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FRACTIONAL LAPLACE TRANSFORM TO SOLVE CONFORMABLE DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we convert some of the conformable fractional differential equations (CFDEs) into ordinary differential equations using the fractional Laplace transform. The fractional Laplace transform introduced by Abdeljawad are investigated by other authors. The fractional Laplace transform method is developed to get the exact solution of conformable fractional differential equations. Our paper's aim is to convert the conformable fractional differential equations into ordinary differential equations. This is done using the fractional Laplace transformation of $(\alpha+\beta)$ or $(\alpha+\beta+\gamma)$ order. Furthermore, a new definition of fractional Laplace transformation is introduced. We do not need the initial value of function at $t=a$.
Keywords: fractional Laplace transform, exact analytical solutions, conformable fractional derivative, conformable fractional differential equation.

1. Introduction

The fractional calculus attracted much research in the last and present centuries. Impact of this fractional calculus in both pure and applied branches of science and engineering started to increase substantially. Last two decades, due to its widespread

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applications in different fields of science and engineering, have attracted the attention of many researchers. They started to deal with the discrete version of fractional calculus benefitting from the theory of time scales. In recent decades, fractional calculus and fractional differential equations have gained significant development in both theory and application because of their powerful potential applications. Fractional differential equations are sometimes called extraordinary differential equations because of their nature and are easily found in various fields of applied sciences [14, 19]. For example, fractional-order differential equations have been established for modeling natural phenomena in various fields such as physics, engineering, mechanics, control theory, economics, medical science, finance etc. [1, 2, 5, 6, 7, 8, 10, 13, 14, 19, 21, 22]. Recently, scientists have proposed many efficient and powerful methods to obtain exact or numerical solutions of fractional differential equations [13, 8]. In addition, many researchers have been trying to form a new definition of fractional derivative. Most of these definitions include an integral form for fractional derivatives.There are many types of differential derivatives in fractional calculus e.g. Grunwald-Letnikov, Riemann-Liouville, Caputo [21], Caputo-Fabrizio [7], Atangana-Baleanu [2], and more recently one, the conformable fractional derivative [13]. Modeling of spring pendulum in fractional sense and its numerical solution are proposed in [5]. Study of the motion of a capacitor microphone in fractional calculus is proposed in [6]. So the scientific and engineering problems which involve fractional calculus are huge and still very effective. Recently, some authors have introduced the concept of non-local derivatives. In [13], Khalil presented a new definition of derivative prominently compatible with the classical derivative. This operator is called "conformable derivative". The chain rule which is an applicable and useful rule in the calculus, is held only for conformable fractional derivatives. This derivative satisfied some conventional properties, for instance, the chain rule. This operator can be used to solve conformable differential equations. In [8], the author showed the exact solutions of time heat differential equations by using the conformable derivative. In [4], Atangana investigated some properties of this derivative, related theorems, and new definitions were introduced. Interesting works related to the operating are given by [23, 12]. The rest of this study is organized as follows. In Section 2, we give some important theorems based on conformable fractional Laplace transformation. In Section 3, we present some conformable fractional theorems of $\alpha-$ order. In Section 4, we present some new Theorems and methods for solving a class of conformable fractional differential equations by fractional Laplace transformations of $(\alpha+\beta)$ and $(\alpha+\beta+\gamma)$ order.

1.1. The conformable fractional derivative

The conformable fractional derivative was introduced first by [13]. After that, [15] has also presented the generalized and fractional derivative. The generalized derivative of a given function u, defined on the range of p, by means of the limit

$$Du(t) = \lim_{\epsilon\to 0} \frac{u\left(p(t,\epsilon)\right) - u(t)}{\epsilon}, \tag{1.1}$$

whenever the limit exists and is finite, it will be called the p-derivative of u at t or the generalized derivative of u at t. Recently, a fractional derivative of order α $(0<\alpha\leq 1)$, is defined by

$$Du(t) = \lim_{\epsilon\to 0} \frac{u\left(p(t,\epsilon,\alpha)\right) - u(t)}{\epsilon}. \tag{1.2}$$

In particular when $a \in \mathbb{R}$ and $p(t,\varepsilon,\alpha) = t + \epsilon(t-a)^{1-\alpha}$ we have, the left conformable fractional derivative of order $0 < \alpha \leq 1$ starting from $a \in \mathbb{R}$ the function $u : [a, +\infty) \to \mathbb{R}$, is defined by

$$({}_tT_\alpha^a u)(t) = \lim_{\epsilon \to 0} \frac{u\left(t + \epsilon(t-a)^{1-\alpha}\right) - u(t)}{\epsilon}. \tag{1.3}$$

When $a = 0$, we have:

$$({}_tT_\alpha^0 u)(t) = {}_tT_\alpha u(t) = \lim_{\epsilon \to 0} \frac{u\left(t + \epsilon t^{1-\alpha}\right) - u(t)}{\epsilon}.$$

If $(T_\alpha^a u)(t)$ exists on $(a, +\infty)$, then $(T_\alpha^a u)(a) = \lim_{t \to a^+} (T_\alpha^a u)(t)$. If $(T_\alpha^a u)(t_0)$ exists and is finite, then we say that u is left $\alpha-$differentiable at t_0.
The right conformable fractional derivative of order $0 < \alpha \leq 1$ terminating at $b \in \mathbb{R}$ of function $u : (-\infty, b] \to \mathbb{R}$, is defined by

$$({}^bT_\alpha u)(t) = -\lim_{\epsilon \to 0} \frac{u\left(t + \epsilon(b-t)^{1-\alpha}\right) - u(t)}{\epsilon}. \tag{1.4}$$

If $({}^bT_\alpha u)(t)$ exists on $(-\infty, b)$, then $({}^bT_\alpha u)(a) = \lim_{t \to b^-} ({}^bT_\alpha u)(t)$. If $({}^bT_\alpha u)(t_0)$ exists and is finite, then we say that u is right $\alpha-$differentiable at t_0. See [18]. Let $\alpha \in (n, n+1]$, and u be an n-differentiable at t, where $t > 0$, then the conformable derivative of u of order α is defined by

$$({}_tT_\alpha^0 u)(t) = \lim_{\epsilon \to 0} \frac{u^{([\alpha]-1)}(t + \epsilon t^{([\alpha]-\alpha)}) - u^{([\alpha]-1)}(t)}{\epsilon}. \tag{1.5}$$

Where $[\alpha]$ is the smallest integer greater than or equal to α. Let $\alpha \in (n, n+1]$, and u is (n+1)-differentiable at $t > 0$. Then

$$({}_tT_\alpha^0 u)(t) = t^{([\alpha]-\alpha)} u^{([\alpha])}(t). \tag{1.6}$$

Theorem 1.1. *Let $0 < \alpha \leq 1$, and f, g be left(right) $\alpha-$differentiable functions. Then,*

1) $\forall c_1, c_2 \in \mathbb{R}$, $\left(T_\alpha^a(c_1 f + c_2 g)\right)(t) = c_1 \left(T_\alpha^a f\right)(t) + c_2 \left(T_\alpha^a g\right)(t)$,
2) $\forall c_1, c_2 \in \mathbb{R}$, $\left({}^bT_\alpha(c_1 f + c_2 g)\right)(t) = c_1 \left({}^bT_\alpha f\right)(t) + c_2 \left({}^bT_\alpha g\right)(t)$,
3) $\forall \lambda \in \mathbb{R}$, $T_\alpha^a\left((t-a)^\lambda\right) = \lambda(t-a)^{\lambda-\alpha}$, ${}^bT_\alpha\left((b-t)^\lambda\right) = \lambda(b-t)^{\lambda-\alpha}$,
4) $T_\alpha^a(C) = 0$, ${}^bT_\alpha(C) = 0$, *where C is a constant.*
5) *(Product rule for left and right CF derivative)*
$\left(T_\alpha^a(fg)\right)(t) = f(t)\left(T_\alpha^a g\right)(t) + g(t)\left(T_\alpha^a f\right)(t)$,
$\left({}^bT_\alpha(fg)\right)(t) = f(t)\left({}^bT_\alpha g\right)(t) + g(t)\left({}^bT_\alpha f\right)(t)$,
6) *(Quotient rule for left and right CF derivative)*
$\left(T_\alpha^a(\frac{f}{g})\right)(t) = \frac{g(t)\left(T_\alpha^a f\right)(t) - f(t)\left(T_\alpha^a g\right)(t)}{g(t)^2}$, $\left({}^bT_\alpha(\frac{f}{g})\right)(t) = \frac{g(t)\left({}^bT_\alpha f\right)(t) - f(t)\left({}^bT_\alpha g\right)(t)}{g(t)^2}$,
$g(t) \neq 0$,
7) $\left(T_\alpha^a f\right)(t) = (t-a)^{1-\alpha} f'(t)$, $\left({}^bT_\alpha f\right)(t) = -(b-t)^{1-\alpha} f'(t)$,
where $f'(t) = \lim_{\epsilon \to 0}\left(\frac{f(t+\epsilon) - f(t)}{\epsilon}\right)$.

