classical and Modern Geometries, **2**, (2014), 75–84.
3. B.B. Chaturvedi and P. Pandey: *Study on special type of a weakly symmetric Kahler manifold*, Diffenetial Geometry-Dynamical System, **17**, (2015), 32–37.
4. B.B. Chaturvedi and P. Pandey: *Some Examples on Weak Symmetries*, General Mathematics Notes, **29**, (2015), 61–66.
5. S.K. Chaubey and U.C. De *Three-Dimensional Riemannian Manifolds and Ricci Solitons*, Quaestiones Mathematicae, (2021) **DOI**:10.2989/16073606.2021.1895352.
6. K. De, U.C. De, A.A. Syied, N.B. Turki and S. Alsaeed: *Perfect Fluid Spacetimes and Gradient Solitons*, Journal of Nonlinear Mathematical Physics, (2022), **DOI**: 10.1007/s44198-022-00066-5.

7. U.C. De, C.A. Mantica, and Y. Suh: *Perfect fluid spacetimes and gradient solitons*, Filomat, **36(3)**, (2022), 829–842.

8. U.C. De, A. Sardar and K. De: *Ricci-Yamabe solitons and 3-dimensional Riemannian manifolds*, Turkish Journal of Mathematics, **46(3)**, (2022), 1078–1088.

9. U.C. De and A.A. Shaikh: *Differential Geometry of Manifolds*, Narosa Publishing House (2007).

10. R.S. Hamilton: *Three-manifolds with positive Ricci curvature*, Journal of Differential Geometry, **17**, (1982), 255–306.

11. A. Haseeb and S.K. Choubey: *Lorentzian Para Sasakian Manifolds and *Ricci Solitons*, Kragujevac Journal of Mathematics, (2021), 167-–179.

12. S.K. Hui, R.S. Lemence and D. Chakraborty: *Ricci Solitons on Ricci Pseudosymmetric $(LCS)_n$-Manifolds*, [math.DG] (2017).

13. S.K. Hui, S.K. Yadav and A. Patra: *Almost Conformal Ricci Solitons on f-Kenmotsu Manifolds*, Khayyam J. Math. , **5(1)**, (2019), 89–104.

14. Y.C. Mandal and S.K. Hui: *Yamabe Solitons with potential vector field as torse forming*, CUBO A Mathematical Journal, **20** (2018), 37–47.

15. K. Matsumoto: *On Lorentzian paracontact manifolds*, Bull. of Yamagata Univ., Nat. Sci. **12** , (1989), 151–156.

16. K. Matsumoto and I. Mihai: *On a certain transformation in a Lorentzian para Sasakian manifold*, Tensor, N.S. **47**, (1988), 189–197.

17. I. Mihai and R. Rosca: *On Lorentzian P-Sasakian manifolds*, Classical Analysis, World Scientific Publi., (1992), 159–169.

18. H.G. Nagaraja and K. Venu: *Ricci Solitons in Kenmotsu Manifold*, Journal of Informatics and Mathematical Sciences, **8(1)**, (2016), 29–36.

19. P. Pandey: *On Weakly Cyclic Generalized Z-Symmetric Manifolds*, Natl. Acad. Sci. Lett., **43**, (2020), 347–350.

20. P. Pandey and B.B. Chaturvedi *On a Lorentzian Complex Space Form*, Natl. Acad. Sci. Lett., **43**, (2020), 351–353.

21. G.P. Pokhariyal: *Study of a new curvature tensor in a Sasakian manifold*, Tensor N.S. **36(2)**, (1982), 222–226.

22. G.P. Pokhariyal and R. S. Mishra: *Curvature tensors and their relativistics significance*, Yokohama Mathematical Journal, **18** (1970), 105–108.

23. S. Roy, S. Dey and A. Bhattacharya: *Yambe Solitons on $(LCS)_n$-manifolds*, arXiv:1909.06551v1 [math.DG] (2019).

24. B. Shanmukha and V. Venkatesha: *Some Ricci solitons on Kenmotsu manifold*, The Journal of Analysis, (2020).

25. A. Sardar, U.C. De and B. Nazafi: *3-Dimensional f -Kenmotsu Manifolds And Solitons*, International Journal of Geometric Methods in Modern Physics, (2022), **doi**:10.1142/S0219887822501997.

26. A.A. Shaikh: *On Lorentzian almost paracontact manifolds with a structure of the concircular type*, Kyungpook Math.Journal, **43**, (2003), 305-314.

27. A.A. Shaikh: *Some Results On $(LCS)_n$ Manifolds*, Journal of Korean Mathematical Society, **46**, (2009), 449-461.

28. A.A. Shaikh and S. Biswas: *On LP-Sasakian manifolds*, Bull. Malaysian Math. Sc. Soc. (Second Series), **27**, (2004), 17–26.

29. A.A. Shaikh and S.K. Hui: *On extended generalized ϕ-recurrent and β-Kenmotsu manifold*, Publ. De L' Inst. Math., (2010), **89(103)**, 77-88.

30. M.D. Siddiqi and U.C. De: *Relativistic perfect fluid spacetimes and Ricci–Yamabe solitons*, Letters in Mathematical Physics, **112(1)**, (2022), DOI: 10.1007/s11005-021-01493-z.

31. A. Singh, R.K. Pandey, A. Prakash and K. Sachin: *On a Pseudo Projective ϕ- Recurrent Sasakian Manifolds*, Journal of mathematics and computer science, **14**, (2015), 309–314.

32. Venkatesha, C.S. Bagewadi and K.T. Pradeep Kumar: *Some Results on Lorentzian Para-Sasakian Manifolds*, International Scholarly Research Network ISRN Geometry (2011), *Doi*: 10.5402/2011/161523.

33. K. Yano and S. Sawaki: *Riemannian manifolds admitting a conformal transformation group*, Journal of Differential Geometry, **2**, (1968), 161–184.

34. A. Yildiz and U.C. De: *On a type of Kenmotsu manifolds*, Differential Geometry—Dynamical Systems, **12** (2010), 289-298.