FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 4 (2024), 643–665
https://doi.org/10.22190/FUMI220808043S
Original Scientific Paper

EXISTENCE RESULT FOR A COUPLED SYSTEM OF HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS IN A BANACH ALGEBRA

Hari Mohan Srivastava[1] and Benaouda Hedia[2]

[1] **Department of Mathematics and Statistics, University of Victoria**
Victoria, V8W 3R4 British Columbia, Canada
[2] **Department of Mathematics and Informatics**
Faculty of Mathematics and Computers Science, University of Tiaret
14000 Tiaret, Algeria

ORCID IDs: Hari Mohan Srivastava https://orcid.org/0000-0002-9277-8092
Benaouda Hedia https://orcid.org/0000-0002-7062-5845

Abstract. In this work, we investigate the existence result for a coupled system of hybrid fractional differential equations in a Banach algebra. Our main result is based on a generalization of Darbo's fixed point theorem in Banach algebra. We apply in our approach the technique of measure of non-compactness, we prove that the Kuratowski measure of noncompactness satisfies a condition (m) which will be useful in our considerations. An example is given to illustrate the feasibility of our main result. An example is provided to illustrate our result.

Keywords: hybrid fractional differential equations, Banach algebra, Darbo's fixed point theorem, Kuratowski measure.

1. Introduction

In this paper, we deal with the existence of solutions to a coupled system of nonlinear hybrid fractional differential equations with Erdélyi-Kober integral boundary

Received August 08, 2022, revised: July 23, 2024 accepted: August 18, 2024
Communicated by Marko Petković
Corresponding Author: Benaouda Hedia. E-mail addresses: harimsri@uvic.ca (H. M. Srivastava), b_hedia@univ-tiaret.dz (B. Hedia)
2020 *Mathematics Subject Classification.* Primary 26A33, 34A38, 34K34; Secondary 34A08, 34B15

conditions.

$$
(1.1)\quad \begin{cases} {}^cD^{\alpha}\left(\frac{x(t)}{\mathfrak{f}_1(t,x(t),y(t))}\right) = \mathfrak{h}_1(t,x(t),y(t)), & t\in I=[0,T],\ 1<\alpha\le 2,\\ {}^cD^{\beta}\left(\frac{y(t)}{\mathfrak{f}_2(t,x(t),y(t))}\right) = \mathfrak{h}_2(t,x(t),y(t)), & t\in I=[0,T],\ 1<\beta\le 2,\\ x(0)=0, \qquad y(0)=0, & \\ \left(\frac{x(t)}{\mathfrak{f}_1(t,x(t),y(t))}\right)_{t=T} = \gamma I_{\kappa}^{\varsigma,\tau}x(\zeta), & \\ \left(\frac{y(t)}{\mathfrak{f}_2(t,x(t),y(t))}\right)_{t=T} = \delta I_{\kappa}^{\varsigma,\tau}y(\omega), & \end{cases}
$$

where ${}^cD^{\alpha}$, ${}^cD^{\beta}$ are the Caputo fractional derivatives of order α, β respectively and $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{h}_1, \mathfrak{h}_2$ are given functions $\mathfrak{f}_1, \mathfrak{f}_2 : I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$, $\mathfrak{h}_1, \mathfrak{h}_2 : I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, γ and δ are real numbers , $I_{\kappa}^{\varsigma,\tau}$ denotes the Erdélyi-Kober fractional integral of order $\tau>0$, $\kappa>0$ and $\varsigma\in\mathbb{R}$, ζ, $\omega\in(0,T)$. Recently, fractional differential equations have a large application in a variety of fields such as physics, mathematics, electrical networks, signal and image processing, aerodynamics, economics and so on. Hence there has been increased attention from both theoretical and the applied points, for more details see [5, 3, 4, 6, 15, 26, 23, 24], etc. The study of coupled systems of fractional differential equations is also important as such systems appear in a variety of problems of applied nature, especially in biosciences. For instance, see [7]. Hybrid fractional differential equations have been also studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Our aim in this paper is to generalize in a consistent way this line of reasoning for the case of nonlinear coupled system of a hybrid fractional differential equations.

In the first part of the paper [14], the author dealt with properties concerning the fractional derivative of the Riemann ζ function. He made use Grünwald-Letnikov fractional derivative to compute the functional equation, this one is rewritten in a simplified form that reduces the computational cost. The aim of the second part examined the link with the distribution of prime numbers. The Dirichlet ζ function suggests the introduction of a complex strip as a fractional counterpart of the critical strip; finally the author showed the fractional derivative of ζ with the distribution of prime numbers in the left half-plane.

$$
\zeta^{\alpha}(s)\sim\sum_{p\in P}\sum_{t=0}^{+\infty}\frac{\ln\alpha\ p^t}{p^{-st}}.
$$

In the paper [13], the author dealt with the fractional calculus of ζ function. In particular, his study is based on the Hurwitz ζ function defined by

$$
\begin{cases} \zeta(s,a)=\sum_{n=0}^{+\infty}\frac{1}{(n+a)^s}, & Res>1,\ a\in\mathbb{R}:\ 0<a\le 1;\\ F(s)=\sum_{n=0}^{+\infty}\frac{f(n)}{n^s}, & f:\mathbb{N}\to\mathbb{C}. \end{cases}
$$

The main tool is the complex generalization of the Grünwald-Letnikov fractional derivative. He began by proving the functional equation together with an integral representation by Bernoulli numbers; finally an application in terms of Shannon entropy is given.

In the paper [12], the aim is the study of the fractional derivative of the Lerch zeta function. The calculus of the fractional derivative of the Lerch zeta function was obtained by using a complex generalization of the Grünwald-Letnikov derivative. This derivation combined with generalized Leibniz rule ensures him to obtain a functional equation for the fractional derivative of the Lerch zeta function. He gave another form of this equation, and simplified and proved an approximate functional equation for the fractional derivative of the Lerch zeta function.

In the paper [20] the authors dealt with in the first part properties of the Caputo derivative in real line. Then they studied the fractional derivative in complex plane by Ortigueira defined by

$$D^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi\, j} \int_C \frac{f(\omega)}{(\omega - z)^{\alpha+1}} dw,$$

where C is any U shaped contour that encircles the half-straight line starting at z that is the branch cut line of $w^{-\alpha-1}$, j is the imaginary part. In the second part they generalized the Caputo derivative in real line to that in complex plane then they studied its properties.

In the paper [21] the authors presented a number of fractional derivative and integral representations for general families of the Hurwitz-Lerch ζ function defined by

$$\sum_{n=0}^{+\infty} \frac{1}{(n+a)^s},\ \Re(s) > 1,\ a \in \mathbb{C} \backslash \mathbb{Z}_0^-,\ \mathbb{Z}_0^- = \{0, -1, -2...\},$$

it can be presented as the following sum-integral representation :

$$\sum_{j=0}^{k-1} \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1} e^{-(a+j)t}}{1 - e^{-kt}} dt.$$

In the article [25] the author used a fractional q-calculus operator to define the subclasses Sn $\alpha(\lambda, \beta, b, q)$ and Gn$\alpha(\lambda, \beta, b, q)$ of normalized analytic functions with complex order and negative coefficients. Using the results he obtained their associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points and growth and distortion theorems. This survey motives the researchers to applique the basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials in several areas, we remind at the end the definition of the q-derivative (or the q-difference) of a function f as follows :

Definition 1.1. The q-derivative (or the q-difference) of a function $f(z)$ is denoted by $D_q f(z)$ and defined in a given subset of $\mathbb{C}$ by

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{1 - qz}, & z \neq 0; \\ f'(0), & z = 0. \end{cases}$$

In the paper[27], the author dealt with a fractional derivative introduced by means of the Fourier transform. The explicit form of the kernel of general derivative operator acting on the functions analytic on a curve in complex plane is deduced and the correspondence to exiting approaches is shown.

As far as we know, there are a few papers which treated this work based on fixed point theorem combined with measure of non-compactness for hybrid fractional differential coupled system in Banach algebra see [11, 16], which constitute our first contribution. The second motivation deals with the result that a Kuratowski measure of compactness υ that will be specified later satisfies the condition, that is,

$$\upsilon(XY) \leq \|X\|\upsilon(Y) + \|Y\|\upsilon(X).$$

As far as we know, this preliminary result doesn't exist anywhere in the literature. Let us now list the difficulties which arise in this situation:

(1) We need to know how to define the measure of non-compactness in Banach product space?

(2) How to choose a judicious measure of non-compactness which is compatible with Darbo's fixed point generalization?

(3) Proving that measure of non-compactness of Kuratowski, the measure chosen, verifies the condition (m) see [9] and this is the crucial step of this paper.

We show that that the conditions imposed are optimal in a natural way, in the sense that on one hand they don't imply each other, and on the other hand one there is no need to add further conditions to prove this result. The paper is divided into four sections. In Section 2. we give some basic notations as well as some preliminary lemmas which will play essential roles in this paper. In Section 3. we present the existence results for the problem (1.1) by using a generalization of Darbo's fixed point theorem combined with measure of non-compactness in a Banach algebra ; finally in the last Section 4., we conclude the paper by giving a concrete example to illustrate the feasibility of our main result.

2. Preliminaries

In this section, we introduce some notations and technical results which will be used throughout this paper. By $C(I, \mathbb{R})$, we denote the Banach space of continuous functions x from I to $\mathbb{R}$ equipped with the supremum norm

$$\|x\| = \sup_{t\in[0,T]} |x(t)|.$$

By $C(I, \mathbb{R})\times C(I, \mathbb{R})$, we denote the Banach space of pair of functions (x, y) equipped with the norm

$$\|(x, y)\|_1 = \|x\| + \|y\|.$$

We begin with some definitions from the theory of fractional calculus.

Definition 2.1. [8] Let $h \in L^1([a,b],\mathbb{R})$. The fractional integral of order $\alpha > 0$ of the function h is defined almost everywhere in $[a,b]$ and given by

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

provided that the right side is pointwise defined on $(0,+\infty)$. Γ is the gamma function. When $a = 0$, we write $I^{\alpha} h(t) = [h * \vartheta_{\alpha}](t)$, where $\vartheta_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\vartheta_{\alpha}(t) = 0$ for $t \leq 0$. the equality holds everywhere if $h \in C([a,b],\mathbb{R})$.