Proof. See [18]. □

2. The fractional Laplace transform

In this section, we investigate the definition of fractional Laplace transform and some interesting properties and rules of fractional Laplace transform. Over the following set of functions [3].

$$A = \left\{ u(t) : \exists M, \tau_1, \tau_2 > 0, |u(t)| < Me^{\frac{|t|}{\tau_j}}, if\ \tau_j \in (-1)^j [0, \infty), j = 1, 2 \right\}.$$

The conformable fractional Laplace transform is defined as follows:[1] The conformable fractional Laplace transform (CFLT) of function $u : [0, \infty) \to \mathbb{R}$ for $t > 0$, of order $0 < \alpha \leq 1$, starting from a of u is defined by

$$L_\alpha^a\{u(t)\} = \int_a^\infty e^{-s\frac{(t-a)^\alpha}{\alpha}} u(t)(t-a)^{\alpha-1} dt = U_\alpha^a(s). \tag{2.1}$$

If a=0, we have

$$L_\alpha^0\{u(t)\} = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} u(t) t^{\alpha-1} dt = U_\alpha^0(s) = U_\alpha(s). \tag{2.2}$$

In particular, if $\alpha = 1$, then Eq, (2.2) is reduced to the definition of the Laplace transform.

$$L\{u(t)\} = \int_0^\infty e^{-st} u(t) dt = U(s). \tag{2.3}$$

Theorem 2.1. *Let $u : [a, \infty) \to \mathbb{R}$ be differentiable real valued function and $0 < \alpha \leq 1$. Then*

$$L_\alpha^a\{{}_tT_\alpha^a(u)(t)\} = sU_\alpha^a(s) - u(a). \tag{2.4}$$

Proof. See [1]. □

Theorem 2.2. *Let u is piecewise continuous on $[0, \infty)$ and $L_\alpha^a\{u(t)\} = U_\alpha^a(s)$, then*

$$L_\alpha^0\{t^{n\alpha} u(t)\} = (-1)^n \alpha^n \frac{d^n}{ds^n}\left[U_\alpha^0(s)\right], \quad n \in \mathbb{N}. \tag{2.5}$$

Proof. See [11]. □

3. The exact solutions for a class of conformable fractional differential equations by fractional Laplace transformation

In this section, we use the fractional Laplace transformation for solving a class of conformable fractional differential equations. This is done by taking fractional Laplace transform to convert the (ODE) of the first order.

Theorem 3.1. *[16]. Let $u : [a, \infty) \to \mathbb{R}$ be twice differentiable on (a, ∞), α, $\beta > 0$ and $\alpha + \beta \leq 1$, then*

$${}_tT_\beta[{}_tT_\alpha u(t)] = (1-\alpha)t^{1-(\alpha+\beta)} u'(t) + t^{2-(\alpha+\beta)} u''(t). \tag{3.1}$$

Theorem 3.2. *[16]. Let $u : [a,\infty) \to \mathbb{R}$ be twice differentiable on (a,∞), $\alpha,\ \beta > 0$ and $\alpha+\beta \le 1$, then*

1) $\displaystyle\int_0^{\infty} e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}\Big(tu''(t)\Big)dt = u(0)+s\int_0^{\infty} t^{\alpha+\beta}u'(t)e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}dt - sU_{(\alpha+\beta)}(s).$

2) $$\int_0^{\infty}\Big(1-\alpha+st^{\alpha+\beta}\Big)u'(t)e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}dt$$
$$= (\alpha-1)u(0) - (\alpha+\beta)sU_{(\alpha+\beta)}(s) + (1-\alpha)sU_{(\alpha+\beta)}(s) - (\alpha+\beta)s^2U'_{(\alpha+\beta)}(s).$$

Theorem 3.3. *[16]. Let $u : [a,\infty) \to \mathbb{R}$ be twice differentiable on (a,∞), $\alpha,\ \beta > 0$ and $\alpha+\beta \le 1$, then*

1) $L^0_{(\alpha+\beta)}\left\{{}_tT_\beta\Big({}_tT_\alpha u(t)\Big)\right\} = \alpha u(0) - (2\alpha+\beta)sU_{(\alpha+\beta)}(s) - (\alpha+\beta)s^2U'_{(\alpha+\beta)}(s).$

2) $$L^0_{(\alpha+\beta)}\left\{{}_tT_\beta\Big({}_tT_\alpha u(t)\Big) + {}_tT_\alpha\Big({}_tT_\beta u(t)\Big)\right\}$$
$$= (\alpha+\beta)u(0) - (3\alpha+3\beta)sU_{(\alpha+\beta)}(s) - (2\alpha+2\beta)s^2U'_{(\alpha+\beta)}(s).$$

Theorem 3.4. *[16]. Let $s,\ \alpha,\ \beta > 0$ be and $\alpha+\beta \le 1$, then we have*

1) $L^0_{\alpha+\beta}\left\{t^{-\beta}\Big({}_tT_\alpha u(t)\Big)\right\} = -u(0)+sU_{(\alpha+\beta)}(s).$

2) $$L^0_{\alpha+\beta}\left\{t^{\alpha}\Big({}_tT_\alpha u(t)\Big)\right\} = L^0_{\alpha+\beta}\left\{tu'(t)\right\} = L^0_{\alpha+\beta}\left\{t^{\alpha+\beta}\Big({}_tT_{\alpha+\beta} u(t)\Big)\right\}$$
$$= -(\alpha+\beta)U_{(\alpha+\beta)}(s) - (\alpha+\beta)sU'_{(\alpha+\beta)}(s).$$

Theorem 3.5. *[17]. Let $s,\ \alpha > 0$, be and $\alpha \le 1$ such that $L^0_\alpha\{g(t)\} = G_\alpha(s)$, then*

$$L^0_\alpha\left\{{}_tT_\alpha g(t) + t^{2-\alpha}g''(t)\right\} = -\alpha sG_\alpha(s) - \alpha s^2G'_\alpha(s). \tag{3.2}$$

Therefore, if $\alpha,\beta > 0$ *be and* $\alpha+\beta \le 1$, we have

$$L^0_{\alpha+\beta}\left\{\Big({}_tT_{(\alpha+\beta)}u(t)\Big) + t^{2-(\alpha+\beta)}u''(t)\right\}$$
$$= L^0_{\alpha+\beta}\left\{t^{1-(\alpha+\beta)}u'(t) + t^{2-(\alpha+\beta)}u''(t)\right\}$$
$$= -(\alpha+\beta)sU_{(\alpha+\beta)}(s) - (\alpha+\beta)s^2U'_{(\alpha+\beta)}(s).$$

Theorem 3.6. *[17]. Let $s,\alpha > 0$ be and $\alpha \le 1$, then*

$$L^0_\alpha\{t^{\alpha+1}u'(t)\} = 2\alpha^2U'_\alpha(s) + \alpha^2sU''_\alpha(s). \tag{3.3}$$

Theorem 3.7. *[17]. Let $s,\ \alpha>0,$ be such that $\alpha\le 1,$ and $L^0_\alpha\{u(t)\}=U_\alpha(s)$, then*

1) $L^0_\alpha\Big\{tu'(t)\Big\}=L^0_\alpha\Big\{t^\alpha T_\alpha u(t)\Big\}=-\alpha U_\alpha(s)-\alpha sU'_\alpha(s).$

2) $L^0_\alpha\Big\{t^{2-\alpha}u^{''}(t)\Big\}=\Big(-1-\alpha\Big)sU_\alpha(s)+u(0)-\alpha s^2U^{'}_\alpha(s).$

3) $L^0_\alpha\Big\{t^2u^{''}(t)\Big\}=\Big(\alpha+\alpha^2\Big)U_\alpha(s)+\alpha s\Big(1+3\alpha\Big)U^{'}_\alpha(s)+\alpha^2s^2U^{''}_\alpha(s).$

Therefore, if $\alpha,\ \beta>0,$ *be such that* $\alpha+\beta\le 1$ *and* $m\in\mathbb{R}$ we have

1) $L^0_{\alpha+\beta}\Big\{t^{1-(\alpha+\beta)}u^{'}(t)\Big\}=L^0_{\alpha+\beta}\left\{T_{\alpha+\beta}u(t)\right\}=-u(0)+sU_{\alpha+\beta}(s).$

2) $L^0_{\alpha+\beta}\Big\{t^{2-(\alpha+\beta)}u^{''}(t)\Big\}=\Big(-1-(\alpha+\beta)\Big)sU_{\alpha+\beta}(s)+u(0)-\Big(\alpha+\beta\Big)s^2U^{'}_{\alpha+\beta}(s).$

3) $$L^0_{\alpha+\beta}\Big\{(m+1)t^{1-(\alpha+\beta)}u^{'}(t)+t^{2-(\alpha+\beta)}u^{''}(t)\Big\}=-mu(0)-\Big(\alpha+\beta-m\Big)sU_{\alpha+\beta}(s)-\Big(\alpha+\beta\Big)s^2U^{'}_{\alpha+\beta}(s).$$

Theorem 3.8. *[17]. Let $\alpha,\ \beta,\ \gamma>0,$ and $\alpha+\beta+\gamma\le 1,$ then*

$$ {}_tT_\gamma\Big({}_tT_\beta\Big({}_tT_\alpha u(t)\Big)\Big)=\Big(1-\alpha\Big)\Big(1-\alpha-\beta\Big)t^{1-(\alpha+\beta+\gamma)}u^{'}(t)+\Big(3-(2\alpha+\beta)\Big)t^{2-(\alpha+\beta+\gamma)}u^{''}(t)+t^{3-(\alpha+\beta+\gamma)}u^{'''}(t).$$

Theorem 3.9. *[17]. Let $s,\ \alpha,\ \beta,\ \gamma>0,$ and $\alpha+\beta+\gamma\le 1,$ then*

$$\begin{aligned}L^0_{\alpha+\beta+\gamma}\Big\{{}_tT_\gamma\Big({}_tT_\beta\Big({}_tT_\alpha u(t)\Big)\Big)\Big\}
&=-\Big(\alpha^2+\alpha\beta\Big)u(0)\\
&+\Big\{\Big(3\alpha+2\beta+\gamma\Big)\Big(\alpha+\beta+\gamma\Big)+\alpha^2+\alpha\beta\Big\}sU_{\alpha+\beta+\gamma}(s)\\
&+\Big(5\alpha+4\beta+3\gamma\Big)\Big(\alpha+\beta+\gamma\Big)s^2U^{'}_{\alpha+\beta+\gamma}(s)\\
&+\Big(\alpha+\beta+\gamma\Big)^2s^3U^{''}_{\alpha+\beta+\gamma}(s).\end{aligned}$$