Definition 2.2. [8] Let $\alpha > 0$ and n be the smallest integer greater than or equal to α and $h \in C^n([a,b],\mathbb{R})$. Then the Caputo fractional derivative of order α of the function h is defined by

$$\begin{aligned} {}^{C}D_{a+}^{\alpha} h(t) &= I_{a+}^{n-\alpha} \frac{d^n h}{dt^n}(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \frac{d^n h}{ds^n}(s) ds, \end{aligned}$$

provided that the right side is pointwise defined on $(0,+\infty)$.

Definition 2.3. [22] Let h be a function such that $h \in L^1([a,b],\mathbb{R})$. The Erdélyi-Kober fractional integral of order $\tau > 0$, with $\kappa > 0$ and $\epsilon \in \mathbb{R}$, is defined by

$$I_{\kappa}^{\varsigma,\tau} h(t) = \frac{\kappa t^{-\kappa(\tau+\varsigma)}}{\Gamma(\tau)} \int_0^t \frac{s^{\kappa\varsigma+\kappa-1}}{(t^{\kappa}-s^{\kappa})^{1-\tau}} h(s) ds,$$

provided the right side is pointwise defined on $(0,+\infty)$.

Remark 2.1. For $\kappa = 1$ the above operator is reduced to the Kober operator

$$I_1^{\varsigma,\tau} h(t) = \frac{t^{-(\tau+\varsigma)}}{\Gamma(\tau)} \int_0^t \frac{s^{\varsigma}}{(t-s)^{1-\tau}} h(s) ds, \qquad \varsigma, \tau > 0.$$

For $\varsigma = 0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight,

$$I_1^{0,\tau} h(t) = \frac{t^{-\tau}}{\Gamma(\tau)} \int_0^t \frac{h(s)}{(t-s)^{1-\tau}} ds, \qquad \tau > 0.$$

For the existence of solutions of our problem (1.1), we need the following auxiliary lemmas.

Lemma 2.1. *[18] Let $\alpha > 0$ and $\beta > 0$ and h a function such that $h \in L^1([a,b],\mathbb{R})$. Then the following semigroup property is valid for fractional integrals :*

(i)

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} h(t) = I_{a^+}^{\alpha+\beta} h(t),$$

(ii)

$$I^{\alpha}_{b^-} I^{\beta}_{b^-} h(t) = I^{\alpha+\beta}_{b^-} h(t).$$

Lemma 2.2. *[18] Let $\alpha > 0$. Then we have for a function $h \in C^n([a,b],\mathbb{R})$*

$$^{c}D^{\alpha} h(t) = 0,$$

has a unique solution $h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$, and $n = [\alpha] + 1$.

Lemma 2.3. *[18] Let $\alpha > 0$. Then we have for a function $h \in C^n([a,b],\mathbb{R})$*

$$I^{\alpha c} D^{\alpha} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$, and $n = [\alpha] + 1$.

Lemma 2.4. *[22] Let $\kappa, \tau > 0$ and $\varsigma, q \in \mathbb{R}$, then we have*

$$I^{\varsigma,\tau}_{\kappa} t^q = \frac{t^q \Gamma\left(\varsigma + (\frac{q}{\kappa}) + 1\right)}{\Gamma\left(\varsigma + (\frac{q}{\kappa}) + \tau + 1\right)}. \tag{2.1}$$

2.1. Measure of non-compactness

The measure of non-compactness is a very useful tool in nonlinear analysis. It was initiated by Kuratowski [19] and Darbo [10] which are applied to the theories of differential and integral equations. In this section, we give some definitions, properties and examples about measure of non-compactness that will be used in this work. More details about these facts can be found in the monograph [2]. Let us now give the definition of the measure of non-compactness in the sense of Kuratowski and its properties. Let E be a Banach space with the norm $\|.\|$, If X is a nonempty subset X of E then $\overline{X}$ and $ConvX$ denote the closure and the convex closure of X, respectively. By diam X we will denote the diameter of a bounded set X, the norm of X is defined by $\|.\|$, i.e. $\|X\| = \sup \|x\| : x \in X$. Further we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily which contains all relatively compact subsets.
Recall that a subset $X \subset E$ is relatively compact provided that the closure $\overline{X}$ is compact.

Definition 2.4. [17] Let E be a Banach space and $\mathfrak{M}_E$ the family of all bounded subsets of E. Then the function: $\upsilon : \mathfrak{M}_E$ to $\mathbb{R}_+$ defined by

$$\upsilon(\Omega) = \inf\{\varsigma > 0 : \Omega \text{admits a finite cover by sets of diameter } \leq \varsigma\},$$

is called the Kuratowski measure of noncompactness.

Let us list some properties of Kuratowski measure of non-compactness that will be useful hereafter.

Lemma 2.5. *[17] Let $A, B \in \mathfrak{M}_E$. The following properties hold :*

(i_1) $\upsilon(A) = 0$ *if and only if* A *is relatively compact,*

(i_2) $\upsilon(A) = \upsilon(\overline{A})$, *where* $\overline{A}$ *denotes the closure of* A,

(i_3) $\upsilon(A + B) \leq \upsilon(A) + \upsilon(B)$,

(i_4) $A \subset B$ *implies* $\upsilon(A) \leq \upsilon(B)$,

(i_5) $\upsilon(aA) = \|a\|\upsilon(A)$ *for all* $a \in E$,

(i_6) $\upsilon(\{a\} \cup A) = \upsilon(A)$ *for all* $a \in E$,

(i_7) $\upsilon(A) = \upsilon(Conv(A))$, *where* $Conv(A)$ *is the smallest convex set that contains* A.

Lemma 2.6. *[17] If $V \subset C(I, E)$ is a bounded and equicontinuous set, then*
i) *the function $\upsilon(V(.))$ is continuous on I and*

$$\upsilon_c(V) = \sup_{0 \leq t \leq T} \upsilon(V(t)),$$

ii)

$$\upsilon\left(\int_0^T x(s)ds : \quad x \in V\right) \leq \int_0^T \upsilon(V(s))ds,$$

where $V(s) = \{x(s) : \ x \in V\}$, $s \in I$.

In the following part, we will assume that the space E has a structure of Banach algebra. In this situation, we denote by xy the product of two elements $x, y \in E$ and by XY the product of two subsets X and Y of E, i.e. $XY = \{xy : x \in X, y \in Y\}$. Now, we recall the following property of measure of non-compactness [2] which will be very useful hereafter.

Definition 2.5. [9] We say that a measure of noncompactness μ defined on the Banach algebra E satisfies the assumption (m) if, for any $X, Y \in \mathfrak{M}_E$, the following property holds,

$$\mu(XY) \leq \|X\|\mu(Y) + \|Y\|\mu(X). \tag{2.2}$$

Let us first give an essential result which will be used in the following.

Theorem 2.1. *The Kuratowski measure of noncompactness υ on $C(I, \mathbb{R})$ satisfies the condition* (2.2).

Proof. Let $X, Y \subset \mathfrak{M}_{C(I,\mathbb{R})}$ a nonempty bounded subset of $C(I,\mathbb{R})$. Let S_i be a partition of bounded subset of $C(I,\mathbb{R})$, with $\text{diam}(S_i) < d$ for each $i = 1, \ldots, n$ and $X = \bigcup_{i=1}^{n} S_i$. Furthermore, let G_i be a partition of bounded subset of $C(I,\mathbb{R})$ with $\text{diam}(G_j) < p$ for each $j = 1, \ldots, m$ and $Y = \bigcup_{j=1}^{n} G_j$. Note that $\text{diam}(S_i)$ and $\text{diam}(G_i)$ indicate respectively the diameter of (S_i) and (G_i).

$$\begin{aligned}
\text{diam}(S_iG_j) &= \sup_{(x,y)\in S_iG_i,\ (x',y')\in S_iG_j} \|xy - x'y'\| \\
&= \sup_{(x,y)\in S_iG_i,\ (x',y')\in S_iG_j} \|xy - xy' + xy' - x'y'\| \\
&\leq \|x\| \sup_{(y,y')\in G_j^2} \|y - y'\| + \|y'\| \sup_{(x,x')\in S_i^2} \|x - x'\| \\
&\leq \|X\|\text{diam}(G_j) + \|Y\|\text{diam}(S_j).
\end{aligned}$$

From the infimum property, we deduce that there exists ϵ such that

$$\text{diam}(S_j) < d < \upsilon(X) + \epsilon,\ \text{diam}(G_j) < p < \upsilon(Y) + \epsilon,$$

it follows then,

$$\begin{aligned}
\text{diam}(S_iG_j) &\leq \|X\|\text{diam}(G_j) + \|Y\|\text{diam}(S_i) \\
&< \|X\|p + \|Y\|d \\
&< \|X\|\upsilon(Y) + \|Y\|\upsilon(X) + \epsilon\|X\| + \varsigma\|Y\|.
\end{aligned}$$

Bearing in the mind definition (2.4) and taking $\epsilon \to 0$ in the last inequality, we get

$$\upsilon(XY) \leq \|X\|\upsilon(Y) + \|Y\|\upsilon(X).$$

□

In [1], the authors proved the following generalization of Darbo's fixed point theorem.

Theorem 2.2. *[9]Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathfrak{N} : \Omega$ to Ω be a continuous operator satisfying*

$$\upsilon(\mathfrak{N}X) \leq \vartheta(\upsilon(X)), \tag{2.3}$$

for any nonempty subset X of Ω, where υ is an arbitrary measure of noncompactness and $\vartheta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing function such that $\lim_{n\to+\infty} \vartheta^n(t) = 0$ for each $t \in \mathbb{R}^+$, where ϑ^n denotes the n-iteration of ϑ. Then $\mathfrak{N}$ has at least one fixed point in Ω.

Moreover, in [1] the authors proved the following lemma which will be useful in our consideration.