Theorem 3.10. *Let $0<\alpha\le\frac{1}{2},0<\beta<\frac{1}{2}$ be and $\alpha>\beta$ then we have*

$$ {}_tT_{\alpha+\beta}\Big({}_tT_{\alpha-\beta}u(t)\Big)={}_tT_\alpha\Big({}_tT_\alpha u(t)\Big)+\beta\Big({}_tT_{2\alpha}u(t)\Big). \tag{3.4}$$

$$ {}_tT_{\alpha-\beta}\Big({}_tT_{\alpha+\beta}u(t)\Big)={}_tT_\alpha\Big({}_tT_\alpha u(t)\Big)-\beta\Big({}_tT_{2\alpha}u(t)\Big). \tag{3.5}$$

Proof. *From Theorem 1.1, we have*

$$
\begin{aligned}
{}_tT_{\alpha+\beta}\Big({}_tT_{\alpha-\beta}u(t)\Big) &= {}_tT_{\alpha+\beta}\Big(t^{1-(\alpha-\beta)}u'(t)\Big)\\
&= t^{1-(\alpha+\beta)}\Big[t^{1-(\alpha-\beta)}u'(t)\Big]'\\
&= (1-\alpha)t^{1-2\alpha}u'(t) + t^{2-2\alpha}u''(t) + \beta t^{1-2\alpha}u'(t).
\end{aligned}
$$

Now, from Theorem 3.1, the proof of Eq (3.4) is clear. Similarly the Eq (3.5) is proof. □

Theorem 3.11. *Suppose that $u(t)$ be twice differentiable on $(0,\infty)$ and $0<\alpha\leq\frac{1}{2}, 0<\beta<\frac{1}{2}$ such that $\alpha>\beta$ then the following CFDE*

$$
{}_tT_{\alpha+\beta}\Big({}_tT_{\alpha-\beta}u(t)\Big) + t^{\alpha}\Big({}_tT_{\alpha}u(t)\Big) + t^{\beta}\Big({}_tT_{\beta}u(t)\Big) = q(t), \tag{3.6}
$$

has its solution given by

$$
u(t) = \Big(L^0_{2\alpha}\Big)^{-1}\left\{\frac{(s+2)^{\frac{\beta}{2\alpha}-\frac{1}{2}}}{s}\Big(\int\frac{(\alpha-\beta)u(0)-Q_{2\alpha}(s)}{2\alpha(s+2)^{\frac{1}{2}+\frac{\beta}{2\alpha}}}ds + C\Big)\right\}. \tag{3.7}
$$

Proof. *By using Theorems 3.4 and 3.10 the equation (3.6) can be written as*

$$
{}_tT_{\alpha}\Big({}_tT_{\alpha}u(t)\Big) + \beta\Big({}_tT_{2\alpha}u(t)\Big) + 2tu'(t) = q(t). \tag{3.8}
$$

Applying the CF Laplace transform ($L^0_{2\alpha}$) to the both sides of equation 3.8 and using Theorems 3.4, 2.1, 3.3 yields

$$
\alpha u(0) - 3\alpha sU_{2\alpha}(s) - 2\alpha s^2 U'_{2\alpha}(s) + \beta\Big(sU_{2\alpha}(s) - u(0)\Big) - 4\alpha U_{2\alpha}(s) - 4\alpha sU'_{2\alpha}(s) = Q_{2\alpha}(s).
$$

So, we can write

$$
U'_{2\alpha}(s) + \Big(\frac{3\alpha-\beta}{2\alpha(s+2)} + \frac{2}{s(s+2)}\Big)U_{2\alpha}(s) = \frac{(\alpha-\beta)u(0)-Q_{2\alpha}(s)}{2\alpha s(s+2)}.
$$

By solving the above ODE of the first order we obtain

$$
\begin{aligned}
U_{2\alpha}(s) &= e^{-\int(\frac{3\alpha-\beta}{2\alpha(s+2)}+\frac{2}{s(s+2)})ds}\Big(\int\Big(\frac{(\alpha-\beta)u(0)-Q_{2\alpha}(s)}{2\alpha s(s+2)}\Big)e^{\int(\frac{3\alpha-\beta}{2\alpha(s+2)}+\frac{2}{s(s+2)})ds}ds + C\Big)\\
&= \frac{1}{s}(s+2)^{\frac{\beta}{2\alpha}-\frac{1}{2}}\Big(\int\frac{(\alpha-\beta)u(0)-Q_{2\alpha(s)}}{2\alpha(s+2)^{\frac{1}{2}+\frac{\beta}{2\alpha}}}ds + C\Big),
\end{aligned}
$$

by taking $(L^0_{2\alpha})^{-1}$ with respect to s, we obtain the explicit solution $u(t)$. □

In particular if $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$, $u(0) = 1$, *the following CFDE*

$$ {}_tT_{\frac{3}{4}}\Big({}_tT_{\frac{1}{4}}u(t)\Big) + 2t^{\frac{1}{2}}\Big({}_tT_{\frac{1}{2}}u(t)\Big) = -\frac{3}{2}e^{-2t}, $$

has the solution given by

$$ \begin{aligned} u(t) &= \Big(L_1^0\Big)^{-1}\left\{\frac{(s+2)^{\frac{\frac{1}{4}}{2(\frac{1}{2})}-\frac{1}{2}}}{s}\Big(\int \frac{(\frac{1}{2}-\frac{1}{4})(1)+\frac{3}{2(s+2)}}{2(\frac{1}{2})(s+2)^{\frac{1}{2}+\frac{\frac{1}{4}}{2(\frac{1}{2})}}}ds + C\Big)\right\} \\ &= L^{-1}\left\{\frac{(s+2)^{\frac{-1}{4}}}{s}\Big(\int \frac{\frac{1}{4}+\frac{3}{2(s+2)}}{(s+2)^{\frac{3}{4}}}ds + C\Big)\right\} \\ &= L^{-1}\left\{\frac{1}{s(s+2)^{\frac{1}{4}}}\Big(\frac{s}{(s+2)^{\frac{3}{4}}} + C\Big)\right\} \\ &= L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{C}{s(s+2)^{\frac{1}{4}}}\right\} \\ &= e^{-2t} + C\Big(-\frac{1}{2}\frac{2^{\frac{1}{4}}\Gamma(\frac{1}{4},2t)}{\pi} + \frac{1}{2}2^{\frac{3}{4}}\Big). \end{aligned} $$

Since $u(0) = 1$, *so, we have* $u(t) = e^{-2t}$.

Theorem 3.12. *Suppose that* $u(t)$ *be twice differentiable on* $(0,\infty)$ *and* $0 < \alpha \leq \frac{1}{2}, 0 < \beta < \frac{1}{2}$ *such that* $\alpha > \beta$ *then the following CFDE*

$$ {}_tT_{\alpha+\beta}\Big({}_tT_{\alpha-\beta}u(t)\Big) + {}_tT_{\alpha}\Big({}_tT_{\alpha}u(t)\Big) + \Big({}_tT_{2\alpha}u(t)\Big) + tu'(t) + u(t) = q(t), \tag{3.9} $$

has its solution given by

$$ u(t) = \Big(L_{2\alpha}^0\Big)^{-1}\left\{(2s+1)^{\frac{\beta-1}{4\alpha}-\frac{1}{2}}s^{\frac{1}{2\alpha}-1}\Big(\int \frac{(2\alpha-\beta-1)u(0)-Q_{2\alpha}(s)}{2\alpha s^{\frac{1}{2\alpha}}(2s+1)^{\frac{1}{2}+\frac{\beta-1}{4\alpha}}}ds + C\Big)\right\}. \tag{3.10} $$

Proof. From Theorem 3.10, the equation (3.9) can be written as

$$ 2\Big({}_tT_{\alpha}\Big({}_tT_{\alpha}u(t)\Big)\Big) + (1+\beta)\Big({}_tT_{2\alpha}u(t)\Big) + tu'(t) + u(t) = q(t), \tag{3.11} $$

Applying the CF Laplace transform ($L_{2\alpha}^0$) *to the both sides of equation 3.11 and using of Theorems 3.4, 2.1, 3.3 yields*

$$ \begin{aligned} &2\Big(\alpha u(0) - 3\alpha sU_{2\alpha}(s) - 2\alpha s^2U_{2\alpha}^{'}(s)\Big) + (1+\beta)\Big(sU_{2\alpha}(s) - u(0)\Big) \\ &\qquad -2\alpha U_{2\alpha}(s) - 2\alpha sU_{2\alpha}^{'}(s) + U_{2\alpha}(s) = Q_{2\alpha}(s), \end{aligned} $$

Therefore we can write

$$U_{2\alpha}^{'}(s)+\left(\frac{6\alpha-\beta-1+\frac{2\alpha-1}{s}}{2\alpha(2s+1)}\right)U_{2\alpha}(s)=\frac{(2\alpha-\beta-1)u(0)-Q_{2\alpha}(s)}{2\alpha s(2s+1)}.$$