Lemma 2.7. *[9]Let $\vartheta : \mathbb{R}^+$ to $\mathbb{R}^+$ be a nondecreasing and upper semicontinuous function. Then the following conditions are equivalent :*
a) $\lim_{n\to+\infty} \vartheta^n(t) = 0$, *for any* $t \geq 0$,
b) $\vartheta(t) < t$ *for any* $t > 0$.

Theorem 2.3. *[7] Assume that $\upsilon_1, \upsilon_2, \upsilon_3, \dots, \upsilon_n$ are the measures of noncompactness in $E_1, E_2, \dots, E_n$, a sequence of Banach spaces, respectively. Moreover suppose that the function $\mathfrak{F} : [0, +\infty)^n$ to $[0, +\infty)$ is convex and $\mathfrak{F}(x_1, x_2 \dots x_n) = 0$ if, and only if $x_i = 0$ for $i = 1, \dots, n$. Then*

$$\widetilde{\upsilon}(V) = \mathfrak{F}(\upsilon(V_1), \upsilon(V_2), \dots, \upsilon(V_n)),$$

defines a measure of noncompactness on $E_1 \times E_2 \times \cdots \times E_n$, where V_i denotes the natural projection of V onto E_i for $i = 1, \dots, n$.

Example 2.1. [7] Let υ be a measure of non-compactness. We define $\mathfrak{F}(x, y) = x + y$ for any $x, y \in [0, +\infty)$. Then $\mathfrak{F}$ satisfies all assumptions cited in Theorem 2.3. Hence $\widetilde{\upsilon}(V) = \upsilon(V_1) + \upsilon(V_2)$ is a measure of noncompactness in the space $E \times E$ where $V_i,\ i = 1, 2$ denote the natural projections of V.

3. Main Result

Definition 3.1. A pair of functions $(x, y) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$, whose α, β-derivatives exists on I is said to be a solution of (1.1) if x and y satisfy the equations,

$$\begin{cases} {}^cD^\alpha \left(\frac{x(t)}{\mathfrak{f}_1(t,x(t),y(t))} \right) = \mathfrak{h}_1(t, x(t), y(t)), \quad t \in I = [0, T],\ 1 < \alpha \leq 2, \\ {}^cD^\beta \left(\frac{y(t)}{\mathfrak{f}_2(t,x(t),y(t))} \right) = \mathfrak{h}_2(t, x(t), y(t)), \quad t \in I = [0, T], \quad 1 < \beta \leq 2, \end{cases}$$

on I and also satisfy the conditions,

$$\begin{cases} x(0) = 0, \qquad y(0) = 0, \\ \left(\frac{x(t)}{\mathfrak{f}_1(t,x(t),y(t))} \right)_{t=T} = \gamma I_\kappa^{\varsigma,\tau} x(\zeta),\ \zeta \in (0, T), \\ \left(\frac{y(t)}{\mathfrak{f}_2(t,x(t),y(t))} \right)_{t=T} = \delta I_\kappa^{\varsigma,\tau} y(\omega)\ \omega \in (0, T). \end{cases}$$

Lemma 3.1. *Let $1 < \alpha \leq 2$ and let $h : I \to \mathbb{R}$ be continuous. A function x is a solution of the fractional equation*

$$(3.1)\begin{cases} x(t) = \frac{t\gamma\kappa\zeta^{-\kappa(\tau+\varsigma)}}{\Gamma(\alpha)\Gamma(\tau)\left[T - \gamma\frac{\zeta\Gamma(\varsigma+\frac{1}{\kappa}+1)}{\Gamma(\varsigma+\frac{1}{\kappa}+\tau+1)}\right]} \int_0^\zeta \int_0^r \frac{r^{\kappa\varsigma+\kappa-1}}{(\zeta^\kappa - r^\kappa)^{1-\tau}} (r-s)^{\alpha-1} h(s) ds dr \\ -\frac{t}{\Gamma(\alpha)\left[T - \gamma\frac{\zeta\Gamma(\varsigma+\frac{1}{\kappa}+1)}{\Gamma(\varsigma+\frac{1}{\kappa}+\tau+1)}\right]} \int_0^T (T-s)^{\alpha-1} h(s) ds \\ +\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \end{cases}$$

if and only if x is a solution of the fractional differential equation

$$ {}^cD^\alpha x(t) = h(t), \qquad 0 < t < T, \tag{3.2}$$

$$ x(0) = 0, \qquad x(T) = \gamma I_\kappa^{\varsigma,\tau} x(\zeta), \tag{3.3}$$

with

$$T - \gamma \frac{\zeta\Gamma(\varsigma + \frac{1}{\kappa} + 1)}{\Gamma(\varsigma + \frac{1}{\kappa} + \tau + 1)} \neq 0.$$

Proof. Assume that x satisfies (3.2). Lemma 2.3 implies

$$x(t) = c_0 + c_1 t + I^\alpha h(t),$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. From the condition (3.3), we deduce that $c_0 = 0$. It follows then

$$x(T) = c_1 T + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) ds$$

and

$$x(\zeta) = c_1 \zeta + \frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta-s)^{\alpha-1} h(s) ds.$$

Again from the condition (3.3) we obtain

$$\begin{aligned} c_1 = & \frac{\gamma\kappa\zeta^{-\kappa(\tau+\varsigma)}}{\Gamma(\alpha)\Gamma(\tau)\left[T - \gamma\frac{\zeta\Gamma(\varsigma+\frac{1}{\kappa}+1)}{\Gamma(\varsigma+\frac{1}{\kappa}+\tau+1)}\right]} \int_0^\zeta \int_0^r \frac{r^{\kappa\varsigma+\kappa-1}}{(\zeta^\kappa - r^\kappa)^{1-\tau}} (r-s)^{\alpha-1} h(s) ds dr \\ & - \frac{1}{\Gamma(\alpha)\left[T - \gamma\frac{\zeta\Gamma(\varsigma+\frac{1}{\kappa}+1)}{\Gamma(\varsigma+\frac{1}{\kappa}+\tau+1)}\right]} \int_0^T (T-s)^{\alpha-1} h(s) ds; \end{aligned}$$

so,

$$\begin{aligned} x(t) = & \frac{t\gamma\kappa\zeta^{-\kappa(\tau+\varsigma)}}{\Gamma(\alpha)\Gamma(\tau)\left[T - \gamma\frac{\zeta\Gamma(\varsigma+\frac{1}{\kappa}+1)}{\Gamma(\varsigma+\frac{1}{\kappa}+\tau+1)}\right]} \int_0^\zeta \int_0^r \frac{r^{\kappa\varsigma+\kappa-1}}{(\zeta^\kappa - r^\kappa)^{1-\tau}} (r-s)^{\alpha-1} h(s) ds dr \\ & - \frac{t}{\Gamma(\alpha)\left[T - \gamma\frac{\zeta\Gamma(\varsigma+\frac{1}{\kappa}+1)}{\Gamma(\varsigma+\frac{1}{\kappa}+\tau+1)}\right]} \int_0^T (T-s)^{\alpha-1} h(s) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds. \end{aligned}$$

Conversely, assume that x satisfies the fractional equation (3.1), since ${}^cD^\alpha$ is the left inverse of I^α, we get

$${}^cD^\alpha x(t) = h(t), \quad \text{for each } t \in [0, T].$$

Also, obviously one has by an easy computation (3.3). □

From lemma 3.1 we infer :

Lemma 3.2. *Let* $1 < \alpha, \beta \leq 2$, $\mathfrak{h}_1, \mathfrak{h}_2 : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *and* $\mathfrak{f}_1, \mathfrak{f}_2 : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$. *Then* (x, y) *is a solution of the integral system*

$$x(t) = \mathfrak{f}_1(t, x(t), y(t)) \times \left(-\frac{t}{\Gamma(\alpha)\left[T - \gamma \frac{\zeta\Gamma(\varsigma + \frac{1}{\kappa} + 1)}{\Gamma(\varsigma + \frac{1}{\kappa} + \tau + 1)}\right]} \int_0^T (T - s)^{\alpha - 1} \mathfrak{h}_1(s, x(s), y(s)) ds \right.$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \mathfrak{h}_1(s, x(s), y(s)) ds$$

$$\left. + \frac{t\beta\kappa\zeta^{-\kappa(\tau+\varsigma)}}{\Gamma(\alpha)\Gamma(\tau)\left[T - \gamma \frac{\zeta\Gamma(\varsigma + \frac{1}{\kappa} + 1)}{\Gamma(\varsigma + \frac{1}{\kappa} + \tau + 1)}\right]} \int_0^\zeta \int_0^r \frac{r^{\kappa\varsigma + \kappa - 1}}{(\zeta^\kappa - r^\kappa)^{1-\tau}} (r - s)^{\alpha - 1} \mathfrak{h}_1(s, x(s), y(s)) ds dr \right)$$

$$y(t) = \mathfrak{f}_2(t, x(t), y(t)) \times \left(-\frac{t}{\Gamma(\beta)\left[T - \delta \frac{\omega\Gamma(\varsigma + \frac{1}{\kappa} + 1)}{\Gamma(\varsigma + \frac{1}{\kappa} + \tau + 1)}\right]} \int_0^T (T - s)^{\beta - 1} \mathfrak{h}_2(s, x(s), y(s)) ds \right.$$

$$+ \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \mathfrak{h}_2(s, x(s), y(s)) ds$$

$$\left. + \frac{t\beta\kappa\omega^{-\kappa(\tau+\varsigma)}}{\Gamma(\beta)\Gamma(\tau)\left[T - \delta \frac{\omega\Gamma(\varsigma + \frac{1}{\kappa} + 1)}{\Gamma(\varsigma + \frac{1}{\kappa} + \tau + 1)}\right]} \int_0^\omega \int_0^r \frac{r^{\kappa\varsigma + \kappa - 1}}{(\omega^\kappa - r^\kappa)^{1-\tau}} (r - s)^{\beta - 1} \mathfrak{h}_2(s, x(s), y(s)) ds dr \right),$$

if and only if (x, y) *is a solution of the coupled fractional differential system*

$$\begin{cases} {}^cD^\alpha \left(\frac{x(t)}{\mathfrak{f}_1(t, x(t), y(t))} \right) = \mathfrak{h}_1(t, x(t), y(t)), & 0 < t < T, \\ {}^cD^\beta \left(\frac{y(t)}{\mathfrak{f}_2(t, x(t), y(t))} \right) = \mathfrak{h}_2(t, x(t), y(t)), & 0 < t < T, \end{cases}$$

added with the boundary conditions,

$$\begin{cases} x(0) = 0, \qquad y(0) = 0, \\ x(T) = \gamma I_\kappa^{\varsigma,\tau} y(\zeta), \ \zeta \in (0, T), \\ y(T) = \delta I_\kappa^{\varsigma,\tau} y(\omega), \ \omega \in (0, T), \end{cases}$$

with

$$T - \gamma \frac{\zeta\Gamma(\varsigma + \frac{1}{\kappa} + 1)}{\Gamma(\varsigma + \frac{1}{\kappa} + \tau + 1)} \neq 0, \qquad T - \delta \frac{\omega\Gamma(\varsigma + \frac{1}{\kappa} + 1)}{\Gamma(\varsigma + \frac{1}{\kappa} + \tau + 1)} \neq 0.$$

Assume that Λ is the following set of functions

$$\Lambda = \left\{ \vartheta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ : \ \vartheta \text{ is nondecreasing such that } \lim_{n \to \infty} \vartheta^n(t) = 0 \text{ for any } t \in \mathbb{R}_+ \right\}.$$

Remark 3.1. The set Λ satisfies the following properties.
1) If $\lambda \in [0,1]$ and $\vartheta \in \Lambda$ then $\lambda\vartheta \in \Lambda$.
2) If $\vartheta_1, \vartheta_2 \in \mathcal{A}$, then $\max(\vartheta_1, \vartheta_2) \in \Lambda$.
3) Let $\vartheta \in \Lambda$, it is easy to see that $\vartheta(t) < t$ for any $t > 0$.

Let us now assume the following assumptions,

(H_1) $\mathfrak{f}_1, \mathfrak{f}_2 \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $\mathfrak{h}_1, \mathfrak{h}_2 \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

(H_2) For any $t \in I$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$:

$$\begin{aligned}
|\mathfrak{f}_1(t, x_1, y_1) - \mathfrak{f}_1(t, x_2, y_2)| &\leq \vartheta_1(|x_1 - x_2| + |y_1 - y_2|),\\
|\mathfrak{f}_2(t, x_1, y_1) - \mathfrak{f}_2(t, x_2, y_2)| &\leq \vartheta_2(|x_1 - x_2| + |y_1 - y_2|),\\
|\mathfrak{h}_1(t, x_1, y_1) - \mathfrak{h}_1(t, x_2, y_2)| &\leq \sigma_1(|x_1 - x_2| + |y_1 - y_2|),\\
|\mathfrak{h}_2(t, x_1, y_1) - \mathfrak{h}_2(t, x_2, y_2)| &\leq \sigma_2(|x_1 - x_2| + |y_1 - y_2|),
\end{aligned}$$

where $\vartheta_1, \vartheta_2, \sigma_1, \sigma_2 \in \Lambda$ and $\vartheta_1, \vartheta_2, \sigma_1$ and σ_2 are continuous functions.
(H$_3$) There exists $r > 0$ satisfying,

$$\begin{aligned}
&T^{\alpha}\tfrac{T|M_1|+1}{\Gamma(\alpha+1)} + \tfrac{T^{\alpha+1}\gamma\ |M_1|\Gamma(\varsigma+1)}{\Gamma(\alpha+1)\Gamma(\varsigma+\tau+1)} \leq \tfrac{r}{2(\vartheta_1(r)+f_1^\star)(\sigma_1(r)+h_1^\star)},\\
&\text{and}\\
&T^{\beta}\tfrac{T|M_2|+1}{\Gamma(\beta+1)} + \tfrac{T^{\beta+1}\delta\ |M_2|\Gamma(\varsigma+1)}{\Gamma(\beta+1)\Gamma(\varsigma+\tau+1)} \leq \tfrac{r}{2(\vartheta_2(r)+f_2^\star)(\sigma_2(r)+h_2^\star)},
\end{aligned} \tag{3.4}$$

where

$$M_1 = \frac{1}{\left[T - \gamma\frac{\zeta\Gamma(\varsigma+\frac{1}{\kappa}+1)}{\Gamma(\varsigma+\frac{1}{\kappa}+\tau+1)}\right]}, \qquad M_2 = \frac{1}{\left[T - \delta\frac{\omega\Gamma(\varsigma+\frac{1}{\kappa}+1)}{\Gamma(\varsigma+\frac{1}{\kappa}+\tau+1)}\right]}.$$

Notice that

$$\mathfrak{h}_1^\star = \sup_{t\in I}\mathfrak{h}_1(t,0,0),\ \ \mathfrak{h}_2^\star = \sup_{t\in I}\mathfrak{h}_2(t,0,0),\ \ \mathfrak{f}_1^\star = \sup_{t\in I}\mathfrak{f}_1(t,0,0),\ \ \mathfrak{f}_2^\star = \sup_{t\in I}\mathfrak{f}_2(t,0,0).$$

Remark 3.2. From assumption (H_2) we deduce,

$$\begin{aligned}
&\upsilon\left(\mathfrak{f}_1(t,\Omega_1,\Omega_2)\right) \leq \vartheta_1\left(\upsilon(\Omega_1)+\upsilon(\Omega_2)\right),\ \upsilon\left(\mathfrak{f}_2(t,\Omega_1,\Omega_2)\right) \leq \vartheta_2\left(\upsilon(\Omega_1)+\upsilon(\Omega_2)\right),\\
&\upsilon\left(\mathfrak{h}_1(t,\Omega_1,\Omega_2)\right) \leq \sigma_1\left(\upsilon(\Omega_1)+\upsilon(\Omega_2)\right),\ \upsilon\left(\mathfrak{h}_2(t,\Omega_1,\Omega_2)\right) \leq \sigma_2\left(\upsilon(\Omega_1)+\upsilon(\Omega_2)\right),
\end{aligned}$$

for any bounded sets $\Omega_1, \Omega_2 \subset C(I, \mathbb{R})$ and for each $t \in I$.

Theorem 3.1. *If $(H_1) - (H_3)$ hold, then the boundary value problem (1.1) has at least one solution.*

Proof. We transform the boundary value problem (1.1) into a fixed point problem. Consider the set

$$\mathfrak{D}_r = \left\{(x,y) \in C(I,\mathbb{R}) \times C(I,\mathbb{R}) : \|(x,y)\| \leq r\right\},$$

where r is defined by (3.4). Clearly, the subset $\mathfrak{D}_r$ is closed, bounded and convex. We define the operator $\mathfrak{N} : C(I,\mathbb{R}) \times C(I,\mathbb{R}) \longrightarrow C(I,\mathbb{R}) \times C(I,\mathbb{R})$

$$\mathfrak{N}(x,y) = \begin{pmatrix} \mathfrak{N}_1(x,y) \\ \mathfrak{N}_2(x,y) \end{pmatrix},$$

with

$$\mathfrak{N}_1(x,y)(t) = \mathfrak{f}_1(t,x(t),y(t)) \times \Bigg(\tfrac{-t\ M_1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}\mathfrak{h}_1(s,x(s),y(s))ds$$

$$+\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\mathfrak{h}_1(s,x(s),y(s))ds$$

$$+\ \tfrac{t\ \gamma\ M_1}{\Gamma(\alpha)} \int_0^\zeta \int_0^r \frac{r^{\kappa\varsigma+\kappa-1}}{(\zeta^\kappa - r^\kappa)^{1-\tau}}(r-s)^{\alpha-1}\mathfrak{h}_1(s,x(s),y(s))dsdr \Bigg),$$

$$\mathfrak{N}_2(x,y)(t) = \mathfrak{f}_2(t,x(t),y(t)) \times \Bigg(\tfrac{-t\ M_2}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1}\mathfrak{h}_2(s,x(s),y(s))ds$$

$$+\tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\mathfrak{h}_2(s,x(s),y(s))ds$$

$$+\ \tfrac{t\ \delta\ M_2}{\Gamma(\beta)} \int_0^\omega \int_0^r \frac{r^{\kappa\varsigma+\kappa-1}}{(\omega^\kappa - r^\kappa)^{1-\tau}}(r-s)^{\beta-1}\mathfrak{h}_2(s,x(s),y(s))dsdr \Bigg).$$

Obviously the fixed points of the operator $\mathfrak{N}$ are solutions of the problem (1.1). Consider the operators $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1$ and $\mathfrak{B}_2$ defined on $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ by

$$\mathfrak{A}_1(x,y)(t) = \mathfrak{f}_1(t,x(t),y(t)),\ \mathfrak{A}_2(x,y)(t) = \mathfrak{f}_2(t,x(t),y(t)).$$

$$\mathfrak{B}_1(x,y)(t) = \Bigg(\tfrac{-t\ M_1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}\mathfrak{h}_1(s,x(s),y(s))ds$$

$$+\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\mathfrak{h}_1(s,x(s),y(s))ds$$

$$+\ \tfrac{t\ \gamma\ M_1}{\Gamma(\alpha)} \int_0^\zeta \int_0^r \frac{r^{\kappa\varsigma+\kappa-1}}{(\zeta^\kappa - r^\kappa)^{1-\tau}}(r-s)^{\alpha-1}\mathfrak{h}_1(s,x(s),y(s))dsdr \Bigg),$$

and

$$\mathfrak{B}_2(x,y)(t) = \Bigg(\tfrac{-t\ M_2}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1}\mathfrak{h}_2(s,x(s),y(s))ds$$

$$+\tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\mathfrak{h}_2(s,x(s),y(s))ds$$

$$+\ \tfrac{t\ \delta\ M_2}{\Gamma(\beta)} \int_0^\omega \int_0^r \frac{r^{\kappa\varsigma+\kappa-1}}{(\omega^\kappa - r^\kappa)^{1-\tau}}(r-s)^{\beta-1}\mathfrak{h}_2(s,x(s),y(s))dsdr \Bigg).$$