By solving the above ODE of first order, we have

$$U_{2\alpha}(s)=(2s+1)^{\frac{\beta-1}{4\alpha}-\frac{1}{2}}s^{\frac{1}{2\alpha}-1}\left(\int\frac{(2\alpha-\beta-1)u(0)-Q_{2\alpha}(s)}{2\alpha s^{\frac{1}{2\alpha}}(2s+1)^{\frac{1}{2}+\frac{\beta-1}{4\alpha}}}ds+C\right).$$

by taking $(L_{2\alpha}^{0})^{-1}$ *with respect to s, we obtain the explicit solution* $u(t)$ *in equation* (3.10). □

In particular if $\alpha=\frac{1}{2},\ \beta=\frac{1}{4},$ *and* $u(0)=1$, *then the following CFDE*

$${}_tT_{\frac{3}{4}}\left({}_tT_{\frac{1}{4}}u(t)\right)+{}_tT_{\frac{1}{2}}\left({}_tT_{\frac{1}{2}}u(t)\right)+{}_tT_{2(\frac{1}{2})}u(t)+tu'(t)+u(t)=\frac{13}{4}e^t+3te^t,$$

has the solution given by

$u(t)$

$$=L^{-1}\left\{(2s+1)^{\frac{\frac{1}{4}-1}{4(\frac{1}{2})}-\frac{1}{2}}s^{\frac{1}{2(\frac{1}{2})}-1}\left(\int\frac{\left(2(\frac{1}{2})-(\frac{1}{4})-1\right)(1)-\frac{13}{4(s-1)}-\frac{3}{(s-1)^2}}{2(\frac{1}{2})s^{\frac{1}{2(\frac{1}{2})}}(2s+1)^{\frac{1}{2}+\frac{(\frac{1}{4})-1}{4(\frac{1}{2})}}}ds+C\right)\right\}$$

$$=L^{-1}\left\{\frac{1}{(2s+1)^{\frac{7}{8}}}\left(\frac{(2s+1)^{\frac{7}{8}}}{s-1}+C\right)\right\}=L^{-1}\left\{\frac{1}{s-1}+\frac{C}{(2s+1)^{\frac{7}{8}}}\right\}=e^t+C\left(\frac{e^{-\frac{t}{2}}2^{\frac{-7}{8}}}{t^{\frac{1}{8}}\Gamma(\frac{7}{8})}\right).$$

Since $u(0)=1$, *therefore* $C=0$, *so we obtain* $u(t)=e^t$.

Example 3.1. Let $0<\alpha\le 1,\quad \gamma=.5772157...$ we apply the conformable fractional Laplace transform to obtain the exact solution of the following differential equation

$$\alpha^2s^2U_\alpha''(s)+\left(3\alpha^2+5\alpha\right)sU_\alpha'(s)+\left(\alpha^2+5\alpha+6\right)U_\alpha(s)=\frac{1}{\alpha s}\left(\ln(\frac{s}{\alpha})+\gamma\right). \tag{3.12}$$

The equation (3.12), can be written as

$$\alpha^2s^2U_\alpha''(s)+\alpha s(1+3\alpha)U_\alpha'(s)$$
$$+(\alpha^2+\alpha)U_\alpha(s)+4\alpha U_\alpha(s)+4\alpha sU_\alpha'(s)+6U_\alpha(s)=\frac{1}{\alpha s}\left(\ln(\frac{s}{\alpha})+\gamma\right). \tag{3.13}$$

By applying $\left(L_\alpha^0\right)^{-1}$ to the both sides of (3.13) and using Theorem 3.7, we have

$$t^2u''(t)-4tu'(t)+6u(t)=\ln(t).$$

By solving the above differential equation we obtain

$$u(t)=C_1t^2+C_2t^3+\frac{5}{36}+\frac{1}{6}\ln(t). \tag{3.14}$$

Now, by taking L_α^0 to the both sides of (3.14), we have

$U_\alpha(s)$

$$=C_1\left\{\frac{\alpha^{\frac{2}{\alpha}}}{s^{1+\frac{2}{\alpha}}}\Gamma\left(1+\frac{2}{\alpha}\right)\right\}+C_2\left\{\frac{\alpha^{\frac{3}{\alpha}}}{s^{1+\frac{3}{\alpha}}}\Gamma\left(1+\frac{3}{\alpha}\right)\right\}+\frac{5}{36s}+\frac{1}{6\alpha s}\left(\ln(\frac{s}{\alpha})+\gamma\right).$$

Theorem 3.13. *Let $0 < \alpha \leq 1$, be then*

$$L_\alpha^0 \left\{ t^\alpha \Big({}_tT_\alpha \Big({}_tT_\alpha u(t) \Big) \Big) \right\} = \alpha u(0) - 2\alpha s U_\alpha(s) - \alpha s^2 U'_\alpha(s). \tag{3.15}$$

Proof. *From Theorem 3.1, we have*

$${}_tT_\alpha[{}_tT_\alpha u(t)] = (1-\alpha)t^{1-2\alpha} u'(t) + t^{2-2\alpha} u''(t).$$

Therefore

$$\begin{aligned} t^\alpha \Big({}_tT_\alpha[{}_tT_\alpha u(t)] \Big) &= (1-\alpha)t^{1-\alpha} u'(t) + t^{2-\alpha} u''(t) \\ &= (1-\alpha)\Big({}_tT_\alpha u(t) \Big) + t^{2-\alpha} u''(t). \qquad (3.16) \end{aligned}$$

By applying L_α^0 to both sides of Eq. (3.16) and using Theorems 2.1, 3.7 the proof is clear. □

Theorem 3.14. *Suppose that $u(t)$ be differentiable on $(0,\infty)$, $0 < \alpha \leq 1$, A, B, $M \in \mathbb{R}$, and $A \neq 0$, then the following CFDE*

$$At^\alpha \Big({}_tT_\alpha[{}_tT_\alpha u(t)] \Big) + B\Big({}_tT_\alpha u(t) \Big) + Mu(t) = q(t), \tag{3.17}$$

is given by

$$u(t) = \Big(L_\alpha^0 \Big)^{-1} \left\{ s^{(\frac{B}{A\alpha}-2)} e^{-(\frac{M}{A\alpha s})} \Big(\int \frac{(A\alpha - B)u(0) - Q_\alpha(s)}{A\alpha s^{(\frac{B}{A\alpha})}} e^{(\frac{M}{A\alpha s})} ds + C \Big) \right\}.$$

Proof. *Applying L_α^0 to the both sides of Eq. (3.17), and using Theorems 3.13, 2.1, we have*

$$A\Big(\alpha u(0) - 2\alpha s U_\alpha(s) - \alpha s^2 U'_\alpha(s) \Big) + B\Big(sU_\alpha(s) - u(0) \Big) + MU_\alpha(s) = Q_\alpha(s).$$

Or equivalently

$$U'_\alpha(s) + \Big(\frac{2}{s} - \frac{B}{A\alpha s} - \frac{M}{A\alpha s^2} \Big) U_\alpha(s) = \frac{(A\alpha - B)u(0) - Q_\alpha(s)}{A\alpha s^2}.$$

By solving the above differential equation we obtain

$$\begin{aligned} U_\alpha(s) &= e^{-\int (\frac{2}{s} - \frac{B}{A\alpha s} - \frac{M}{A\alpha s^2})ds} \Big(\int \frac{(A\alpha - B)u(0) - Q_\alpha(s)}{A\alpha s^2} e^{\int (\frac{2}{s} - \frac{B}{A\alpha s} - \frac{M}{A\alpha s^2})ds} ds + C \Big) \\ &= s^{(\frac{B}{A\alpha}-2)} e^{-(\frac{M}{A\alpha s})} \Big(\int \frac{(A\alpha - B)u(0) - Q_\alpha(s)}{A\alpha s^{(\frac{B}{A\alpha})}} e^{(\frac{M}{A\alpha s})} ds + C \Big). \end{aligned}$$

Thus, solution $u(t)$ results from the CF inverse transform. □

Theorem 3.15. *Suppose that $u(t)$ be twice differentiable on $(0,\infty)$, $0 < \alpha \leq 1$, A, B, M, $N \in \mathbb{R}$, and $A, B \neq 0$, then the following CFDE*

$$At^\alpha \Big({}_tT_\alpha[{}_tT_\alpha u(t)] \Big) + Bt^\alpha \Big({}_tT_\alpha u(t) \Big) + MT_\alpha u(t) + Nu(t) = q(t), \tag{3.18}$$

is given by

$$u(t)$$

$$= (L^0_\alpha)^{-1}\left\{\frac{(s^{\frac{N}{B\alpha}-1}(As+B)^{\frac{BM-AN-AB\alpha}{AB\alpha}})}{\alpha}\left(\int \frac{(A\alpha-M)u(0)-Q_\alpha(s)}{s^{\frac{N}{B\alpha}}}(As+B)^{\frac{AN-BM}{AB\alpha}}ds+C\right)\right\}$$

Proof. *Applying L^0_α to the both sides of Eq. (3.18) and using Theorems 3.13, 3.7 and 2.1 we have*

$$A\Big(\alpha u(0)-2\alpha sU_\alpha(s)-\alpha s^2U'_\alpha(s)\Big)+B\Big(-\alpha U_\alpha(s)-\alpha sU'_\alpha(s)\Big)$$
$$+M\Big(sU_\alpha(s)-u(0)\Big)+NU_\alpha(s)=Q_\alpha(s).$$

Therefore we can write

$$U'_\alpha(s)+\Big(\frac{2A\alpha-M}{A\alpha s+B\alpha}+\frac{B\alpha-N}{A\alpha s^2+B\alpha s}\Big)U_\alpha(s)=\frac{(A\alpha-M)u(0)-Q_\alpha(s)}{A\alpha s^2+B\alpha s}.$$