Now, we shall show that $\mathfrak{N}$ satisfies all assumptions of Theorem 2.2.
We break the proof into several steps.
Step 1. $\mathfrak{N}$ maps $\mathfrak{D}_r$ into itself.
Let $(x,y)\in\mathfrak{D}_r$, from (H_2), we have

$$\begin{aligned}
&|\mathfrak{N}_1(x,y)(t)| \le |\mathfrak{f}_1(t,x(t),y(t))|\ \Big[\tfrac{1}{\Gamma(\alpha)}t\ |M_1| \int_0^T (T-s)^{\alpha-1}|\mathfrak{h}_1(s,x(s),y(s))|ds \\
&+\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|\mathfrak{h}_1(s,x(s),y(s))|ds \\
&+\tfrac{1}{\Gamma(\alpha)}t\ \gamma\ |M_1| \int_0^r (r-s)^{\alpha-1}I_\kappa^{\varsigma,\tau}|\mathfrak{h}_1(s,x(s),y(s))|(\xi)ds|\Big] \\
&\le |\mathfrak{f}_1(t,x(t),y(t) - \mathfrak{f}_1(t,0,0)| + |f_1(t,0,0)| \\
&\times\Big[\tfrac{1}{\Gamma(\alpha)}t\ |M_1| \int_0^T (T-s)^{\alpha-1}\big(|\mathfrak{h}_1(s,x(s),y(s)) - \mathfrak{h}_1(s,0,0)| + |h_1(s,0,0)|\big)ds \\
&+\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\big(|\mathfrak{h}_1(s,x(s),y(s)) - \mathfrak{h}_1(s,0,0)| + \mathfrak{h}|_1(s,0,0)|\big)ds \\
&+\tfrac{1}{\Gamma(\alpha)}t\ \gamma\ |M_1|\int_0^r (r-s)^{\alpha-1}I_\kappa^{\varsigma,\tau}(|\mathfrak{h}_1(s,x(s),y(s)) - \mathfrak{h}_1(s,0,0)| + |h_1(s,0,0)|)(\xi)ds\Big] \\
&\le (\vartheta_1(\|x\|+\|y\|)+\mathfrak{f}_1^\star)\Big[\tfrac{1}{\Gamma(\alpha)}t\ |M_1|(\sigma_1(\|x\|+\|y\|)+\mathfrak{h}_1^\star) \int_0^T (T-s)^{\alpha-1}ds \\
&+\tfrac{1}{\Gamma(\alpha)}(\sigma_1(\|x\|+\|y\|)+\mathfrak{h}_1^\star) \int_0^t (t-s)^{\alpha-1}ds \\
&+\tfrac{1}{\Gamma(\alpha)}t\ \gamma\ |M_1|(\sigma_1(\|x\|+\|y\|)+\mathfrak{h}_1^\star) \int_0^r (r-s)^{\alpha-1}I_\kappa^{\varsigma,\tau}(1)(\zeta)ds\Big] \\
&\le (\vartheta_1(r)+\mathfrak{f}_1^\star)(\sigma_1(r)+\mathfrak{h}_1^\star)\Big[\tfrac{T^{\alpha+1}|M_1|+T^\alpha}{\Gamma(\alpha+1)} + \tfrac{T^{\alpha+1}\gamma|M_1|\Gamma(\varsigma+1)}{\Gamma(\alpha+1)\Gamma(\varsigma+\tau+1)}\Big].
\end{aligned}$$

Thanks to the condition (H_3), we get

$$\|\mathfrak{N}_1(x,y)\| \le \frac{1}{2}r,$$

As in the previous step, one has,

$$|\mathfrak{N}_2(x,y)(t)| \quad \le (\vartheta_2(r)+\mathfrak{f}_2^\star)(\sigma_2(r)+\mathfrak{h}_2^\star)\Big[T^\beta\tfrac{T|M_2|+1}{\Gamma(\beta+1)} + \tfrac{T^{\beta+1}\delta\ M_2\Gamma(\varsigma+1)}{\Gamma(\beta+1)\Gamma(\varsigma+\tau+1)}\Big].$$

Using condition H_3, it follows,

$$\|\mathfrak{N}_2(x,y)\| \leq \frac{1}{2}r;$$

finally

$$\|\mathfrak{N}(x,y)\|_1 \leq r.$$

and then $\mathfrak{N}$ maps $\mathfrak{D}_r$ into itself.

Step 2. $\mathfrak{N}$ is continuous on $C(I,\mathbb{R}) \times C(I,\mathbb{R})$.
Indeed, let $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ be a sequence such that $(x_n,y_n) \to (x,y)$ in $C(I,\mathbb{R}) \times C(I,\mathbb{R})$. Then

$$\begin{aligned} |\mathfrak{A}_1(x_n,y_n)(t) - \mathfrak{A}_1(x,y)(t)| &= |\mathfrak{f}_1(t,x_n(t),y_n(t)) - \mathfrak{f}_1(t,x(t),y(t))| \\ &\leq \vartheta_1(|x_n(t)-x(t)| + |y_n(t)-y(t)|) \\ &\leq \vartheta_1(\|x_n - x\| + \|y_n - y\|). \end{aligned}$$

From the fact that $\vartheta_1(t) < t$ for any $t > 0$, we conclude that $\vartheta_1(0) = 0$ and $\lim_{t\to 0}\vartheta_1(t) = 0$. Consequently, ϑ_1 is continuous at $t = 0$, this means that

$$\lim_{n\to+\infty} \|\mathfrak{A}_1(x_n,y_n) - \mathfrak{A}_1(x,y)\| = 0;$$

finally $\mathfrak{A}_1$ is continuous on $\mathfrak{D}_r$.
Also, one has

$$\begin{aligned} & |\mathfrak{B}_1(x_n,y_n)(t) - \mathfrak{B}_1(x,y)(t)| \\ \leq\ & \frac{1}{\Gamma(\alpha)}t\ |M_1| \int_0^T (T-s)^{\alpha-1}|\left(|\sigma_1(|x_n(s)-x(s)| + |y_n(s)-y(s))|\right) ds \\ +\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\sigma_1\left(|x_n(s)-x(s)| + |y_n(s)-y(s)|\right) ds \\ +\ & \frac{1}{\Gamma(\alpha)}t\gamma|M_1| \int_0^r (r-s)^{\alpha-1} I_\kappa^{\varsigma,\tau}\left(|\sigma_1(|x_n(s)-x(s)| + |y_n(s)-y(s)|)\right)(\xi) ds \\ \leq\ & \frac{1}{\Gamma(\alpha)}|M_1|\sigma_1(\|x_n-x\| + \|y_n-y\|)\frac{T^{\alpha+1}+1}{\Gamma(\alpha+1)} \\ +\ & \frac{T^{\alpha+1}\sigma_1(\|x_n-x\| + \|y_n-y\|)}{\Gamma(\alpha+1)} \\ +\ & \gamma|M_1|\sigma_1(\|x_n-x\| + \|y_n-y)\|\frac{\Gamma(\varsigma+1)}{\Gamma(\varsigma+\tau+1)}\frac{T^{\alpha+1}+1}{\Gamma(\alpha+1)}. \end{aligned}$$

Arguing as the previous step, one has,

$$\|\mathfrak{B}_1(x_n,y_n) - \mathfrak{B}_1(x,y)\| \to 0 \quad \text{as } n \to \infty,$$

it follows that $\mathfrak{N}_1$ is continuous.
The continuity of $\mathfrak{N}_2 = \mathfrak{A}_2\mathfrak{B}_2$ is proved similarly. Consequently, $\mathfrak{N} = \begin{pmatrix} \mathfrak{N}_1 \\ \mathfrak{N}_2 \end{pmatrix}$ is

continuous.

Step 3. Next, we will prove that $\mathfrak{N}$ maps bounded sets into equi-continuous sets in $\mathfrak{D}_r$.

Let $t_1, t_2 \in I$, $t_1 < t_2$, and $B_{\eta^*} = \{(x,y) \in \mathfrak{D}_r,\ \|(x,y)\|_1 \leq \eta^*\}$ be a bounded set of $\mathfrak{D}_r$, let $(x,y) \in B_{\eta^*}$, it follows then,

$$\begin{aligned}
|\mathfrak{A}_1(x,y)(t_1) - \mathfrak{A}_1(x,y)(t_2)| \quad &= |\mathfrak{f}_1(t_1, x(t_1), y(t_1)) - \mathfrak{f}_1(t_2, x(t_2), y(t_2))| \\
&\leq |\mathfrak{f}_1(t_1, x(t_1), y(t_1)) - \mathfrak{f}_1(t_1, x(t_2), y(t_2))| \\
&+ |\mathfrak{f}_1(t_1, x(t_2), y(t_2)) - \mathfrak{f}_1(t_2, x(t_2), y(t_2))| \\
&\leq \vartheta_1(|x(t_2) - x(t_1)| + |y(t_2) - y(t_1)|) \\
&+ |\mathfrak{f}_1(t_1, x(t_2), y(t_2)) - \mathfrak{f}_1(t_2, x(t_2), y(t_2))|.
\end{aligned}$$

Arguing as in step 2 and from the fact that x and y are continuous, we deduce that

$$\vartheta_1(|x(t_2) - x(t_1)| + |y(t_2) - y(t_1)|)_{t_1 \to t_2} \to 0.$$

On another hand,

$$\begin{aligned}
&|\mathfrak{f}_1(t_1, x(t_2), y(t_2)) - \mathfrak{f}_1(t_2, x(t_2), y(t_2))| \\
&\leq \sup\left\{\mathfrak{f}_1(t,u,v) - \mathfrak{f}_1(s,u,v)|, (r,s) \in [0,T]^2,\ |r-s| \leq \epsilon,\ (u,v) \in [-\eta^*, \eta^*]^2\right\},
\end{aligned}$$

since $t_2 \to t_1$ and bearing in the mind that f is uniformly continuous on any bounded subset of $[0,T] \times \mathbb{R} \times \mathbb{R}$, one has

$$\sup_{\epsilon \to 0}\left\{\mathfrak{f}_1(t,u,v) - \mathfrak{f}_1(s,u,v)|, (r,s) \in [0,T]^2,\ |r-s| \leq \epsilon,\ (u,v) \in [-\eta^*, \eta^*]^2\right\} \to 0;$$

finally $\mathfrak{A}_1(B_{\eta^*})$ is equi-continuous.