By solving the above differential equation of the first order we obtain

$$U_\alpha(s)$$

$$=\Big(e^{-\int(\frac{2A\alpha-M}{A\alpha s+B\alpha}+\frac{B\alpha-N}{A\alpha s^2+B\alpha s})ds}\Big)\Big(\int\frac{(A\alpha-M)u(0)-Q_\alpha(s)}{A\alpha s^2+B\alpha s}e^{\int(\frac{2A\alpha-M}{A\alpha s+B\alpha}+\frac{B\alpha-N}{A\alpha s^2+B\alpha s})ds}ds+C\Big)$$

$$=\frac{\Big(s^{\frac{N}{B\alpha}-1}(As+B)^{\frac{BM-AN-AB\alpha}{AB\alpha}}\Big)}{\alpha}\Big(\int\frac{(A\alpha-M)u(0)-Q_\alpha(s)}{s^{\frac{N}{B\alpha}}}(As+B)^{\frac{AN-BM}{AB\alpha}}ds+C\Big),$$

By taking $(L^0_\alpha)^{-1}$ with respect to s, we obtain the explicit solution $u(t)$. □

For example by substituting $\alpha=A=M=1$ and $B=N=2$, the following CFDE

$$t\Big({}_tT_1[{}_tT_1u(t)]\Big)+2t\Big({}_tT_1u(t)\Big)+T_1u(t)+2u(t)=\Big(\frac{1}{2}e^{-2t}-te^{-2t}\Big),$$

has the solution given by

$$u(t)$$

$$=\Big(L\Big)^{-1}\left\{\frac{1}{s+2}\Big(\int\Big(\frac{-1}{2s(s+2)}+\frac{1}{s(s+2)^2}\Big)ds+C\Big)\right\}$$
$$=\Big(L\Big)^{-1}\left\{\frac{1}{s+2}\Big(\frac{1}{2(s+2)}+C\Big)\right\}=\frac{1}{2}te^{-2t}+Ce^{-2t}.$$

Theorem 3.16. *Suppose that $u(t)$ be twice differentiable on $(0,\infty)$ and $0<\alpha\le\frac{1}{2}, 0<\beta<\frac{1}{2}$ such that $\alpha>\beta$ then the following CFDE*

$$t^\alpha\Big({}_tT_{\alpha-\beta}[{}_tT_{\alpha+\beta}u(t)]\Big)+t^\alpha\Big({}_tT_\alpha[{}_tT_\alpha u(t)]\Big)+t^\alpha\Big({}_tT_\alpha u(t)\Big)+T_\alpha u(t)+u(t)$$

$$= q(t), \tag{3.19}$$

is given by

$$u(t)$$

$$= (L_\alpha^0)^{-1}\left\{\frac{s^{\frac{1-\alpha}{\alpha}}(2s+1)^{-\frac{1+\beta+2\alpha}{2\alpha}}}{\alpha}\left(\int \frac{(2\alpha-1+\beta)u(0)-Q_\alpha(s)}{s^{\frac{1}{\alpha}}}(2s+1)^{\frac{1+\beta}{2\alpha}}ds + C\right)\right\}.$$

Proof. *From Eq. (3.5), we have* ${}_tT_{\alpha-\beta}\Big({}_tT_{\alpha+\beta}u(t)\Big) = {}_tT_\alpha\Big({}_tT_\alpha u(t)\Big) - \beta\Big({}_tT_{2\alpha}u(t)\Big)$. *Therefore,*

$$t^\alpha\left({}_tT_{\alpha-\beta}\left({}_tT_{\alpha+\beta}u(t)\right)\right) = t^\alpha\left({}_tT_\alpha\left({}_tT_\alpha u(t)\right)\right) - \beta t^\alpha\left({}_tT_{2\alpha}u(t)\right)$$
$$= t^\alpha\left({}_tT_\alpha\left({}_tT_\alpha u(t)\right)\right) - \beta\left({}_tT_\alpha u(t)\right).$$

So, the Eq. (3.19), can be written as

$$2t^\alpha\left({}_tT_\alpha[{}_tT_\alpha u(t)]\right) + t^\alpha\left({}_tT_\alpha u(t)\right) + (1-\beta)T_\alpha u(t) + u(t) = q(t).$$

Now, by setting $A = 2,\ B = 1,\ M = 1-\beta,\ N = 1,$ *in Eq. (3.18), the desired result for Eq. (3.19), is valid.* □

Theorem 3.17. *Assume* $u(t)$ *be twice differentiable on* $(0,\infty)$ *and* $0 < \alpha \leq 1,\ 0 < \beta \leq 1,$ *then, the following CFDE*

$$t^\alpha\left({}_tT_\beta[{}_tT_\alpha u(t)]\right) + tu'(t) + u(t) = q(t), \tag{3.20}$$

is given by

$$u(t) = \left(L_\beta^0\right)^{-1}\left\{\frac{s^{\frac{1}{\beta}-1}}{(s+1)^{\frac{\alpha+1}{\beta}}}\left(\int \frac{\alpha u(0)-Q_\beta(s)}{s^{\frac{1}{\beta}}}(s+1)^{\frac{\alpha+1}{\beta}-1}ds + C\right)\right\}.$$

Proof. *From Eq. (3.1), the Eq. (3.20), can be written as*

$$(1-\alpha)t^{1-\beta}u'(t) + t^{2-\beta}u''(t) + tu'(t) + u(t) = q(t), \tag{3.21}$$

applying L_β^0 *to the both sides of Eq. (3.21), and using Theorem 3.7, we have*

$$\left(1-\alpha\right)\left(sU_\beta(s)-u(0)\right)$$
$$+ u(0) - (1+\beta)sU_\beta(s) - \beta s^2U'(s) - \beta U_\beta(s) - \beta sU'(s) + U_\beta(s) = Q_\beta(s).$$

Or equivalently

$$U_\beta'(s) + \left(\frac{\alpha+\beta}{\beta(s+1)} + \frac{\beta-1}{\beta s(s+1)}\right)U_\beta(s) = \frac{\alpha u(0)-Q_\beta(s)}{\beta s(s+1)}.$$

By solving the above ODE of the first order, we obtain

$$U_\beta(s) = e^{-\int(\frac{\alpha+\beta}{\beta(s+1)}+\frac{\beta-1}{\beta s(s+1)})ds}\left(\int \frac{\alpha u(0)-Q_\beta(s)}{\beta s(s+1)} e^{\int(\frac{\alpha+\beta}{\beta(s+1)}+\frac{\beta-1}{\beta s(s+1)})ds} ds + C\right)$$

$$= \frac{s^{\frac{1}{\beta}-1}}{(s+1)^{\frac{\alpha+1}{\beta}}}\left(\int \frac{\alpha u(0)-Q_\beta(s)}{s^{\frac{1}{\beta}}}(s+1)^{\frac{\alpha+1}{\beta}-1} ds + C\right).$$

Thus, solution $u(t)$ results from the CF inverse transform. □

In particular if $\alpha = \frac{1}{2}$, $\beta = 1$ and $u(0) = 1$, then the following CFDE,

$$\sqrt{t}\Big({}_tT_1\Big({}_tT_{\frac{1}{2}} u(t)\Big)\Big) + tu'(t) + u(t) = \frac{1}{2}e^{-t},$$

is given by

$$u(t) = \Big(L_1^0\Big)^{-1}\left\{\frac{s^{\frac{1}{1}-1}}{(s+1)^{\frac{\frac{1}{2}+1}{1}}}\left(\int \frac{\frac{1}{2}(1)-\frac{1}{2(s+1)}}{s^{\frac{1}{1}}}(s+1)^{\frac{\frac{1}{2}+1}{1}-1} ds + C\right)\right\}$$

$$= \Big(L\Big)^{-1}\left\{\frac{1}{(s+1)^{\frac{3}{2}}}\Big(\sqrt{s+1}+C\Big)\right\} = e^{-t}\Big(1+2C\sqrt{\frac{t}{\pi}}\Big).$$

Theorem 3.18. *Assume $u(t)$ be twice differentiable on $(0,\infty)$ and $0 < \alpha \le 1$, $0 < \beta \le 1$, then, the following CFDE*

$$t^\alpha\Big({}_tT_\beta[{}_tT_\alpha u(t)]\Big) + tu'(t) + t^\beta u(t) + u(t) = q(t), \tag{3.22}$$

is given by

$u(t)$

$$= (L_\beta^0)^{-1}\left\{\frac{e^{\left(\frac{\alpha-\beta+2}{\sqrt{3}\beta}\right)\arctan\left(\frac{2s+1}{\sqrt{3}}\right)}}{(s^2+s+1)^{\frac{\alpha+\beta}{2\beta}}}\left(\int \frac{\alpha u(0)-Q_\beta(s)}{\beta(s^2+s+1)^{\frac{\beta-\alpha}{2\beta}}} e^{\left(\frac{\beta-\alpha-2}{\sqrt{3}\beta}\right)\arctan\left(\frac{2s+1}{\sqrt{3}}\right)} ds + C\right)\right\}.$$

Proof. *From Eq. (3.1), the Eq. (3.22), can be written as*

$$(1-\alpha)t^{1-\beta}u'(t) + t^{2-\beta}u''(t) + tu'(t) + t^\beta u(t) + u(t) = q(t), \tag{3.23}$$

applying L_β^0 to the both sides of Eq. (3.23), and using Theorems 2.2 and 3.7, we have

$$\Big(1-\alpha\Big)\Big(sU_\beta(s)-u(0)\Big)$$

$$+u(0) - (1+\beta)sU_\beta(s) - \beta s^2 U'(s) - \beta U_\beta(s) - \beta s U'(s) - \beta U_\beta'(s) + U_\beta(s) = Q_\beta(s).$$

Or equivalently

$$U_\beta'(s) + \left(\frac{(\alpha+\beta)s}{\beta(s^2+s+1)} + \frac{\beta-1}{\beta(s^2+s+1)}\right)U_\beta(s) = \frac{\alpha u(0)-Q_\beta(s)}{\beta(s^2+s+1)}.$$

By solving the above ODE of the first order we obtain

$$U_\beta(s)$$

$$= e^{-\int\left(\frac{(\alpha+\beta)s}{\beta(s^2+s+1)}+\frac{\beta-1}{\beta(s^2+s+1)}\right)ds}\left(\int \frac{\alpha u(0)-Q_\beta(s)}{\beta(s^2+s+1)} e^{\int\left(\frac{(\alpha+\beta)s}{\beta(s^2+s+1)}+\frac{\beta-1}{\beta(s^2+s+1)}\right)ds} ds + C\right)$$

$$= \frac{e^{\left(\frac{\alpha-\beta+2}{\sqrt{3}\beta}\right)\arctan\left(\frac{2s+1}{\sqrt{3}}\right)}}{(s^2+s+1)^{\frac{\alpha+\beta}{2\beta}}}\left(\int \frac{\alpha u(0)-Q_\beta(s)}{\beta(s^2+s+1)^{\frac{\beta-\alpha}{2\beta}}} e^{\left(\frac{\beta-\alpha-2}{\sqrt{3}\beta}\right)\arctan\left(\frac{2s+1}{\sqrt{3}}\right)} ds + C\right).$$

By applying $\left(L_\beta^0\right)^{-1}$ *we obtain the desired result.* □

In particular if $\alpha = \frac{1}{2}, \beta = 1$ *and* $u(0) = 1$, *then the following CFDE*

$$t^{\frac{1}{2}}\left({}_tT_1[{}_tT_{\frac{1}{2}}u(t)]\right) + tu'(t) + t^1u(t) + u(t) = \frac{3}{2}e^t + 3te^t.$$

has the solution given by

$$u(t)$$

$$= L^{-1}\left\{\frac{e^{\frac{\sqrt{3}}{2}\arctan(\frac{2s+1}{\sqrt{3}})}}{(s^2+s+1)^{\frac{3}{4}}}\left(\int \frac{\frac{1}{2}-\frac{3}{2(s-1)}-\frac{3}{(s-1)^2}}{(s^2+s+1)^{\frac{1}{4}}} e^{-\frac{\sqrt{3}}{2}\arctan(\frac{2s+1}{\sqrt{3}})} ds + C\right)\right\}$$

$$= L^{-1}\left\{\frac{e^{\frac{\sqrt{3}}{2}\arctan(\frac{2s+1}{\sqrt{3}})}}{(s^2+s+1)^{\frac{3}{4}}}\left(\frac{(s^2+s+1)^{\frac{3}{4}}e^{-\frac{\sqrt{3}}{2}\arctan(\frac{2s+1}{\sqrt{3}})}}{s-1} + C\right)\right\}$$

$$= L^{-1}\left\{\frac{1}{s-1}\right\} + CL^{-1}\left\{\frac{e^{\frac{\sqrt{3}}{2}\arctan(\frac{2s+1}{\sqrt{3}})}}{(s^2+s+1)^{\frac{3}{4}}}\right\}$$

$$= e^t + CL^{-1}\left\{\frac{e^{\frac{\sqrt{3}}{2}\arctan(\frac{2s+1}{\sqrt{3}})}}{(s^2+s+1)^{\frac{3}{4}}}\right\}.$$

On the other hand, since u(0)=1, then C=0 and we obtain $u(t) = e^t$.

Theorem 3.19. *Suppose that* $u(t)$ *be twice differentiable on* $(0,\infty)$ *and* $0 < \alpha \leq \frac{1}{2}, 0 < \beta < \frac{1}{2}$ *such that* $\alpha > \beta$ *then the following CFDE*

$$ {}_tT_{\alpha+\beta}\left({}_tT_{\alpha-\beta}u(t)\right) + u(t) = q(t), \tag{3.24}$$

has the solution given by

$$u(t) = \left(L_{2\alpha}^0\right)^{-1}\left\{\frac{s^{(\frac{\beta-3\alpha}{2\alpha})}}{e^{(\frac{1}{2\alpha s})}}\int \frac{(\alpha-\beta)u(0)-Q_{2\alpha}(s)}{2\alpha s^{(\frac{\alpha+\beta}{2\alpha})}} e^{(\frac{1}{2\alpha s})} ds + C\right\}.$$

Proof. *From Theorem 3.10, the Eq. (3.24), can be written as*

$$ {}_tT_\alpha\Big({}_tT_\alpha u(t)\Big) + \beta\Big({}_tT_{2\alpha} u(t)\Big) + u(t) = q(t), \tag{3.25} $$

by applying $L^0_{2\alpha}$ to the both sides of Eq. (3.25) and using Theorems 2.1, 3.3, we obtain

$$ \alpha u(0) - 3\alpha s U_{2\alpha}(s) - 2\alpha s^2 U'_{2\alpha}(s) + \beta\Big(sU_{2\alpha}(s) - u(0)\Big) + U_{2\alpha}(s) = Q_{2\alpha}(s). $$

Or equivalently

$$ U'_{2\alpha}(s) + \left(\frac{3\alpha-\beta}{2\alpha s} - \frac{1}{2\alpha s^2}\right) U_{2\alpha}(s) = \frac{(\alpha-\beta)u(0) - Q_{2\alpha}(s)}{2\alpha s^2}. $$

By solving the above ODE of the first order we have

$$ U_{2\alpha}(s) = e^{-\int(\frac{3\alpha-\beta}{2\alpha s} - \frac{1}{2\alpha s^2})ds} \left(\int \frac{(\alpha-\beta)u(0) - Q_{2\alpha}(s)}{2\alpha s^2} e^{\int(\frac{3\alpha-\beta}{2\alpha s} - \frac{1}{2\alpha s^2})ds} ds + C\right) $$

$$ = \frac{s^{(\frac{\beta-3\alpha}{2\alpha})}}{e^{(\frac{1}{2\alpha s})}} \left(\int \frac{(\alpha-\beta)u(0) - Q_{2\alpha}(s)}{2\alpha s^{(\frac{\alpha+\beta}{2\alpha})}} e^{(\frac{1}{2\alpha s})} ds + C\right). $$

By applying $\Big(L^0_{2\alpha}\Big)^{-1}$ we obtain the desired result. □

For example we can solve the following ODE of the second order, by using fractional Laplace transform.

$$ tu''(t) + \frac{3}{4}u'(t) + u(t) = \frac{1}{8\sqrt{t}} + \sqrt{t}, \quad u(0) = 0, \tag{3.26} $$

The Eq. (3.26), can be written as

$$ (1-\frac{1}{2})t^{1-(\frac{1}{2}+\frac{1}{2})}u'(t) + t^{2-(\frac{1}{2}+\frac{1}{2})}u''(t) + \frac{1}{4}u'(t) + u(t) = \frac{1}{8\sqrt{t}} + \sqrt{t}. $$

On the other hand using Theorems 1.1 and 3.1, yields

$$ {}_tT_{\frac{1}{2}}[{}_tT_{\frac{1}{2}} u(t)] + \frac{1}{4}\Big({}_tT_{2(\frac{1}{2})} u(t)\Big) + u(t) = \frac{1}{8\sqrt{t}} + \sqrt{t}. $$

From Theorem 3.10, we get

$$ {}_tT_{(\frac{1}{2}+\frac{1}{4})}\Big({}_tT_{(\frac{1}{2}-\frac{1}{4})} u(t)\Big) + u(t) = \frac{1}{8\sqrt{t}} + \sqrt{t}. $$

Therefore, we obtain

$$ u(t) = \Big(L^0_{2(\frac{1}{2})}\Big)^{-1} \left\{ \frac{s^{(\frac{\frac{1}{4}-3(\frac{1}{2})}{2(\frac{1}{2})})}}{e^{(\frac{1}{2(\frac{1}{2})s})}} \left(\int \frac{(\frac{1}{2}-\frac{1}{4})(0) - \frac{1}{2}\sqrt{\frac{\pi}{s}}(\frac{1}{4}+\frac{1}{s})}{2(\frac{1}{2})s^{(\frac{(\frac{1}{2})+(\frac{1}{4})}{2(\frac{1}{2})})}} e^{(\frac{1}{2(\frac{1}{2})s})} ds + C\right) \right\} $$

$$ = L^{-1}\left\{ \frac{e^{-\frac{1}{s}}}{s^{\frac{5}{4}}} \left(\int \frac{-\frac{1}{2}\sqrt{\frac{\pi}{s}}(\frac{1}{4}+\frac{1}{s})}{s^{\frac{3}{4}}} e^{\frac{1}{s}} ds + C\right) \right\} $$

$$= L^{-1}\left\{\frac{e^{-\frac{1}{s}}}{s^{\frac{5}{4}}}\left(\frac{1}{2}\sqrt{\frac{\pi}{s}}s^{\frac{1}{4}}e^{\frac{1}{s}}+C\right)\right\}$$

$$= L^{-1}\left\{\frac{1}{2s}\sqrt{\frac{\pi}{s}}\right\}+CL^{-1}\left\{\frac{e^{-\frac{1}{s}}}{s^{\frac{5}{4}}}\right\} = \sqrt{t}+Ct^{\frac{1}{8}}J(\frac{1}{4},2\sqrt{t}).$$