A similar argument shows that the operator $\mathfrak{A}_2$ maps bounded sets of $\mathfrak{D}_r$ into equi-continuous ones.

Now, we prove that the operator $\mathfrak{B}_1$ maps bounded set B_{η^*} of $\mathfrak{D}_r$ into an equi-continuous one. Let us consider a nonempty bounded B_{η^*} subset of $\mathfrak{D}_r$. Then, for $(x,y) \in B_{\eta^*}$ and $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$, one has

$$\begin{aligned}
|\mathfrak{B}_1(x,y)(t_2) - \mathfrak{B}_1(x,y)(t_1)| &\leq \frac{1}{\Gamma(\alpha)}(t_2 - t_1)|M_1| \int_0^T (T-s)^{\alpha-1}|\mathfrak{h}_1(s, x(s), y(s))|ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1}\mathfrak{h}_1(s, x(s), y(s))ds \\
- \quad & \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1})\mathfrak{h}_1(s, x(s), y(s))ds \\
+ \quad & \frac{1}{\Gamma(\alpha)}(t_2 - t_1)\gamma|M_1| \int_0^r (r-s)^{\alpha-1} I_\kappa^{\varsigma,\tau}(|\mathfrak{h}_1(s, x(s), y(s))|)(\xi)ds \\
\leq \quad & \frac{1}{\Gamma(\alpha)}(t_2 - t_1)|M_1| \int_0^T (T-s)^{\alpha-1}(|\mathfrak{h}_1(s, x(s), y(s)) - \mathfrak{h}_1(s,0,0)| + |\mathfrak{h}_1(s,0,0)|)ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\left(|\mathfrak{h}_1(s, x(s), y(s)) - \mathfrak{h}_1(s,0,0))| + |\mathfrak{h}_1(s,0,0)|\right)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\,(|\mathfrak{h}_1(s,x(s),y(s))-\mathfrak{h}_1(s,0,0))|+|\mathfrak{h}_1(s,0,0)|)\,ds\\
&+\ \frac{(t_2-t_1)\gamma|M_1|}{\Gamma(\alpha)}\int_0^{r}(r-s)^{\alpha-1}\,(I_\kappa^{\varsigma,\tau}|\mathfrak{h}_1(s,x(s),y(s))-\mathfrak{h}_1(s,0,0)|+|\mathfrak{h}_1(s,0,0)|)\,(\xi)ds\\
&\le\ \frac{1}{\Gamma(\alpha)}(t_2-t_1)|M_1|\,(\sigma_1(\|x\|+\|y\|)+\mathfrak{h}_1^*)\int_0^T(T-s)^{\alpha-1}ds\\
&+\ |\,(\sigma_1(\|x\|+\|y\|)+\mathfrak{h}_1^*)\,\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\
&+\ |\,(\sigma_1(\|x\|+\|y\|)+\mathfrak{h}_1^*)\,\frac{1}{\Gamma(\alpha)}\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})ds\\
&+\ \frac{1}{\Gamma(\alpha)}(t_2-t_1)\ \gamma|M_1|(|\,(\sigma_1(\|x\|+\|y\|)+\mathfrak{h}_1^*))\int_0^r(r-s)^{\alpha-1}|I_\kappa^{\varsigma,\tau}(1)(\xi)|ds\\
&\le\ \frac{T^\alpha(t_2-t_1)|M_1|\,(\sigma_1(\eta^*)+\mathfrak{h}_1^*)}{\Gamma(\alpha)}+\frac{|\,(\sigma_1(\eta^*)+\mathfrak{h}_1^*)\,|\,(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}\\
&+\frac{|\,(\sigma_1(\eta^*)+\mathfrak{h}_1^*)\,|\,(t_2^\alpha-(t_2-t_1)^\alpha-t_1^\alpha)}{\Gamma(\alpha+1)}+\frac{T^\alpha\Gamma(\varsigma+1)(t_2-t_1)\gamma|M_1|(\sigma_1(\eta^*)+\mathfrak{h}_1^*|)}{\Gamma(\alpha+1)\Gamma(\varsigma+\tau+1)}.
\end{aligned}
$$

As $t_1\to t_2$, the right hand side of the above inequality goes to zero, which implies that $\mathfrak{B}_1(B_{\eta^*})$ is equi-continuous, arguing samely as previously one has $\mathfrak{B}_2(B_{\eta^*})$ is equicontinous. Therefore, the operator $\mathfrak{N}$ maps a bounded set into an equicontinuous one ; finally we deal with the proof of the inequality (2.3).

Setting $\mathfrak{V}=\overline{co}(\mathfrak{N}\mathfrak{D}_r)$. Clearly $\mathfrak{V}$ is a bounded, convex and closed subset of $\mathfrak{D}_r$. One remarks that $\mathfrak{N}\mathfrak{V}\subset\mathfrak{N}\mathfrak{D}_r)\subset\mathfrak{V}$, it follows that step 1 and step 2 infer that $\mathfrak{N}:\mathfrak{V}\to\mathfrak{V}$ is bounded and continuous, and therefore the function $t\to\upsilon(t)=(\widetilde{\upsilon}(\mathfrak{V}(t))$ is continuous and bounded on I. From Example 2.1 combined with properties of the measure of noncompactness $\widetilde{\upsilon}$, we have for each $t\in I$,

$$
\begin{aligned}
\widetilde{\upsilon}(\mathfrak{N}(\mathfrak{V})(t)) &= \upsilon(\mathfrak{N}_1(\mathfrak{V})(t))+\upsilon(\mathfrak{N}_2(\mathfrak{V})(t))\\
&\le \upsilon(\mathfrak{A}_1(\mathfrak{V}(t)))\cdot\mathfrak{B}_1(\mathfrak{V}(t)))+\upsilon(\mathfrak{A}_2(\mathfrak{V}(t))\cdot\mathfrak{B}_2(\mathfrak{V}(t)))\\
&\le \|\mathfrak{A}_1(\mathfrak{V}(t))\|\upsilon(\mathfrak{B}_1(\mathfrak{V}(t))+\|\mathfrak{B}_1(\mathfrak{V}(t))\|\upsilon(\mathfrak{A}_1(\mathfrak{V}(t))\\
&+ \|\mathfrak{A}_2(\mathfrak{V}(t))\|\upsilon(\mathfrak{B}_2(\mathfrak{V}(t))+\|\mathfrak{B}_2(\mathfrak{V}(t))\|\upsilon(\mathfrak{A}_2(\mathfrak{V}(t))).
\end{aligned}
$$

We estimate

$$
\begin{gathered}
\|\mathfrak{A}_1(\mathfrak{V}(t))\|,\ \upsilon(\mathfrak{B}_1(\mathfrak{V}(t)),\ \|\mathfrak{B}_1(\mathfrak{V}(t))\|,\ \upsilon(\mathfrak{A}_1(\mathfrak{V}(t))\\
\|\mathfrak{A}_2(\mathfrak{V}(t))\|,\ \upsilon(\mathfrak{B}_2(\mathfrak{V}(t)),\ \|\mathfrak{B}_2(\mathfrak{V}(t))\|,\ \upsilon(\mathfrak{A}_2(\mathfrak{V}(t))).
\end{gathered}
$$

$$
\|\mathfrak{A}_1(\mathfrak{V}(t))\|\le(\vartheta_1(r)+f_1^\star)(\sigma_1(r)+h_1^\star),\ \|\mathfrak{A}_2(\mathfrak{V}(t))\|\le(\vartheta_2(r)+f_2^\star)(\sigma_2(r)+h_2^\star),
$$

we denote by $\mathfrak{V}_1,\ \mathfrak{V}_2$, the natural projection of $\mathfrak{V}\subset\mathfrak{D}_r$ over $C(I,\mathbb{R})$.

$$
\begin{aligned}
&\upsilon(\mathfrak{A}_1(\mathfrak{V}(t)))=\upsilon(f_1(t,\mathfrak{V}_1(t),\mathfrak{V}_2(t))),\\
&\le\vartheta_1(\upsilon(\mathfrak{V}_1(t))+\upsilon(\mathfrak{V}_2(t))),
\end{aligned}
$$

$$\begin{aligned}&\upsilon(\mathfrak{A}_2(\mathfrak{V}(t))) = \upsilon(f_2(t,\mathfrak{V}_1(t),\mathfrak{V}_2(t))),\\ &\leq \vartheta_2(\upsilon(\mathfrak{V}_1(t)) + \upsilon(\mathfrak{V}_2(t))).\end{aligned}$$

Then it follows

$$\widetilde{\upsilon}_c(\mathfrak{A}_1(\mathfrak{V})) \leq \vartheta_1(\widetilde{\upsilon}_c(\mathfrak{V})),\ \widetilde{\upsilon}_c(\mathfrak{A}_2(\mathfrak{V})) \leq \vartheta_2(\widetilde{\upsilon}_c(\mathfrak{V})).$$

$$\|\mathfrak{B}_1(\mathfrak{V}(t))\| \leq N_1(\sigma_1(r) + h_1^\star),\ \|\mathfrak{B}_2(\mathfrak{V}(t))\| \leq N_2(\sigma_2(r) + h_2^\star),$$

where

$$N_1 = \frac{T^{\alpha+1}|M_1|}{\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha+1}\gamma|M_1|\Gamma(\varsigma+1)}{\Gamma(\alpha+1)\Gamma(\varsigma+\tau+1)},$$

$$N_2 = \frac{T^{\beta+1}|M_2|}{\Gamma(\beta+1)} + \frac{T^\beta}{\Gamma(\beta+1)} + \frac{T^{\beta+1}\delta|M_2|\Gamma(\varsigma+1)}{\Gamma(\beta+1)\Gamma(\varsigma+\tau+1)},$$