Theorem 3.20. *Assume $u(t)$ be twice differentiable on $(0,\infty)$ and $\alpha,\beta>0$ be such that $\alpha+\beta\leq 1$, A, B, M, $N\in\mathbb{R}$, and $A+B\neq 0$, then the following CFDE*

$$At^{\alpha+\beta}\Big({}_tT_{\alpha+\beta}\Big({}_tT_{\alpha+\beta}u(t)\Big)\Big)+B\Big({}_tT_{\beta}\Big({}_tT_{\alpha}u(t)\Big)\Big)+MT_{\alpha+\beta}u(t)+Nu(t)=q(t), \tag{3.27}$$

has the solution given by

$$u(t)=\Big(L^0_{\alpha+\beta}\Big)^{-1}\Bigg\{\Bigg(\frac{e^{\left(\frac{-N}{(A+B)(\alpha+\beta)s}\right)}}{s^{\left(\frac{2A(\alpha+\beta)+2B\alpha+B\beta-M}{(A+B)(\alpha+\beta)}\right)}}\Bigg)$$
$$\times\Bigg(\int\frac{(A(\alpha+\beta)+B\alpha-M)u(0)-Q_{\alpha+\beta}(s)}{(A+B)(\alpha+\beta)s^{2-\left(\frac{2A(\alpha+\beta)+2B\alpha+B\beta-M}{(A+B)(\alpha+\beta)}\right)}}e^{\left(\frac{N}{(A+B)(\alpha+\beta)s}\right)}ds+C\Bigg)\Bigg\}.$$

Proof. Applying $L^0_{\alpha+\beta}$ to the both sides of Eq. (3.27) and using Theorems 2.1, 3.3, 3.13, we have

$$A\Big((\alpha+\beta)u(0)-2(\alpha+\beta)sU_{\alpha+\beta}(s)-(\alpha+\beta)s^2U'_{\alpha+\beta}(s)\Big)$$
$$+B\Big(\alpha u(0)-(2\alpha+\beta)sU_{\alpha+\beta}(s)-(\alpha+\beta)s^2U'_{\alpha+\beta}(s)\Big)$$
$$+M\Big(sU_{\alpha+\beta}(s)-u(0)\Big)+NU_{\alpha+\beta}(s)=Q_{\alpha+\beta}(s).$$

Or equivalently

$$U'_{\alpha+\beta}(s)+\left(\frac{2A(\alpha+\beta)+2B\alpha+B\beta-M-\frac{N}{s}}{(A+B)(\alpha+\beta)s}\right)U_{\alpha+\beta}(s)$$
$$=\frac{\Big((A+B)\alpha+A\beta-M\Big)u(0)-Q_{\alpha+\beta}(s)}{(A+B)(\alpha+\beta)s^2}.$$

By solving the above ODE of the first order we have

$$U_{\alpha+\beta}(s)$$
$$=\frac{e^{\left(\frac{-N}{(A+B)(\alpha+\beta)s}\right)}}{s^{\left(\frac{2A(\alpha+\beta)+2B\alpha+B\beta-M}{(A+B)(\alpha+\beta)}\right)}}\Bigg(\int\frac{(A(\alpha+\beta)+B\alpha-M)u(0)-Q_{\alpha+\beta}(s)}{(A+B)(\alpha+\beta)s^{2-\left(\frac{2A(\alpha+\beta)+2B\alpha+B\beta-M}{(A+B)(\alpha+\beta)}\right)}}e^{\left(\frac{N}{(A+B)(\alpha+\beta)s}\right)}ds+C\Bigg).$$

Thus, solution $u(t)$ results from the CF inverse transform. □

For example, by substituting $A = B = 2,\ M = N = 3,\ \alpha = \beta = \frac{1}{2}$, the CFDE,

$$2t\Big({}_tT_1\Big({}_tT_1u(t)\Big)\Big)+2\Big({}_tT_{\frac{1}{2}}\Big({}_tT_{\frac{1}{2}}u(t)\Big)\Big)+3T_{\frac{1}{2}+\frac{1}{2}}u(t)+3u(t)=3t^2+16t,$$

is given by

$$u(t)$$

$$=L^{-1}\left\{\frac{e^{-\frac{3}{4s}}}{s}\left(\int\left(-\frac{6}{4s^4}-\frac{4}{s^3}\right)e^{\frac{3}{4s}}ds+C\right)\right\}$$

$$=L^{-1}\left\{\frac{e^{-\frac{3}{4s}}}{s}\left(\frac{2e^{\frac{3}{4s}}}{s^2}+C\right)\right\}=L^{-1}\left\{\frac{2}{s^3}\right\}+CL^{-1}\left\{\frac{e^{\frac{-3}{4s}}}{s}\right\}=t^2+CJ(0,\sqrt{3t}).$$

Theorem 3.21. *Let $s,\ \alpha,\ \beta,\ \gamma>0$, and $0<\alpha+\beta+\gamma\le 1$, then*

$$L^0_{\alpha+\beta+\gamma}\left\{t^{-\gamma}\Big({}_tT_\beta\Big({}_tT_\alpha u(t)\Big)\Big)\right\}$$

$$=\alpha u(0)-(2\alpha+\beta+\gamma)sU_{\alpha+\beta+\gamma}(s)-(\alpha+\beta+\gamma)s^2U'_{\alpha+\beta+\gamma}(s).$$

Proof. *By Theorem 3.1 and using integration by parts, we have*

$$L^0_{\alpha+\beta+\gamma}\left\{t^{-\gamma}\Big({}_tT_\beta\Big({}_tT_\alpha u(t)\Big)\Big)\right\}$$

$$=L^0_{\alpha+\beta+\gamma}\left\{t^{-\gamma}\Big((1-\alpha)t^{1-(\alpha+\beta)}u'(t)+t^{2-(\alpha+\beta)}u''(t)\Big)\right\}$$

$$=(1-\alpha)\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u'(t)dt+\int_0^\infty e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}\Big(tu''(t)\Big)dt$$

$$=\int_0^\infty\Big(-\alpha+st^{\alpha+\beta+\gamma}\Big)e^{-s\frac{t^{(\alpha+\beta+\gamma)}}{(\alpha+\beta+\gamma)}}u'(t)dt.$$

Now, from Theorem 2.2 and using integration by parts, we obtain

$$\int_0^\infty\Big(-\alpha+st^{\alpha+\beta+\gamma}\Big)e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}u'(t)dt$$

$$=\Big(-\alpha+st^{\alpha+\beta+\gamma}\Big)e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}u(t)\Big]_0^\infty-\Big(\alpha+\beta+\gamma\Big)s\int_0^\infty t^{\alpha+\beta+\gamma-1}e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}u(t)dt$$

$$-\alpha s\int_0^\infty t^{\alpha+\beta+\gamma-1}e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}u(t)dt+s^2\int_0^\infty\Big(t^{\alpha+\beta+\gamma}u(t)\Big)e^{-s\frac{t^{\alpha+\beta+\gamma}}{\alpha+\beta+\gamma}}t^{\alpha+\beta+\gamma-1}dt$$

$$=\alpha u(0)-\Big(2\alpha+\beta+\gamma\Big)sU_{\alpha+\beta+\gamma}(s)-\Big(\alpha+\beta+\gamma\Big)s^2U'_{\alpha+\beta+\gamma}(s),$$

which completes the proof. □

Theorem 3.22. *Consider $\alpha,\ \beta,\ \gamma > 0$, such that $0 < \alpha+\beta+\gamma \le 1$ and k, m are constants, then the following CFDE,*

$$t^{-\gamma}\Big({}_tT_\beta\Big({}_tT_\alpha u(t)\Big)\Big) + ktu^{'}(t) + mu(t) = q(t), \tag{3.28}$$

is given by

$$u(t) = \Big(L^0_{\alpha+\beta+\gamma}\Big)^{-1}\Bigg\{\Bigg(\frac{s^{\frac{m}{\alpha+\beta+\gamma}-1}}{(s+k)^{\frac{\alpha+m}{\alpha+\beta+\gamma}}}\Bigg)\Bigg(\int \frac{\alpha u(0) - Q_{\alpha+\beta+\gamma}(s)}{(\alpha+\beta+\gamma)s^{\frac{m}{\alpha+\beta+\gamma}}}(s+k)^{\frac{\alpha+m}{\alpha+\beta+\gamma}-1}ds + C\Bigg)\Bigg\}.$$

Proof. *By applying $L^0_{\alpha+\beta+\gamma}$ to the both sides of Eq. (3.28) and using Theorems 3.4 and 3.21, we have*

$$\alpha u(0) - (2\alpha+\beta+\gamma)sU_{\alpha+\beta+\gamma}(s) - (\alpha+\beta+\gamma)s^2U^{'}_{\alpha+\beta+\gamma}(s) - k(\alpha+\beta+\gamma)U_{\alpha+\beta+\gamma}(s) - k(\alpha+\beta+\gamma)sU^{'}_{\alpha+\beta+\gamma}(s) + mU_{\alpha+\beta+\gamma}(s) = Q_{\alpha+\beta+\gamma}(s).$$

So, we have

$$U^{'}_{\alpha+\beta+\gamma}(s) + \Big(\frac{\alpha s + (\alpha+\beta+\gamma)(s+k) - m}{(\alpha+\beta+\gamma)s(s+k)}\Big)U_{\alpha+\beta+\gamma}(s) = \frac{\alpha u(0) - Q_{\alpha+\beta+\gamma}(s)}{(\alpha+\beta+\gamma)s(s+k)}.$$

By solving the above ODE of first order, we obtain

$$U_{\alpha+\beta+\gamma}(s) = \Big(e^{-\int(\frac{\alpha s+(\alpha+\beta+\gamma)(s+k)-m}{(\alpha+\beta+\gamma)s(s+k)})ds}\Big)\Big(\int \frac{\alpha u(0) - Q_{\alpha+\beta+\gamma}(s)}{(\alpha+\beta+\gamma)s(s+k)}e^{\int(\frac{\alpha s+(\alpha+\beta+\gamma)(s+k)-m}{(\alpha+\beta+\gamma)s(s+k)})ds}ds + C\Big)$$

$$= \Bigg(\frac{s^{\frac{m}{\alpha+\beta+\gamma}-1}}{(s+k)^{\frac{\alpha+m}{\alpha+\beta+\gamma}}}\Bigg)\Bigg(\int \frac{\alpha u(0) - Q_{\alpha+\beta+\gamma}(s)}{(\alpha+\beta+\gamma)s^{\frac{m}{\alpha+\beta+\gamma}}}(s+k)^{\frac{\alpha+m}{\alpha+\beta+\gamma}-1}ds + C\Bigg).$$

Thus, solution $u(t)$ results from the CF inverse transform. □

By substituting $\alpha=\beta=\gamma=\frac{1}{3},\ u(0)=1,\ k=1,$ and $m=2$, the following CFDE,

$$t^{-\frac{1}{3}}({}_tT_{\frac{1}{3}}({}_tT_{\frac{1}{3}}u(t))) + 1tu^{'}(t) + 2u(t) = \frac{2}{3}e^{-2t} + 2te^{-2t}.$$

has the solution given by

$$u(t) = L^{-1}\Bigg\{\Bigg(\frac{s}{(s+1)^{\frac{7}{3}}}\Bigg)\Bigg(\int(\frac{1}{3s^2} - \frac{2}{3s^2(s+2)} - \frac{2}{s^2(s+2)^2})(s+1)^{\frac{4}{3}}ds + C\Bigg)\Bigg\}$$

$$= L^{-1}\Bigg\{\frac{s}{(s+1)^{\frac{7}{3}}}\Bigg(\frac{(s+1)^{\frac{7}{3}}}{s(s+2)} + C\Bigg)\Bigg\}$$

$$= L^{-1}\Big\{\frac{1}{s+2}\Big\} + CL^{-1}\Bigg\{\frac{s}{(s+1)^{\frac{7}{3}}}\Bigg\}$$

$$= e^{-2t} + C\Bigg(\frac{-3t^{\frac{1}{3}}e^{-t}\sqrt{3}\Gamma(\frac{2}{3})(3t-4)}{8\pi}\Bigg).$$

Theorem 3.23. *Suppose* $\alpha,\ \beta,\ \gamma > 0,$ *such that* $0 < \alpha+\beta+\gamma \leq 1,\ w,\ k,\ m \in \mathbb{R}$ *and* $k \neq 0$, *then the following CFDE,*

$$t^{-\gamma}\Big({}_tT_\beta({}_tT_\alpha u(t))\Big) + w\Big({}_tT_{\alpha+\beta+\gamma}u(t)\Big) + ktu^{'}(t) + mu(t) = q(t), \tag{3.29}$$

is given by

$$u(t) = \Big(L^0_{\alpha+\beta+\gamma}\Big)^{-1}\Bigg\{\Bigg(\frac{s^{\frac{m}{(\alpha+\beta+\gamma)k}-1}}{(s+k)^{\frac{k\alpha-kw+m}{(\alpha+\beta+\gamma)k}}}\Bigg) \times\Bigg(\int \frac{(\alpha-w)u(0)-Q_{\alpha+\beta+\gamma}(s)}{(\alpha+\beta+\gamma)s^{\frac{m}{(\alpha+\beta+\gamma)k}}}(s+k)^{\frac{k\alpha-kw+m}{(\alpha+\beta+\gamma)k}-1}ds + C\Bigg)\Bigg\}.$$

By applying $L^0_{\alpha+\beta+\gamma}$ *to the both sides of Eq. (3.29) and using Theorems 1.1, 2.2 and 3.21, we have*

$$\alpha u(0) - (2\alpha+\beta+\gamma)sU_{\alpha+\beta+\gamma}(s) - (\alpha+\beta+\gamma)s^2U^{'}_{\alpha+\beta+\gamma}(s) - wu(0) + wsU_{\alpha+\beta+\gamma}(s) - k(\alpha+\beta+\gamma)U_{\alpha+\beta+\gamma}(s) - k(\alpha+\beta+\gamma)sU^{'}_{\alpha+\beta+\gamma}(s) + mU_{\alpha+\beta+\gamma}(s) = Q_{\alpha+\beta+\gamma}(s).$$

Therefore, we have

$$U^{'}_{\alpha+\beta+\gamma}(s) + \Bigg(\frac{(\beta+\gamma-w)s+(\alpha+\beta+\gamma)k-m}{(\alpha+\beta+\gamma)s(s+k)}\Bigg)U_{\alpha+\beta+\gamma}(s) = \frac{(\alpha-w)u(0)-Q_{\alpha+\beta+\gamma}(s)}{(\alpha+\beta+\gamma)s(s+k)}.$$

By solving the above ODE of the first order, we have

$$U_{\alpha+\beta+\gamma}(s) = \Bigg(e^{-\int(\frac{(\beta+\gamma-w)s+(\alpha+\beta+\gamma)k-m}{(\alpha+\beta+\gamma)s(s+k)})ds}\Bigg) \times\Bigg(\int \frac{(\alpha-w)u(0)-Q_{\alpha+\beta+\gamma}(s)}{(\alpha+\beta+\gamma)s(s+k)}e^{\int(\frac{(\beta+\gamma-w)s+(\alpha+\beta+\gamma)k-m}{(\alpha+\beta+\gamma)s(s+k)ds})ds}ds + C\Bigg)$$

$$= \Bigg(\frac{s^{\frac{m}{(\alpha+\beta+\gamma)k}-1}}{(s+k)^{\frac{k\alpha-kw+m}{(\alpha+\beta+\gamma)k}}}\Bigg)\Bigg(\int \frac{(\alpha-w)u(0)-Q_{\alpha+\beta+\gamma}(s)}{(\alpha+\beta+\gamma)s^{\frac{m}{(\alpha+\beta+\gamma)k}}}(s+k)^{\frac{k\alpha-kw+m}{(\alpha+\beta+\gamma)k}-1}ds + C\Bigg).$$

Therefore, we obtain

$$u(t) = (L^0_{\alpha+\beta+\gamma})^{-1}\Bigg\{\Bigg(\frac{s^{\frac{m}{(\alpha+\beta+\gamma)k}-1}}{(s+k)^{\frac{k\alpha-kw+m}{(\alpha+\beta+\gamma)k}}}\Bigg) \times\Bigg(\int \frac{(\alpha-w)u(0)-Q_{\alpha+\beta+\gamma}(s)}{(\alpha+\beta+\gamma)s^{\frac{m}{(\alpha+\beta+\gamma)k}}}(s+k)^{\frac{k\alpha-kw+m}{(\alpha+\beta+\gamma)k}-1}ds + C\Bigg)\Bigg\}.$$

For example by substituting $\alpha = \beta = \gamma = \frac{1}{3}$, $u(0) = 1$, $k = 1$, $w = 1$, $m = 2$ *in Eq.* (3.29) *we have*

$$t^{-\frac{1}{3}}({}_tT_{\frac{1}{3}}({}_tT_{\frac{1}{3}}u(t))) + 1({}_tT_{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}u(t)) + 1tu^{'}(t) + 2u(t) = \frac{11}{3}e^t + 2te^t.$$

Therefore, we obtain

$$\begin{aligned} u(t) = (L_1^0)^{-1}\Bigg\{ \Bigg(& \frac{s^{\frac{2}{(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})(1)}-1}}{(s+1)^{\frac{(1)(\frac{1}{3})-(1)(1)+2}{(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})(1)}}} \Bigg) \Bigg(\\ & \int \frac{(\frac{1}{3}-1)(1) - \frac{11}{3(s-1)} - \frac{2}{(s-1)^2}}{(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})s^{\frac{2}{(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})(1)}}} (s+1)^{\frac{(1)(\frac{1}{3})-(1)(1)+2}{(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})(1)}-1} ds + C \Bigg) \Bigg\} \\ = L^{-1} & \Bigg\{ \Bigg(\frac{s}{(s+1)^{\frac{4}{3}}} \Bigg) \Bigg(\int (\frac{-2}{3s^2} - \frac{11}{3(s-1)s^2} - \frac{2}{(s-1)^2s^2})(s+1)^{\frac{1}{3}} ds + C \Bigg) \Bigg\} \\ = L^{-1} & \Bigg\{ \frac{s}{(s+1)^{\frac{4}{3}}} \Bigg(\frac{(s+1)^{\frac{4}{3}}}{s(s-1)} + C \Bigg) \Bigg\} \\ = L^{-1} & \Bigg\{ \frac{1}{s-1} \Bigg\} + CL^{-1}\Bigg\{ \frac{s}{(s+1)^{\frac{4}{3}}} \Bigg\} \\ = e^t & + C\Bigg(\frac{-e^{-t}\sqrt{3}\Gamma(\frac{2}{3})(3t-1)}{2\pi t^{\frac{2}{3}}} \Bigg). \end{aligned}$$

Since $u(0) = 1$, *so we obtain,* $u(t) = e^t$.

4. Conclusion

The conformable fractional differential equation is a new kind of fractional derivative that needs to be more investigated. The product and chain rules are valid in the conformable fractional derivative. In this paper, some new results are reported by fractional Laplace transform which is useful in the theory of conformable fractional differential equations. By using fractional Laplace transform we can convert some of the ordinary differential equations and conformable fractional differential equations into the ordinary differential equations of the first order.

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