$$\begin{aligned}\upsilon(\mathfrak{B}_1\mathfrak{V}(t)) \leq &\left(\frac{t}{\Gamma(\alpha)\left|T-\gamma\frac{\zeta\Gamma(\varsigma+\frac{1}{\kappa}+1)}{\Gamma(\varsigma+\frac{1}{\kappa}+\tau+1)}\right|}\int_0^T (T-s)^{\alpha-1}\sigma_1\left(\upsilon(\mathfrak{V}_1(s)+\mathfrak{V}_2(s))\right)ds\right.\\
&+ \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma_1\left(\upsilon(\mathfrak{V}_1(s)+\mathfrak{V}_2(s))\right)ds\\
&+ \frac{t\gamma\kappa\zeta^{-\kappa(\tau+\varsigma)}}{\Gamma(\alpha)\Gamma(\tau)\left|T-\gamma\frac{\zeta\Gamma(\varsigma+\frac{1}{\kappa}+1)}{\Gamma(\varsigma+\frac{1}{\kappa}+\tau+1)}\right|}\\
&\left.\int_0^\zeta\int_0^r \frac{r^{\kappa\varsigma+\kappa-1}}{(\zeta^\kappa-r^\kappa)^{1-\tau}}(r-s)^{\alpha-1}\sigma_1\left(\upsilon(\mathfrak{V}_1(s)+\mathfrak{V}_2(s))\right)dsdr\right)\\
&\leq \sigma_1\left(\widetilde{\upsilon}_c(\mathfrak{V})\right)\left(T^\alpha\tfrac{T|M_1|+1}{\Gamma(\alpha+1)}\right) + \sigma_1\left(\widetilde{\upsilon}_c(\mathfrak{V})\right)\left(\tfrac{|M_1|\gamma\kappa T^{\alpha+1}\Gamma(\varsigma+\frac{\alpha}{\kappa}+1)}{\Gamma(\alpha+1)\Gamma(\varsigma+\frac{\alpha}{\kappa}+\tau+1)}\right);\end{aligned}$$

so,

$$\widetilde{\upsilon}_c(\mathfrak{B}_1\mathfrak{V}) \leq \sigma_1\left(\widetilde{\upsilon}_c(\mathfrak{V})\right)\left(T^\alpha\frac{T|M_1|+1}{\Gamma(\alpha+1)} + \frac{|M_1|\gamma\kappa T^\alpha\Gamma(\varsigma+\frac{\alpha}{\kappa}+1)}{\Gamma(\alpha+1)\Gamma(\varsigma+\frac{\alpha}{\kappa}+\tau+1)}\right).$$

In the same way, we obtain,

$$\widetilde{\upsilon}_c(\mathfrak{B}_2(\mathfrak{V})) \leq \sigma_2\left(\widetilde{\upsilon}_c(\mathfrak{V})\right)\left(T^\beta\frac{T|M_2|+1}{\Gamma(\beta+1)} + \frac{|M_2|\delta\kappa T^\beta\Gamma(\varsigma+\frac{\beta}{\kappa}+1)}{\Gamma(\beta+1)\Gamma(\varsigma+\frac{\beta}{\kappa}+\tau+1)}\right).$$

Setting,

$$\begin{aligned}\Pi \quad = \quad & [((\vartheta_1(r)+f_1^\star)\sigma_1(r)+h_1^\star)\\
&\left(T^\alpha\frac{T|M_1|+1}{\Gamma(\alpha+1)} + \frac{|M_1|\gamma\kappa T^\alpha\Gamma(\varsigma+\frac{\alpha}{\kappa}+1)}{\Gamma(\alpha+1)\Gamma(\varsigma+\frac{\alpha}{\kappa}+\tau+1)}\right)\sigma_1\\
+ \quad & N_1(\sigma_1(r)+h_1^\star)\vartheta_1 + N_2(\sigma_2(r)+h_2^\star)\vartheta_2\\
+ \quad & (\vartheta_2(r)+f_2^\star)(\sigma_2(r)+h_2^\star)\\
&\left.\left(T^\beta\frac{T|M_2|+1}{\Gamma(\beta+1)} + \frac{|M_2|\delta\kappa T^\beta\Gamma(\varsigma+\frac{\beta}{\kappa}+1)}{\Gamma(\beta+1)\Gamma(\varsigma+\frac{\beta}{\kappa}+\tau+1)}\right)\sigma_2\right].\end{aligned}$$

Bearing in mind that $\vartheta_1,\ \vartheta_2,\ \sigma_1,\ \sigma_2$ belong to Λ, and taking account Remark 3.1, it follows that $\Pi \in \Lambda$.

Consequently,

$$\widetilde{v}_c(\mathfrak{N}(\mathfrak{V})) \le \Pi\left(\widetilde{v}_c(\mathfrak{V})\right);$$

finally by using Theorem 2.2, the operator $\mathfrak{N}$ has at least one fixed point in $\mathfrak{D}_r$. □

The paper concludes with an example to illustrate the feasibility of our main result.

4. Example

Consider the following hybrid fractional differential coupled system with Erdélyi-Kober integral boundary conditions,

$$(4.1)\begin{cases} {}^cD^{3/2}\left(\frac{x(t)}{1/2+\ln(|x(t)+y(t)|+1)}\right) + 10^{-2}\arctan|x(t)+y(t)| = 0,\ t\in I=[0,1],\\ {}^cD^{3/2}\left(\frac{x(t)}{1/4+\tanh|x(t)+y(t)|}\right) + 10^{-2}\arctan|x(t)+y(t)| = 0, \quad t\in I=[0,1],\\ x(0)=0, \qquad y(0)=0,\\ x(1) = I_{1/2}^{1/2,1}x(1/2),\\ y(1) = I_{1/2}^{1/2,1}y(1/2). \end{cases}$$

Notice that, the equation (1.1) is an abstract form of (4.1), where $\alpha=\beta=3/2$, $\gamma=\delta=1$, $\varsigma=1/2$, $\kappa=1/2$, $\tau=1$, $\zeta=\omega=1/2$,$\mathfrak{f}_1(t,u,v)=1/2+\ln(|u+v|+1)$, $\mathfrak{f}_2(t,u,v)=1/4+\tanh|u+v|$ and $\mathfrak{h}(t,u,v)=\mathfrak{h}_1(t,u,v)=\mathfrak{h}_2(t,u,v)=10^{-2}\arctan|u+v|$. It is clear that $\mathfrak{f}_1,\mathfrak{f}_2\in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $\mathfrak{h}_1,\mathfrak{h}_2\in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $|\mathfrak{f}_1(t,0,0)|=1/2$, $|\mathfrak{f}_2(t,0,0)|=1/4$ and $|\mathfrak{h}_1(t,0,0)|=|\mathfrak{h}_2(t,0,0)|=0$. Thus, we conclude that condition (H_1) is satisfied.
We recall that :
1) A function $\vartheta:\mathbb{R}^+\to\mathbb{R}^+$ is said to be subadditive if

$$\vartheta(x+y)\le\vartheta(x)+\vartheta(y), \qquad \text{for any } x,y\in\mathbb{R}^+.$$

2) If $\vartheta:\mathbb{R}^+\to\mathbb{R}^+$ is subadditive and $y\le x$, then

$$\vartheta(x)-\vartheta(y)\le\vartheta(x-y).$$

3) If $\vartheta:\mathbb{R}^+\to\mathbb{R}^+$ is subadditive, then

$$|\vartheta(x)-\vartheta(y)|\le\vartheta(|x-y|), \qquad \text{for any } x,y\in\mathbb{R}^+.$$

4) If $\vartheta:\mathbb{R}^+\to\mathbb{R}^+$ is a concave function and $\vartheta(0)=0$. Then ϑ is subadditive.

Remark 4.1. It's clear that the positive value functions ϑ_1, ϑ_2 and σ defined on $\mathbb{R}^+$ by: $\vartheta_1(t)=\ln(1+t)$, $\vartheta_2(t)=\tanh t$ and $\sigma(t)=\arctan t$ are concave.

Since $\ln(1+t) < t$, $\tanh t < t$ and $\arctan t < t$ for $t > 0$ and $\vartheta_1(t) = \ln(1+t)$, $\vartheta_2(t) = \tanh t$, and $\sigma(t) = \arctan t$ are continuous. From Lemma 2.7 we obtain $\lim_{n\to\infty} \vartheta_1^n(t) = \lim_{n\to\infty} \vartheta_2^n(t) = \lim_{n\to\infty} \sigma^n(t) = 0$ for any $t \in \mathbb{R}^+$. Moreover, $\vartheta_1(t) = \ln(1+t)$, $\vartheta_2(t) = \tanh t$, and $\sigma(t) = \arctan t$ are nondecreasing and, therefore, $\vartheta_1,\ \vartheta_2,\ \sigma \in \Lambda$.
Now, we are ready to show that our functions verify hypothesis (H_2). Let x_1, x_2, y_1 and y_2 be in $\mathbb{R}$ and $t \in \mathbb{R}^+$,

$$\begin{aligned}
&|h(t,x_1,y_1) - h(t,x_2,y_2)| = 10^{-2}\Big|\arctan|x_1+y_1| - \arctan|x_2+y_2|\Big| \\
&\le 10^{-2}\arctan\Big|\ |x_1+y_1| - |x_2+y_2|\Big| \\
&\le 10^{-2}\arctan\left(|x_1-x_2| + |y_1-y_2|\right) \\
&\le \sigma\Big(x_1-x_2| + |y_1-y_2|\Big),
\end{aligned}$$

where we have used the nondecreasing character of the inverse tangent function with the fact that $\Big||x_1+y_1| - |x_2+y_2|\Big| \le |x_1-x_2| + |y_1-y_2|$. From Remark 3.1, we deduce that $10^{-2}\arctan t$ is also in Λ.
Also, let x_1, x_2, y_1 and y_2 in $\mathbb{R}$ and $t \in \mathbb{R}^+$. Assume that $|x_1+y_1| > |x_2+y_2|$ (same argument works for $|x_2+y_2| > |x_1+y_1|$), then

$$\begin{aligned}
&|\mathfrak{f}_1(t,x_1,y_1) - \mathfrak{f}_1(t,x_2,y_2)| = \Big|\ln|1+x_1+y_1| - \ln|1+x_2+y_2|\Big| \\
&= \left|\ln\left(\tfrac{1+|x_1+y_1|}{1+|x_2+y_2|}\right)\right| = \left|\ln\left(\tfrac{1+|x_2+y_2|}{1+|x_2+y_2|} + \tfrac{|x_1+y_1|-|x_2+y_2|}{1+|x_2+y_2|}\right)\right| \\
&= \left|\ln\left(1+\tfrac{|x_1+y_1|-|x_2+y_2|}{1+|x_2+y_2|}\right)\right| \le \ln\Big(1+|x_1+y_1| - |x_2+y_2|\Big) \\
&\le \ln\Big(1+|x_1-x_2| + |y_1-y_2|\Big) \le \vartheta_1\Big(|x_1-x_2| + |y_1-y_2|\Big),
\end{aligned}$$

where we have used the nondecreasing character of $\vartheta_1(t) = \ln(1+t)$ for $t \in \mathbb{R}^+$ and the fact that

$$|x_1+y_1| - |x_2+y_2| \le |x_1-x_2| + |y_1-y_2|.$$

Also for x_1, x_2, y_1 and y_2 be in $\mathbb{R}$ and $t \in \mathbb{R}^+$ then

$$\begin{aligned}
&|\mathfrak{f}_2(t,x_1,y_1) - \mathfrak{f}_2(t,x_2,y_2)| = \Big|\tanh|x_1+y_1| - \tanh|x_2+y_2|\Big| \\
&\le \tanh\Big||x_1+y_1| - |x_2+y_2|\Big| \le \tanh\Big(|x_1-x_2| + |y_1-y_2|\Big) \\
&\le \vartheta_2\Big(|x_1-x_2| + |y_1-y_2|\Big).
\end{aligned}$$

This proves that assumption (H_2) of our main result is satisfied. To end the proof, it siffices to prove that the assumption (H_3) is satisfied.

$$M_1 = M_2 = \frac{7}{6}.$$

Condition (H_3) becomes

$$\begin{cases} \frac{28}{6\sqrt{\pi}} + \frac{4}{\sqrt{\pi}} + \frac{28}{6\sqrt{\pi}} \leq \frac{r}{2\left(\ln(1+r)+\frac{1}{2}\right)\left(10^{-2}\arctan(r)\right)}, \\ \frac{28}{6\sqrt{\pi}} + \frac{4}{\sqrt{\pi}} + \frac{28}{6\sqrt{\pi}.} \leq \frac{r}{2(\tanh(r)+1/4)\left(10^{-2}\arctan(r)\right)}, \end{cases}$$

which implies

$$\begin{cases} \frac{80}{6\sqrt{\pi}} \leq \frac{r}{2\left(\ln(1+r)+\frac{1}{2}\right)\left(10^{-2}\arctan(r)\right)}, \\ \frac{80}{6\sqrt{\pi}} \leq \frac{r}{2(\tanh(r)+1/4)\left(10^{-2}\arctan(r)\right)}. \end{cases}$$

The assumption (H_3) is equivalent then to

$$\begin{cases} \left(\ln(1+r)+\frac{1}{2}\right)10^{-2}\arctan(r) \leq \frac{6\sqrt{\pi}}{80}\frac{r}{2}, \\ (\tanh(r)+1/4)\,10^{-2}\arctan(r) \leq \frac{6\sqrt{\pi}}{80}\frac{r}{2}. \end{cases}$$

It is easy to see that these inequalities are satisfied by $r = 1$, moreover,

$$\frac{80}{6\sqrt{\pi}}(10^{-2}\arctan(r)) = \frac{80}{6\sqrt{\pi}}(10^{-2}\arctan(1)) = 0.0598 < \frac{1}{2}.$$

The condition (H_3) is then satisfied, which shows that all assumptions of the Theorem 3.1 hold. Consequently the hybrid fractional differential coupled system (4.1) has at least one solution (x, y) in $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ such that $\|(x,y)\| \leq 1$.

5. Conclusion

In this work, we deal with the problem concerning the existence of solution for a hybrid fractional differential coupled system modeled by the problem (1.1) with Erdélyi-Kober integral boundary conditions. Our contribution in this paper is to make use of the measure of non-compactness combined with a generalization of Darbo's fixed point theorem in contrary to a standard fixed point theorem due to Dhage for hybrid differential equation in Banach algebra, we show then that the Kuratowski measure of non-compactness satisfies a condition denoted (2.2) throughout the paper, we are the first who have proved this property.
The assumed hypotheses have the following goals:
i) In this paper we have assumed a condition (H_1) to ensure the continuity of the operator solution $\mathfrak{N}$.
ii) Hypothesis (H_2) being supposed to prove the boundedness and equi-continuity to make use of Lemma 2.6.
iii) Condition (H_3) guaranteed that $\mathfrak{N}$ maps $\mathfrak{D}_r$ into itself.
These conditions are optimal in the sense that no condition implies the other. We deal in our approach with a generalization of Darbo's fixed point theorem combined with tools from classical functional analysis and measure of non-compactness.

Acknowledgments

Use of AI tools declaration: The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.
Data Availability Statements: The study protocol (hypotheses, methods and analysis plan) was preregistered on June 2024. The lemmas and theorems and analysis tools needed for this manuscript are explicitly acknowledged.
Acknowledgment: The authors would like to express their thanks to the editor and anonymous referees for his/her suggestions and comments that improved the quality of the paper.
Funding: The authors would like to thank university of Tiaret, especially the laboratory of mathematics and computers science for support this work.
The authors were supported in part by Project of research COOL03UN140120220002 supported by high ministry of education...
Conflict of interest: The authors declare no conflict of interest.

REFERENCES

1. A. Aghajani, J. Banas and N. Sebzali: *Some generalizations of Darbo fixed point theorem and applications*. Bull. Belg. Math. Soc. **20** (2013), 345–358.
2. J. Banas and L. Olszowy: *On a class of measures of noncompactness in Banach algebras and their application to nonlinear integral equations*. Z. Anal. Anwend. **28** (2009), 475–498.
3. M. Beddani and B. Hedia: *Existence result for a fractional differential equation involving special derivative*. Moroccan J. of Pure and Appl. Anal. **8**(1) (2022), 67–77.
4. M. Beddani and B. Hedia: *Existence result for fractional differential equation on unbounded domain*. Kragujev. J. Math. **48**(5) (2024), 755–766.
5. M. Beddani and B. Hedia: *Solution sets for fractional differential inclusions*. J. Fractional Calc. Appl. **10**(2) July (2019), 273–289.
6. F. Z. Berrabah, M. Boukrouche and B. Hedia: *Fractional Derivatives with Respect to Time for Non-Classical Heat Problem*. J. Math. Phys. Anal. Geom. **17** (2021), 30–53.
7. F. Z. Berrabah, B. Hedia and J. Henderson: *Fully Hadamard and Erdélyi-Kober-type integral boundary value problem of a coupled system of implicit differential equations*. Turk. J. Math. **43** (2019), 1308–1329.
8. F. Z. Berrabah, B. Hedia and M. Kirane: *Lyapunov- and Hartman–Wintner-type inequalities for a nonlinear fractional BVP with generalized ψ-Hilfer derivative*. Math. Method. Appl. Sci. **44** (2021), 2637–2649.
9. J. Caballero, M. Darwish and K. Sadarangani: *Existence of solutions for a fractional hybrid boundary value problem via measure of noncompactness in Banach algebras*. Topol. Methods. Nonlinear. Anal. **43**(2) (2014), 535–548.
10. G. Darbo: *Punti uniti in transformazioni a condominio non compatto. Rendiconti del Seminario Matematico della Università di Padova*. **24** (1955), 84–92.

11. A. El Sayed, S. Alissa and N. H. Al Maoued: *On A Coupled System of Hybrid Fractional-order Differential Equations in Banach Algebras.* Adv. Dyn. Syst. Appl. **16**(1), 91–112.

12. E. Guariglia: *Fractional calculus of the Lerch zeta function.* Mediterr. J. Math. **19**(109) (2022).

13. E. Guariglia: *Fractional calculus, zeta functions and Shannon entropy.* Open Math. **19**(1) (2021), 87–100.

14. E. Guariglia: *Riemann zeta fractional derivative-functional equation and link with primes.* Adv. Differ. Equ. **2019**(1(261)) (2019).

15. B. Hedia: *Non-local Conditions for Semi-linear Fractional Differential Equations with Hilfer Derivative, Springer proceeding in mathematics and statistics 303.* ICFDA 2018, Amman, Jordan, July 16–18 (2018), 69–83.

16. B. Hedia and K. Benia: *Hybrid implicit fractional integro-differential equations with Hadamard integral boundary conditions.* Dyn. Contin. Discrete. Impuls. Syst. **28** (2021), 207–222.

17. B. Hedia, F. Z. Berrabah and J. Henderson : *Existence results for differential evolution equations with nonlocal conditions in Banach space.* Malaya. J. Mat. **6** (2018), 457–466.

18. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo: *Theory and Applications of Fractional Differential Equations.* Elsevier B. V. Amsterdam, (2006).

19. K. Kuratowski: *Sur les espaces complets.* Fundam. Math. **15** (1930), 301–309.

20. C. Li , X. Dao and P. Guo : *Fractional derivatives in complex planes.* Nonlinear Anal. **71**(5-6) (2009), 1857–1869.

21. S. D. Lin and H. M. Srivastava: *Some families of the Hurwitz-Lerch zeta functions and associated fractional derivative and other integral representations.* Appl. Math. Comput. **154**(1) (2004), 725–733.

22. A. Mathai and H. J. Haubold: *Erdélyi-Kober Fractional Calculus, Springer Briefs in Mathematical Physics.* Germany: Springer **23** (2018).

23. H. M. Srivastava: *An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions.* J. Adv. Engrg. Comput. **5** (2021), 135–166.

24. H. M. Srivastava: *Fractional-order derivatives and integrals: Introductory overview and recent developments.* Kyungpook Math. J. **60** (2020), 73–116.

25. H. M. Srivastava: *Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis.* Iran. J. Sci. Technol. Trans. A Sci. **44**(1) (2020), 327–344.

26. H. M. Srivastava: *Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations.* J. Nonlinear Convex Anal. **22** (2021), 1501–1520.

27. P. Zvada: *Operator of fractional derivative in the complex plane.* Comm. Math. Phys. **192**(2) (1998), 261–285.